CRITICAL POINTS, EXTREME VALUES

GOAL FOR THE NEXT FEW CLASSES:

FIND MIN/MAX OF $f(x,y)$ (OR $g(x,y,z)$)
ON A REGION $D$.

$D$ CAN BE MANY DIFFERENT REGIONS:

$0 \leq x-y \leq 6$

$0 \leq y \leq 1$

$0 \leq x \leq 1$

$X^2 + y^2 \leq 4$

$x \geq 0$

$X + y + z \leq 1$

$X, Y, Z \geq 0$

TWO SOURCES OF DIFFICULTY:

- THE FUNCTION $f$
- THE REGION $D$

STEP 1: FINDING THE CRITICAL POINTS FOR $f(x,y)$ (OR $g(x,y,z)$).
DEF: \( P \) is a critical point for \( f \) if \( \nabla f \big|_P = \vec{0} \) or at least one partial derivative is not defined.

(More precisely: \( f \) is not differentiable at \( P \))

DEF: \((a,b)\) (respectively \((a,b,c)\)) is a local max for \( f(x,y) \) (\( f(x,y,z) \)) if there is a disk (a sphere) around \((a,b)\) \((a,b,c)\) such that for all points \((a',b')\) in the disk

\[
f(a,b) \geq f(a',b')
\]

respectively: for all \((a',b',c')\) in the sphere

\[
f(a,b,c) \geq f(a',b',c')
\]

Local minimum: same with \( \leq \).

Example: Grouse Mtn Peak is a local max for altitude as a function of NS/WE coordinate.

Example: The center of the Sun is a local max for temperature in the Solar System.
**Theorem:** A local min/max can only occur at a critical point.

**Why:**

A local min/max, if \( \nabla f \neq 0 \) then along the direction \( \hat{u} \) of \( \nabla f \), the function increases, but then for some small \( h \)

we must have \( f(p+h\hat{u}) > f(p) \)

and \( f(p-h\hat{u}) < f(p) \) so \( p \) cannot be a max or min.

**E.g. Find the critical points of**

\[ f(x,y) = x^3 + x^2y^2 - y^4 \]

\[ f_x = 3x^2 + 2xy^2 \quad f_y = 2yx^2 - 4y^3 \]

\[ = x(3x + 2y^2) \quad y(2x^2 - 4y^2) \]

\[ \begin{cases} x(3x + 2y^2) = 0 \Rightarrow x = 0 \text{ or } x = -\frac{2}{3}y^2 \\ y(2x^2 - 4y^2) = 0 \Rightarrow y = 0 \text{ or } \sqrt{2}x + 2y = 0 \text{ or } \sqrt{2}x - 2y = 0 \end{cases} \]
INTERSECTIONS: \( x = 0, \ y = 0 \)

AND \[ \begin{cases} x = -\frac{2}{3} y^2 \\ x = \sqrt{2} y \end{cases} \sim \begin{cases} \frac{2}{3} y^2 = \sqrt{2} y \\ x = \sqrt{2} y \end{cases} \sim \]

\[ \begin{cases} y \left( \frac{2}{3} y - \sqrt{2} \right) = 0 \\ - \end{cases} \sim \begin{cases} y = \frac{3}{\sqrt{2}} \\ x = -3 \end{cases} \]

\[ \begin{cases} x = -\frac{2}{3} y^2 \\ x = \sqrt{2} y \end{cases} \sim \begin{cases} y = \frac{-3}{\sqrt{5}} \\ x = -3 \end{cases} \]

So critical points: \((0, 0), (-3, \frac{2}{\sqrt{2}}), (-3, \frac{-3}{\sqrt{5}})\)

OK, WE FOUND THE CRITICAL POINTS. WHAT NOW? WE LOOK AT HIGHER DERIVATIVES.
A critical point can be a:

- LOCAL MAX – SADDLE POINT
- LOCAL MIN – NEITHER

SADDLE POINT:
LOCAL MAX ALONG A DIRECTION,
LOCAL MIN ALONG ANOTHER

\[ f(x, y) = x^2 - y^2 \]

\( (0, 0) \) is a saddle point.

LOCAL MIN IN THIS DIRECTION
LOCAL MAX IN THIS DIRECTION