1) Find the distance from \( P = (5, 3, 3) \) to the line through the point \( Q' = (4, 5, 3) \) and parallel to the planes
\[ H_1: x + 2 = 0 \quad H_2: y + 2z = 15 \]

2) Fund the distance between the lines
\[ \ell_1: x = y - 1 = z - 1 \]
\[ \ell_2: x = -y + 1 = z + 1 \]
MULTIVARIABLE FUNCTIONS

WHAT'S A (REAL-VALUED) FUNCTION?

It's a "black box" which takes elements in some set \( D \) and produces real numbers. In other words, it is a way to assign to each \( \alpha \) in \( D \) a unique output \( f(\alpha) \) in \( \mathbb{R} \).

We are used to
INPUT: IN \( \mathbb{R} \)  OUTPUT: IN \( \mathbb{R} \)

NEW SETTING
INPUT: IN \( \mathbb{R}^2, \mathbb{R}^3 \) OUTPUT: IN \( \mathbb{R} \)
We call the set of inputs of $f$ the domain $\text{D}(f)$, and the set of outputs of $f$ the range $\text{R}(f)$.

These are "built-in" the function $f$; it is often common though that we only have a "formula" for $f$ and we need to describe a domain where $f$ is defined as a function.

E.g. \[ f(x, y, z) = \sqrt{4-x^2-y^2-3z^2} \]

**Domain of $f(x, y, z)$?**

$\mathbb{D} \subseteq \mathbb{R}^3$, we need a positive (or 0) number under the root. This is the only requirement.

$4-x^2-y^2-3z^2 \geq 0$ \quad "FULL ELLIPSOID"

$x^2 + y^2 + 3z^2 \leq 4$

$\frac{x^2}{2^2} + \frac{y^2}{2^2} + \frac{z^2}{(1/\sqrt{3})^2} \leq 1$
RANGE OF $f(x,y,z)$?

Max value under $\sqrt{ }$ is 4
Min value under $\sqrt{ }$ is 0
So range is $[0,2]$.

**Graphs:**

**In 1 variable**

$$y = f(x)$$

$\uparrow$

New coordinate
The graph lives in
$$D \times \mathbb{R} \subseteq \mathbb{R}^2$$

$\uparrow$

Domain $\uparrow$ New variable

**Vertical line test:**

Unicity of output means
A vertical line meets graph at most
In 1 point.

**In 2 variables**

$$D \times \mathbb{R} \subseteq \mathbb{R}^3$$

$(x,y)$ Point in domain

2 extra variable, so the graph

**Not a graph**
IS A SURFACE IN $\mathbb{R}^3$.

\[(x, y, \delta(x, y))\]

\[z = \delta(x, y)\]

**Vertical Line Test**

IS THE SAME

**Not a Graph**

IN 3 DIMENSIONS

WE CANNOT REALLY DRAW STUFF IN 4 DIMENSIONS... $\mathbb{D} \times \mathbb{R} \subset \mathbb{R}^4$

IN GENERAL, EVEN IN THE 2 DIMENSIONS CASE THE GRAPH CAN BE VERY HARD TO DRAW OR UNDERSTAND. WE NEED SOMETHING SIMPLER.
LEVEL CURVES (OR SURFACES)

For a function \( f(x, y) \) of two variables the subset of \( \mathbb{R}^2 \) defined by \( f(x, y) = k \) for a fixed \( k \) is called a level curve for \( f \).

In 3 dimensions the subset of \( \mathbb{R}^3 \) given by \( f(x, y, z) = k \) is called a level surface for \( f \).

**Contour plot of** \( f(x, y) \):

Plot of several equally spaced level curves

\[ f(x, y) = 1, \quad f(x, y) = 2, \quad f(x, y) = 3, \ldots, \quad f(x, y) = 10 \]

Very useful on Wolfram Alpha. Try: contourplot \( \left( \frac{1}{\sqrt{x^2 - y^2}} \right) \)

* More precisely of \( D \subseteq \mathbb{R}^2 \)
E.G. \[ f(x, y) = \frac{1}{\sqrt{x^2 - y^2}} \]

i) FIND THE DOMAIN OF \( f(x, y) \)

TWO REQUIREMENTS:

* NO NEGATIVE VALUE UNDER \( \sqrt{\text{ }} \)
* NO 0 AT DENOMINATOR

So \( x^2 - y^2 > 0 \), \( x^2 > y^2 \sim \sqrt{x^2} > \sqrt{y^2} \)

\( \sim |x| > |y| \)

ii) SKETCH THE LEVEL CURVES

\[ f(x, y) = 1 \quad , \quad f(x, y) = 3 \]

\[ \frac{1}{\sqrt{x^2 - y^2}} = 3 \sim \sqrt{x^2 - y^2} = \frac{1}{3} \sim x^2 - y^2 = \frac{1}{9} \]

\[ \frac{1}{\sqrt{x^2 + x^2}} = 1 \sim \sqrt{x^2 + y^2} = 1 \sim x^2 - y^2 = 1 \]

HYPERBOLAS
iii) Find the range of $f(x,y)$ must be $>0$

$$f(x,y) = k \cdot \frac{1}{\sqrt{x^2 - y^2}} = k \cdot \frac{1}{\sqrt{k^2 - y^2}}$$

$$x^2 - y^2 = \frac{1}{k^2} \quad \left(\frac{1}{k}, 0\right) \text{ is a solution.}$$

So the range is $(0, +\infty)$.

Equivalently: $f(z,0) = \frac{1}{\sqrt{z^2}} = \frac{1}{|z|}$

Is a continuous function whose range is $(0, +\infty)$, so $\mathcal{R}(f)$ must be $(0, +\infty)$ as well.
E.g. \( f(x, y) = \frac{\sqrt{x^2 + y^2 - 4}}{3x - y} \)

i) Domain of \( f(x, y) \)?

We need the stuff under root to not be negative, and the denominator to not be 0.

So \( x^2 + y^2 - 4 \geq 0 \) \& remove inside of circle of radius 2.

\( 3x - y \neq 0 \) \& remove line \( y = 3x \).

Defined at edge, \( = 0 \).

Defined, \( < 0 \).

ii) Range of \( f(x, y) \)?

\( f(x, y) \) can be, \( 0, > 0, < 0 \) and goes to \( \pm \infty \), so \( \mathcal{R}(f) = (-\infty, \infty) = \mathbb{R} \).