On orderability of fibred knot groups

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(*Received* 3 *September* 2001; *revised* 10 *June* 2002)

Abstract

It is known that knot groups are right-orderable, and that many of them are not bi-orderable. Here we show that certain fibred knots in S^3 (or in a homology sphere) do have bi-orderable fundamental group. In particular, this holds for fibred knots, such as 4_1, for which the Alexander polynomial has all roots real and positive. This is an application of the construction of orderings of groups, which are moreover invariant with respect to a certain automorphism.

1. *Introduction*

A group is *right-orderable* (RO) if its set of elements can be given a strict total ordering which is invariant under right multiplication: $x < y$ implies $xz < yz$. A right-orderable group is easily seen to be left orderable, by a different ordering (compare inverses), but if it has an ordering which is simultaneously left and right invariant, it is said to be *orderable*, or "bi-orderable" for emphasis. See [**8**] and [**9**]. Our application to knot theory is the following.

Theorem 1·1. *If K is a fibred knot in S^3, or in any homology 3-sphere, such that all the roots of its Alexander polynomial $\Delta_K(t)$ are real and positive, then its knot group $\pi_1(S^3 \setminus K)$ is bi-orderable.*

It is a special case of a more general result regarding fibrations. In the next section we discuss the behavior of group ordering under extensions and apply this to fundamental groups of manifolds which fibre over S^1. A key problem is to find bi-orderings of a group, invariant under some automorphism(s). A final section is devoted to solving this problem, provided the group is free and the automorphism, on the homology level as a linear mapping, has all eigenvalues real and positive.

2. *Extensions and fibrations*

Consider a normal subgroup K of a group G, with quotient H, i.e. an exact sequence

$$1 \longrightarrow K \xrightarrow{i} G \xrightarrow{p} H \longrightarrow 1.$$

Moreover, suppose K and H are right-ordered. We can define an ordering of G by declaring that $g < g'$ if and only if either $p(g) <_H p(g')$ or else $p(g) = p(g')$ and $1 <_K g'g^{-1}$.

PROPOSITION 2·1. *If K and H are right-ordered, then the ordering described above is a right-ordering of G. If K and H are bi-ordered, this ordering is a bi-ordering of G if and only if the ordering of K is invariant under conjugation by elements of G.*

Proof. Routine, and left to the reader.

Of particular relevance to this paper are HNN extensions. If K is a group, and $\varphi: K \to K$ an automorphism, the corresponding HNN extension G has presentation consisting of the generators of K plus a new generator t, and the relations of K together with relations $t^{-1}kt = \varphi(k)$ for all generators $k \in K$. Here we have an exact sequence $1 \to K \to G \to \mathbb{Z} \to 1$.

COROLLARY 2·2. *If K is RO, so is the HNN extension G. If K is bi-ordered, then G is bi-orderable if the automorphism φ preserves order*: $k < k' \Leftrightarrow \varphi(k) < \varphi(k')$.

Now consider a fibration $p: E \to B$ with fibre F. There is an exact homotopy sequence

$$\cdots \longrightarrow \pi_2(B) \longrightarrow \pi_1(F) \xrightarrow{i_*} \pi_1(E) \xrightarrow{p_*} \pi_1(B) \longrightarrow 1.$$

COROLLARY 2·3. *If $i_*\pi_1(F)$ and $\pi_1(B)$ are right-orderable, then so is $\pi_1(E)$. If $\pi_1(B)$ is bi-orderable and $i_*\pi_1(F)$ has a bi-ordering invariant under conjugation by $\pi_1(E)$, then $\pi_1(E)$ is bi-orderable.*

Now consider the special case of a manifold X^n which fibres over S^1, with fibre Y^{n-1}. One may represent X as a product $Y \times [0,1]$ modulo the identification $(y,1) \sim (f(y),0)$. Here $f: Y \to Y$ is a homeomorphism; one calls f or its induced mappings on homology or homotopy groups, the associated *monodromy*. We may regard X as the mapping torus of f.

PROPOSITION 2·4. *If the manifold X fibres over S^1, with fibre Y, and $\pi_1(Y)$ is RO, then so is $\pi_1(X)$. If $\pi_1(Y)$ is bi-orderable, by an ordering preserved by the monodromy $f_*: \pi_1(Y) \to \pi_1(Y)$, then $\pi_1(X)$ is bi-orderable.*

Proof. This is an application of Corollary 2·2 and the fact that $\pi_1(X)$ is an HNN extension of $\pi_1(Y)$, corresponding to the automorphism f_*.

COROLLARY 2·5. *If the 3-manifold X^3 fibres over S^1, then $\pi_1(X)$ is right-orderable, unless the fibre is a projective plane $\mathbb{P}^2$.*

Proof. The group of every surface except $\mathbb{P}^2$ is right-orderable. See [**12**].

This raises the question: given a bi-orderable group and an automorphism, can one find a biordering of the group, which is also invariant under the automorphism? The answer may be yes or no, and in general, this seems a difficult problem. If the automorphism has a finite nontrivial orbit, the answer is no, that is, no invariant ordering exists, by an easy argument. However, there is one reasonably general situation in which we can establish sufficient conditions for a “yes” answer.

Suppose F is a free group with finite basis $\{x_1, \dots, x_k\}$ and $\varphi: F \to F$ is an automorphism. Consider the abelianization $\varphi_{ab}: \mathbb{Z}^k \to \mathbb{Z}^k$, which may be considered a k by k matrix of integers, using the images of the x_i as a basis. The eigenvalues of φ_{ab} are, of course, the roots of the characteristic polynomial $\det(\varphi_{ab} - tI)$, where I is the identity k by k matrix. They are, in general, a set of k complex numbers, possibly with multiplicities.

THEOREM 2·6. *Given the free group automorphism $\varphi: F \to F$ it is possible to bi-order F by an ordering which is invariant under φ provided the eigenvalues of the abelianization φ_{ab} are all real and positive (multiplicities allowed).*

The proof of this is rather involved, and will be the subject of the final section. It has an immediate application to fibrations.

THEOREM 2·7. *Suppose X^n fibres over S^1 with fibre Y^{n-1}. If $\pi_1(Y)$ is a free group and the homology monodromy $H_1 f: H_1(Y) \to H_1(Y)$ has only real positive eigenvalues, then $\pi_1(X)$ is bi-orderable.*

3. *Fibred knots*

We recall that a *link* is a pair (S^3, L), where L is a smooth compact 1-manifold in the 3-sphere; if L has a single component it is called a *knot*. The corresponding link group is $\pi_1(S^3 \setminus L)$. A fibred knot or link L is one for which $S^3 \setminus L$ fibres over S^1, with fibres open surfaces, each of whose closures has L as boundary. The Alexander polynomial of a fibred knot is also the characteristic polynomial of its homology monodromy (see e.g [**11**]). Thus Theorem 1·1 follows from Theorem 2·7; fibred knots with positive-root polynomials have bi-ordered groups. To put this in perspective consider the following.

PROPOSITION 3·1. *Classical link groups are right-orderable.*

Proof. This has been noted by Howie and Short [**5**]. The argument is to first observe that, since a free product of groups is RO if and only if each of them is RO, we can assume the link complement is irreducible. In [**5**] it is shown that if M is orientable, irreducible and has positive first Betti number, then $\pi_1(M)$ is locally indicable, meaning that any finitely generated subgroup has $\mathbb{Z}$ as a quotient. According to a theorem of Burns and Hale, [**3**], locally indicable groups are right-orderable.

PROPOSITION 3·2. *Torus knot groups are not bi-orderable. The same holds for satellites of torus knots, e.g. complex algebraic knots in the sense of Milnor* [**6**]*, and for groups of nontrivial cables of arbitrary knots.*

Proof. The group of a torus knot has a presentation $\langle x, y : x^p = y^q \rangle$ where p and q are relatively prime integers greater than 1. In this group, it is easily established that x and y do not commute, but that their powers $x^p = y^q$ are central (in fact generate the center of the knot group). That this group is not bi-orderable follows from the lemma below. Any satellite of a torus knot contains the torus knot's group as a subgroup, and therefore its group could not be bi-ordered either. The same is true of a (p, q)-cable, as long as $|p|$ and $|q|$ are greater than 1.

LEMMA 3·3. *Suppose G is a group containing elements g and h which do not commute, but some power of g commutes with h. Then G is not bi-orderable.*

Proof. Suppose there were an ordering "$<$" on G invariant under multiplication on both sides, and therefore under conjugation. Suppose $g^{-1}hg < h$. Then $g^{-2}hg^2 < g^{-1}hg$ by invariance and by transitivity $g^{-2}hg^2 < h$. An easy induction shows that $g^{-n}hg^n < h$, for all n, contradicting the assumption that some power of g commutes with h. If $h < g^{-1}hg$ a similar contradiction arises.

The question had been raised, whether there exist *any* knot groups (other than $\mathbb{Z}$, the group of the trivial knot) which are bi-orderable. This was the motivation for the present paper and was answered by Theorem 1·1.

Corollary 3·4. *The group of the figure-eight knot 4_1 is bi-orderable.*

Proof. The knot 4_1 has $\Delta = t^2 - 3t + 1$ whose roots are $(3 \pm \sqrt{5})/2$.

To our knowledge, this is the first known nontrivial bi-ordered knot group. It is interesting to note that 4_1 can be realized as the link of an isolated singularity of a real algebraic variety in R^4, c.f. [**10**], but not that of a complex curve in C^2. On the other hand its knot group is better behaved, in terms of orderings, than those of complex algebraic knots.

We recall that a knot polynomial Δ is the Alexander polynomial of some fibred knot in S^3 if and only if it is monic. In other words, a polynomial

$$\Delta = a_0 + a_1 t + \cdots + a_r t^r$$

is a fibred knot polynomial if and only if r is even and:

$$\forall i, a_i = a_{r-i}, \quad \Delta(1) = \Sigma a_i = \pm 1, \quad a_0 = a_r = 1.$$

The condition of having, in addition, all roots real and positive seems to be rather uncommon. We count 121 nontrivial prime fibred knots of fewer than 11 crossings. According to [**2**] in that range, the fibred knot conditions on its Alexander polynomial are not only necessary, but also sufficient, that the knot be fibred. Besides 4_1, only two other prime knots of fewer than 11 crossings have polynomials which satisfy the conditions of the Theorem 1.1, namely $8_{12} : \Delta = t^4 - 7t^3 + 13t^2 - 7^t + 1$ and $10_{137} : \Delta = (t^2 - 3t + 1)^2$.

On the other hand, there are infinitely many fibred knot polynomials with the property of having only positive real roots. It is known [**7**] that, in general, each can be realized by infinitely many fibred knots. The only one of degree 2 is $\Delta = t^2 - 3t + 1$. In degree 4, it is not difficult to show that the class of such polynomials is exactly those of the form $\Delta = t^4 - at^3 + (2a-1)t^2 - at + 1$, for integers $a > 5$. To see this, note that any polynomial with all real positive roots must be alternating, thus our degree 4 polynomial has the form $\Delta = t^4 - at^3 + bt^2 - at + 1$ with a, b positive integers and where moreover $b = 2a - 2 \pm 1$. Symmetrizing, we have that $t^2\Delta(t) = \hat{\Delta}(t + t^{-1})$, where $\hat{\Delta}(u) = u^2 - au + b - 2$. We also see that Δ has all roots t real and positive if and only if all roots u of $\hat{\Delta}(u)$ are real and > 2. This happens only when $b = 2a - 1$ and $a > 5$.

4. *Invariant orderings*

It is well known [**8**] that free groups are bi-orderable. Our aim in this section is to find a bi-ordering which is also invariant with respect to a given automorphism, and in particular to prove Theorem 2·6.

First we consider the analogous problem of ordering a k-dimensional real vector space V, by an ordering which is to be invariant with respect to vector addition and under an invertible linear transformation $L: V \to V$. If L has a finite orbit (other than that of the zero vector), this will be impossible: this corresponds to an eigenvalue which is a (complex) root of unity. It is also clear that negative real eigenvalues pose a problem, and we do not know how to deal with complex eigenvalues in general. On the other hand, suppose all the eigenvalues $\lambda_1, ..., \lambda_k$ of L are real and positive. It is standard linear algebra that, although L may not be diagonalizable, there exists a basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$ with respect to which L has a matrix which is upper triangular, with the eigenvalues on the diagonal. Let $(x_1, \ldots, x_k)$ denote the coordinates of a vector $\mathbf{x}$ in such a basis, that is $\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k$. We now order the vectors $\mathbf{x}, \mathbf{y} \in V$ by (reverse) lexicographic ordering using these coordinates. In other words, $\mathbf{x} < \mathbf{y} \Leftrightarrow x_i < y_i$ (under the usual ordering of $\mathbb{R}$) at the last i for which the coordinates differ. It is a routine exercise to verify the following, noting that $L(\mathbf{v}_i) = \lambda_i\mathbf{v}_i +$ some fixed linear combination of $\mathbf{v}_j, j > i$.

PROPOSITION 4·1. *If $L: V \to V$ is a linear transformation of a real vector space, and the eigenvalues of L are all real and positive, then the (reverse) lexicographic ordering of V in a basis as described above, is invariant under vector addition and under L, that is $\mathbf{x} < \mathbf{y}$ if and only if $L(\mathbf{x}) < L(\mathbf{y})$.*

COROLLARY 4·2. *If $L: V \to V$ is a linear transformation of a real vector space, and the eigenvalues of L are all real and positive, then each tensor power $V^{\otimes p}$ can be bi-ordered (as an additive group) by an ordering invariant under the induced linear mapping $L^{\otimes p}: V^{\otimes p} \to V^{\otimes p}$.*

Proof. The eigenvalues of $L^{\otimes p}$ are products of eigenvalues of L and therefore real and positive.

COROLLARY 4·3. *If $h: H \to H$ is an automorphism of a free abelian group $H \cong \mathbb{Z}^k$, all of whose eigenvalues are real and positive, then one can bi-order H by an ordering that is invariant under h. Moreover, the tensor powers $H^{\otimes p}$ can be bi-ordered by an ordering invariant under $h^{\otimes p}$.*

Proof. Just apply Proposition 4·1 and Corollary 4·2 to the real vector space $V = H \otimes \mathbb{R}$ and $L = h \otimes 1$, then restrict.

Of course, in the above context of abelian groups, bi-orderability is equivalent to right-orderability. We are now ready to turn attention to the proof of Theorem 2·6, involving free nonabelian groups. The appropriate ordering will be defined using the so-called free Lie algebra, which involves the lower central series. It is a well-known technique for ordering residually nilpotent-torsion-free groups, with the added feature of attention to the automorphism. We recall that the lower central series of a group G is defined by $G_1 = G$ and $G_{k+1} = [G, G_k]$, the subgroup generated by commutators $[g, h] = ghg^{-1}h^{-1}, g \in G, h \in G_k$. The quotients G_k/G_{k+1} are abelian groups, finitely generated if G is. Suppose we know that

$$(*)\ G_k/G_{k+1} \text{ is torsion-free, and } \cap_{k=1}^{\infty} G_k = \{e\}.$$

Choose an arbitrary bi-ordering $<_k$ for each of the groups G_k/G_{k+1}, which is certainly possible since they are free abelian. Then for any distinct elements $g, h \in G$ let $k = k(g, h)$ be the unique integer such that $hg^{-1} \in G_k \setminus G_{k+1}$, so the class $[hg^{-1}]$ in G_k/G_{k+1} is not the identity. If $[1] <_k [hg^{-1}]$, define $g < h$ in G, otherwise say $h < g$.

PROPOSITION 4·4. *If G satisfies $(*)$, then G is bi-ordered by $<$, as defined above. If*

$\varphi: G \to G$ is an automorphism, and each of the orderings $<_k$ is invariant under the induced mapping $\varphi_k: G_k/G_{k+1} \to G_k/G_{k+1}$, then $<$ is φ-invariant.

Proof. The proof is routine.

We now turn to proving Theorem 2·6. Let $\mathbb{Z}F$ be the group ring of the free group F, with integer coefficients, $\epsilon: \mathbb{Z}F \to \mathbb{Z}$ the augmentation map sending $\sum_i n_i g_i$ to $\sum_i n_i$. We denote by I the two-sided ideal $I = \ker(\epsilon)$ and I^k, the kth power of I in $\mathbb{Z}F$.

According to [**4**], section 4.5, $z \in F_k$ if and only if $z - 1 \in I^k$, where F_k denotes the kth term of the lower central series of F. This implies that the map

$$F_k/F_{k+1} \xrightarrow{\sigma} I^k/I^{k+1}$$

given by $[z] \to [z-1]$ is a well-defined injective homomorphism of abelian groups. Here, $[\cdot]$ denotes the class in the appropriate quotient.

Suppose $z_1, \ldots, z_n$ generate F, let $H = F/[F,F]$. The additive group I^k/I^{k+1} has a basis of elements of the form $[(z_{i_1} - 1) \cdots (z_{i_k} - 1)]$. We may identify I^k/I^{k+1} with the tensor power $H^{\otimes k}$, via the mapping

$$[(z_{i_1} - 1) \cdots (z_{i_k} - 1)] \longrightarrow a_{i_1} \otimes \cdots \otimes a_{i_k},$$

where a_i is the image of z_i under the canonical homomorphism $F \to H$.

LEMMA 4·5. *Let $\varphi: F \to F$ be a homomorphism of the free group F and let $\varphi_{ab}: H \to H$ be its abelianization. Then the following diagram is commutative*:

$$\begin{array}{ccccc} F_k/F_{k+1} & \stackrel{\sigma}{\hookrightarrow} & I^k/I^{k+1} & \cong & H^{\otimes k} \\ \downarrow \varphi_k & & \downarrow \varphi'_k & & \downarrow \varphi_{ab}^{\otimes k} \\ F_k/F_{k+1} & \stackrel{\sigma}{\hookrightarrow} & I^k/I^{k+1} & \cong & H^{\otimes k} \end{array}$$

Here φ_k and φ'_k are the maps induced by φ and $\varphi_{ab}^{\otimes k}$ is the tensor power of φ_{ab}.

Proof. Commutativity of the left-hand square is obvious, so we only have to verify the right-hand square commutes. By definition

$$\varphi'_k[(z_{i_1} - 1) \cdots (z_{i_k} - 1)] = [(\varphi(z_{i_1}) - 1) \cdots (\varphi(z_{i_k}) - 1)].$$

According to the fundamental theorem of the Fox free calculus (see [**1**], prop. 3.4):

$$\varphi(w) - 1 = \sum_{j=1}^{n} \epsilon\left(\frac{\partial \varphi(w)}{\partial z_j}\right)(z_j - 1) + O(2),$$

where $O(2) \in I^2$ and so

$$(\varphi(z_{i_1}) - 1) \cdots (\varphi(z_{i_k}) - 1) =$$

$$\left(\sum_{j_1=1}^{n} \epsilon\left(\frac{\partial \varphi(z_{i_1})}{\partial z_{j_1}}\right)(z_{j_1} - 1)\right) \cdots \left(\sum_{j_k=1}^{n} \epsilon\left(\frac{\partial \varphi(z_{i_k})}{\partial z_{j_k}}\right)(z_{j_k} - 1)\right) + O(k+1),$$

where $O(k+1) \in I^{k+1}$. Using the identification of I^k/I^{k+1} with $H^{\otimes k}$ we see that

$$\varphi'_k[(z_{i_1}-1)\cdots(z_{i_k}-1)] = \left(\sum_{j_1=1}^{n} \epsilon\left(\frac{\partial\varphi(z_{i_1})}{\partial z_{j_1}}\right) a_{j_1}\right) \otimes\cdots\otimes \left(\sum_{j_k=1}^{n} \epsilon\left(\frac{\partial\varphi(z_{i_k})}{\partial z_{j_k}}\right) a_{j_k}\right).$$

It is well known that the matrix of $\varphi_{ab}\colon H \to H$ in the basis $\{a_1, \ldots a_n\}$ is the Jacobian matrix $(\epsilon(\partial\varphi(z_i)/\partial z_j))$. In other words,

$$\sum_{j=1}^{n} \epsilon\left(\frac{\partial\varphi(z_i)}{\partial z_j}\right) a_j = \varphi_{ab}(a_i).$$

This implies the identity

$$\varphi'_k[(z_{i_1}-1)\cdots(z_{i_k}-1)] = \varphi_{ab}(a_{i_1}) \otimes\cdots\otimes \varphi_{ab}(a_{i_k})$$

which proves the lemma.

Theorem 2·6 follows from Lemma 4·5, Corollary 4·3 and Proposition 4·4, and all the results of the paper are proven.

We conclude with the question: which other knot or link groups are bi-orderable, in classical and higher dimensions?

Acknowledgements. We wish to thank Steve Boyer, Daniel Lines, Luis Paris, Hamish Short and Bert Wiest for helpful discussions regarding this project. The second author thanks the Canadian Natural Sciences and Engineering Research Council for support of this research, and the Laboratoire de Topologie, Université de Bourgogne for their hospitality during the collaboration of the authors.

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