BRAIDS, ORDERINGS AND ZERO DIVISORS

DALE ROLFSEN AND JUN ZHU

ABSTRACT. We begin with the observation that the group algebras $\mathbb{C}B_n$ of Artin's braid groups have no zero divisors or nontrivial units. This follows from the recent discovery of Dehornoy that braids can be totally ordered by a relation $<$ which is invariant under left multiplication. We then show that there is no ordering of $B_n,\ n \geq 3$ which is simultaneously left and right invariant. Nevertheless, we argue that the subgroup of pure braids does possess a total ordering which is invariant on both sides. This follows from a general theorem regarding orderability of certain residually nilpotent groups. As an application, we show that the pure braid groups have no generalized torsion elements, although full braid groups do have such elements.

1. Introduction and statement of results

The Artin braid group B_∞ has countably many generators $\sigma_1, \sigma_2, \cdots$ and relations

$$\sigma_j\sigma_k = \sigma_k\sigma_j \quad \text{if } |j-k| > 1, \quad \sigma_j\sigma_k\sigma_j = \sigma_k\sigma_j\sigma_k \quad \text{if } |j-k| = 1.$$

The n-strand braid group B_n is the subgroup generated by $\sigma_j,\ j < n$. There is a natural homomorphism $B_n \to S_n$, where S_n is the permutation group, sending $\sigma_j^2 \to 1$; its kernel is the *pure* braid group $P_n,\ n \leq \infty$. An equivalent geometric description of braids as strings in space, see [Ar], [Bi1], allows us to study knots and links via braid groups. Many well known knot invariants, such as the Alexander and Jones polynomials and their generalizations, quantum invariants, etc., can be obtained from representations of braid groups; see [Bi2], [Jo], for example. Representations of the braid groups are also important in mathematical physics. The group algebras $\mathbb{C}B_n$ arise naturally in the representation theory, and as explained in [Bi2], they also arise in the study of Vassiliev invariants. The following shows these algebras are well-behaved; in particular multiplication in $\mathbb{C}B_n$ is cancellative.

Theorem 1. *The group algebra $\mathbb{C}B_\infty$ has no zero divisors.*

The same is true, of course, for $\mathbb{C}B_n$, $\mathbb{C}P_n$ and the group rings $\mathbb{Z}B_n$ and $\mathbb{Z}P_n$ since they are subrings of $\mathbb{C}B_\infty$. As will be explained in the next section, Theorem 1 is an immediate application of a remarkable construction of Dehornoy.

Theorem 2. *$\mathbb{C}B_\infty$ has no nontrivial units; its invertible elements are exactly the monomials $z\beta$ with $0 \neq z \in \mathbb{C},\ \beta \in B_\infty$.*

The pure and full braid groups share many properties. They are torsion-free, and for finite n both P_n and B_n are of finite cohomological dimension, residually

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finite, Hopfian. On the other hand, as we will show in the remainder of this paper, their orderability properties are quite different.

Theorem 3. *The pure braid group P_∞ has a (strict total) ordering which respects both left and right multiplication.*

Theorem 4. *If $3 \leq n \leq \infty$, the full braid group B_n cannot be given an ordering which respects both left and right multiplication.*

Theorem 3 is an application of the following general result, which does not seem to be in the literature, although it may be known to some experts in the area. We are indebted to A. Rhemtulla for pointing out that it is implicit in Chapter 2 of [MR].

Theorem 5. *Let G be a group whose lower central series $G = \gamma_1(G) \supset \gamma_2(G) \supset \cdots$ satisfies (1) $\cap_{k=1}^{\infty}\gamma_k(G) = 1$ and (2) $\gamma_k(G)/\gamma_{k+1}(G)$ is torsion-free for every k. Then G has a strict total ordering which respects both left and right multiplication.*

Proof of these theorems, and some related results, will occupy the remainder of the paper.

2. Ordered groups and their group rings

Let G be a group, and consider a relation $<$ on G which is a strict total ordering. That is, for every $g, h \in G$, exactly one of $g < h$, $h < g$ or $g = h$ holds, and $g < h, h < k$ implies $g < k$. The ordering will be called *right invariant* if for all $g, h, k \in G, g < h$ implies $gk < hk$. A group G which possesses a right invariant ordering is said to be *right orderable*. The concept of left orderable is defined similarly, and it is easy to see that a right orderable group is also left orderable, but generally by a different ordering. If some strict total ordering of G simultaneously respects multiplication on *both* sides, then G is said to be *orderable*. Orderable groups have been studied for over fifty years; good general references are [MR] and [Pa], the latter emphasizing applications to group rings. They are rather special, for example it is easily seen that right-orderable groups must be torsion-free (but not conversely). Following is another algebraic reason to be interested in orderings.

Proposition 6. *Suppose G is a left orderable (or right orderable) group and that R is a ring with no zero divisors. Then the group ring RG has no zero divisors. Moreover, the only units of RG are the monomials, rg, with r invertible in R.*

We will include a proof of this well-known result for the reader's convenience, but first some comments on zero divisors are appropriate. If g is a torsion element of a group G, meaning $g \neq 1$ but $g^k = 1$ for some $k > 1$, consider the following product in the integral group ring $\mathbb{Z}G$:

$$(1-g)(1+g+g^2+\cdots+g^{k-1}) = (1-g^k) = 0.$$

This shows that groups having torsion also have nontrivial zero divisors in their group rings. An old conjecture attributed to Kaplanski, and still unsolved in general, is that if G is torsion-free, then $\mathbb{Z}G$ has no zero divisors.

Proof of Proposition 6: Assume G is left-orderable and consider a product in the group ring:

$$(r_1g_1 + \cdots + r_pg_p)(s_1h_1 + \cdots + s_qh_q) = \sum_{i,j}(r_is_j)(g_ih_j)$$

with all r_i and s_j nonzero ring elements, $g_i, h_j \in G$.

If such a product is zero, it means each term is cancelled by some other term(s) in the product. We assume the notation chosen so that $h_1 < h_2 < \cdots < h_q$. Among all pq terms of the product, consider a term $(r_i s_j)(g_i h_j)$ such that $g_i h_j$ is minimal in the ordering. In particular, $g_i h_j \leq g_i h_1$, but by left invariance, the opposite inequality also holds, so that $g_i h_j = g_i h_1$ and so $h_j = h_1$. We've shown that if $g_k h_l$ is any minimal product among these terms, we must have $l = j = 1$, and therefore $g_k = g_i$. In other words, $g_i h_j$ is the *unique* minimal product in the expansion, and since $r_i s_j \neq 0$ the product in the group ring cannot be zero. To prove the last sentence of the proposition, observe there is also a unique maximal element in the product, and if either p or q is greater than one, it must be different from the minimal one. It follows that the product cannot equal 1 in the group ring. □

Proof of Theorems 1 and 2: These follow from Proposition 6 and the following theorem of Dehornoy. □

Dehornoy's Theorem [De1,2]: *B_∞ is left-orderable.*

Again for the reader's convenience, we briefly describe Dehornoy's ordering [D1,2]. In general, for a left-invariant ordering, the set of "positive" elements $P = \{g \in G; 1 < g\}$ satisfies: (1) $P^2 \subset P$, that is P is closed under multiplication, and (2) G is the *disjoint* union $G = P \cup P^{-1} \cup \{1\}$. On the other hand, if one can find *any* subset P of a group G satisfying conditions (1) and (2), then it is easy to verify that the recipe, $x < y$ if and only if $x^{-1}y \in P$, defines a left-invariant ordering on G. In the case of the braid group B_∞, Dehornoy defines the set P to be the set of all braids which can be expressed as a word in the generators σ_j in such a way that the generator with the lowest subscript appearing has only positive exponent. Verification that this choice of P satisfies (1) is easy, but Dehornoy's proof of (2) uses difficult technical results of set theory and "left-distributive" systems. An alternative proof of Dehornoy's theorem, using topological techniques, appears in [FGRRW].

3. Generalized torsion

Definition: An element g in a group G is said to be a *generalized torsion* element if $g \neq 1$ and there exist $x_i \in G,\ 1 \leq i \leq k$, such that

$$(x_1 g x_1^{-1})(x_2 g x_2^{-1}) \cdots (x_k g x_k^{-1}) = 1.$$

Lemma 7. *An orderable group contains no generalized torsion element.*

Proof. Suppose $1 \neq g$ in the orderable group G. Assume $1 < g$, the case $g < 1$ being similar. Then use left and right invariance to conclude that any conjugate of g is also greater than 1 and therefore

$$(x_1 g x_1^{-1})(x_2 g x_2^{-1}) \cdots (x_k g x_k^{-1}) > 1,$$

so g cannot be a generalized torsion element. □

Once Theorem 3 is proved, we have the following consequence.

Corollary 8. *The pure braid groups have no generalized torsion.* □

Theorem 9. *The braid groups $B_n,\ 3 \leq n \leq \infty$, contain generalized torsion elements.*

Proof. It suffices to consider B_3. Let $\Delta = \sigma_1\sigma_2\sigma_1$ and $a = \sigma_2^{-1}\sigma_1$. It is then an easy job to verify that $\Delta a \Delta^{-1} a = 1$. □

Proof of Theorem 4: Follows directly from Theorem 9 and Lemma 7. □

It is interesting to note that the Klein bottle group $< x, y; xyx^{-1} = y^{-1} >$ is another example which is torsion free but not generalized torsion free. It embeds in B_3 by sending x to Δ and y to a.

4. Lower central series

Recall that the lower central series $\gamma_i(G)$ of a group G is defined inductively by $\gamma_1(G) = G$ and $\gamma_{k+1}(G) = [\gamma_k(G), G]$, where $[A, B]$ denotes the subgroup generated by all commutators $aba^{-1}b^{-1}$ for $a \in A$ and $b \in B$. If some $\gamma_i(G) = \{1\}$, G is said to be *nilpotent.* If $\cap_{k=1}^{\infty}\gamma_k(G) = \{1\}$, G is said to be *residually nilpotent.*

We will need the following lemmas from Passman [Pa].

Lemma 10. *If G is a torsion-free nilpotent group, then G is orderable.* □

Lemma 11. *If G has a family of normal subgroups H_r with $\cap_r H_r = \{1\}$ and such that each quotient G/H_r is orderable, then G is orderable.* □

Lemma 12. *A group G is orderable (or right-orderable) if and only if the same is true of all its finitely-generated subgroups.* □

Proof of Theorem 5: Let G denote a residually nilpotent group such that $\gamma_k(G)/\gamma_{k+1}(G)$ is torsion-free for every k. Denote $H_k = \gamma_k(G)$. By the last lemma, it suffices to show that each quotient G/H_k is orderable. To this end, we need to show that G/H_k is torsion-free, since it is obviously nilpotent. We use induction. Clearly $G/H_1 = 1$ is torsion free. Assume G/H_k is torsion free and consider the exact sequence

$$1 \longrightarrow H_k/H_{k+1} \longrightarrow G/H_{k+1} \longrightarrow G/H_k \longrightarrow 1$$

Suppose $g \in G/H_{k+1}$ satisfies $g^p = 1$. Then the image of g in G/H_k is also a torsion element. By inductive assumption, that image has to be the identity, so exactness implies $g \in H_k/H_{k+1}$, which has no torsion by hypothesis. Therefore $g = 1$. □

Proof of Theorem 3: According to Falk and Randall [FR], the pure braid groups P_n satisfy the hypothesis of Theorem 5, and hence are orderable. Lemma 12 implies P_∞ is also orderable. □

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Department of Mathematics, University of British Columbia, Vancouver, B.C. Canada V6T 1Z2

E-mail address: rolfsen@@math.ubc.ca

Department of Mathematics, University of British Columbia, Vancouver, B.C. Canada V6T 1Z2

E-mail address: zhuj@@math.ubc.ca