Invariant ordering of surface groups and 3-manifolds which fibre over $S^1$

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(Received )

Abstract

It is shown that, if $\Sigma$ is a closed orientable surface and $\varphi: \Sigma \to \Sigma$ a homeomorphism, then one can find an ordering of $\pi_1(\Sigma)$ which is invariant under left- and right-multiplication, as well as under $\varphi_*: \pi_1(\Sigma) \to \pi_1(\Sigma)$, provided all the eigenvalues of the map induced by $\varphi$ on the integral first homology groups of $\Sigma$ are real and positive. As an application, if $M^3$ is a closed orientable 3-manifold which fibres over the circle, then its fundamental group is bi-orderable if the associated homology monodromy has all eigenvalues real and positive. This holds, in particular, if the monodromy is in the Torelli subgroup of the mapping class group of $\Sigma$.

1. Introduction

It is well-known that the fundamental group $\pi_1(\Sigma)$ of a closed orientable surface is bi-orderable, that is, the elements of the group may be given a total linear ordering which is invariant under multiplication on both sides. If $\varphi: \Sigma \to \Sigma$ is an automorphism of the surface, we show that $\pi_1(\Sigma)$ can be given a bi-ordering which is invariant under $\varphi_*: \pi_1(\Sigma) \to \pi_1(\Sigma)$ provided all the eigenvalues of the homology map induced by $\varphi$ are real and positive. This generalizes a similar result of [PR] for free groups to the somewhat more complicated case of surface groups, or more generally to certain one-relator groups. The proof depends crucially on a theorem of Labute [Lab].

We apply this result to 3-manifolds $M^3$ which fibre over the circle as follows. Suppose $M^3 \to S^1$ is a fibration, with fibre a closed oriented surface $\Sigma$, and monodromy $\varphi: \Sigma \to \Sigma$. $M^3$ may be regarded as the mapping torus $E_\varphi$ of $\varphi$. From the homotopy exact sequence of the fibration,

$$1 \to \pi_1(\Sigma) \to \pi_1(M^3) \to \pi_1(S^1) \to 1$$
and the orderability of \( \pi_1(\Sigma) \) and \( \pi_1(S^1) = \mathbb{Z} \), one can conclude (for any \( \varphi \)) that \( \pi_1(M^3) \) is left-orderable (i.e. has an ordering invariant under left-multiplication). The fundamental group of \( M^3 \) is an HNN extension of \( \pi_1(\Sigma) \), in other words, it is isomorphic to the group \( \pi_1(\Sigma) \), with an extra generator \( t \), subject to the relations \( t^{-1} xt = \varphi_*(x) \), for all generators \( x \) of \( \pi_1(\Sigma) \). To construct a bi-ordering for \( \pi_1(M^3) \), one needs a bi-ordering of \( \pi_1(\Sigma) \) which is invariant under \( \varphi_* \). Thus \( \pi_1(M^3) \) is bi-orderable if all the eigenvalues of the homology map induced by \( \varphi \) are real and positive.

2. The main result

Let \( G \) be a group. Define the descending central series of \( G \) by

\[
G_1 = G, \quad G_n = [G, G_{n-1}]
\]

where \([G, G_{n-1}]\) is the group generated by commutators \([g, h] = ghg^{-1}h^{-1}, \ g \in G, \ h \in G_{n-1}\). We set

\[
L_n(G) = G_n/G_{n+1} \text{ and } gr(G) = \bigoplus_{n=1}^{\infty} L_n(G).
\]

Then \( L_n(G) \) are abelian groups and \( gr(F) \) has a Lie algebra structure, by defining the Lie product \((u, v) \mapsto [u, v] = uvu^{-1}v^{-1} \in L_{n+m}, \) for \( u \in L_n(G), \ v \in L_m(G) \).

Let \( F \) be a free group generated by \( x_1, \cdots, x_h \) and \( R \in F \). Let \( e(R) = \sup\{n \mid R \in F_n\} \).

We will assume the following condition:

(\*) \( e(R) > 1 \) and \( R \) is primitive, i.e. \( R \) is not a power modulo \( F_{e(R)+1} \).

Suppose \( G = F/\langle \langle R \rangle \rangle \) is the corresponding single relator group. We make the following additional hypothesis:

\[
(\ast\ast) \quad \bigcap_{n=1}^{\infty} G_n = \{1\}.
\]

Let \( G^{ab} \) be the abelianization of \( G \). By (\*), \( G^{ab} \) is free abelian of rank \( h \). More precisely the canonical map \( F^{ab} \to G^{ab} \) is an isomorphism.

Now let \( \varphi \) be an isomorphism of \( G \), \( \varphi_{ab} \) be the induced isomorphism on \( G^{ab} \). We consider the hypothesis.

(\*\*\*) \( \varphi_{ab} \) has all its eigenvalues real and positive (possibly with multiplicity).

**Theorem 2.1.** Let \( G \) be the single relator group \( F/\langle \langle R \rangle \rangle \) satisfying hypothesis (\*) and (\*\*\*), and suppose \( \varphi \) is an isomorphism of \( G \) satisfying (\*\*\*). Then there is a bi-ordering of \( G \) which is invariant under \( \varphi \).

This will be proved in Section 5.

**Corollary 2.2.** Assuming the hypotheses of Theorem 2.1, the HNN extension of \( G \) by \( \mathbb{Z} \) defined by \( \varphi \) is bi-orderable.

**Proof** If \( 1 \to A \to B \to C \to 1 \) is an exact sequence of groups, with \( A \) and \( C \) bi-orderable, then \( B \) is biorderable provided conjugation of \( B \) upon \( A \) preserves a bi-ordering of \( A \). The ordering is defined by taking \( b_1 < b_2 \) in \( B \) if either \( b_1^{-1}b_2 \) lies in \( A \) and is greater than the identity there, or else its image is greater than the identity in \( C \).

**Remark:** Hypotheses (\*) and (\*\*\*) are verified for \( G \) the fundamental group of a closed orientable surface of genus \( g \). Here \( h = 2g, \ F = \langle x_1, \cdots, x_g, y_1, \cdots, y_g \rangle \) and \( R = [x_1, y_1] \cdots [x_g, y_g] \).
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**Corollary 2.3.** Let $\Sigma_g$ be a closed oriented surface of genus $g$, \( \varphi \) a homeomorphism of $\Sigma_g$ such that the induced isomorphism on $H_1(\Sigma_g; \mathbb{Z})$ has all eigenvalues real and positive. Let $E_\varphi$ be the mapping torus of $\Sigma_g$ associated to $\varphi$ (this is a 3-manifold fibering over $S^1$). Then the fundamental group of $E_\varphi$ is bi-orderable. This is true in particular if $\varphi$ belongs to the Torelli subgroup of the mapping class group of $\Sigma_g$ (that is, $\varphi_* = \text{id}$ at the homological level).

**Corollary 2.4.** If $M$ is a 3-manifold which fibres over the circle, with fibre a torus (possibly with punctures), then $\pi_1(M)$ is virtually bi-orderable. In fact, it has a bi-orderable subgroup of index at most six.

**Proof** The monodromy matrix $A$ is a $2 \times 2$ matrix with determinant 1 (if the fibre is a punctured torus, the monodromy is the block sum of $A$ with a number of identity matrices). By considering the characteristic polynomial $\chi_A(t) = t^2 - \text{trace}(A)t + 1$, we see that the eigenvalues of $A$ are real if $|\text{trace}(A)| > 2$, and otherwise are roots of unity of order 2, 3, 4 or 6. Accordingly the matrix $A^p$, with $p = 1, 2, 3, 4$ or 6, will have real positive eigenvalues. This is the monodromy matrix of a $p$-fold cover of $M$.

**Example:** Let $T_1, \ldots, T_5$ denote the Dehn twists along the curves labelled 1, $\ldots$, 5 on the genus two surface pictured in Figure 1. Define $\varphi = T_1T_3(T_5)^2T_2^{-1}T_4^{-1}$. According to [CB], p.79, the characteristic polynomial of $\varphi_*$ is $t^4 - 9t^3 + 21t^2 - 9t + 1$. It is irreducible over $\mathbb{Z}$ and has all its roots real and positive, so $\varphi_*$ satisfies (***), and the corresponding 3-manifold $E_\varphi$ has bi-orderable fundamental group. Moreover, $\varphi$ is pseudo-Anosov and therefore $E_\varphi$ is hyperbolic.

**Remark:** It was mentioned in the introduction that for any homeomorphism $\varphi : \Sigma \to \Sigma$, the fibred manifold $E_\varphi$ has left-orderable fundamental group. We note that if $\varphi$ is periodic, even at the fundamental group level, then $\pi_1(E_\varphi)$ cannot be bi-orderable. If there were a bi-ordering on $\pi_1(E_\varphi)$, which is the HNN extension determined by $\varphi_* : \pi_1(\Sigma) \to \pi_1(\Sigma)$, then the ordering would be invariant under conjugation and therefore $\varphi_*$-invariant. However, if $\varphi_* \neq 1$ but $\varphi_*^p = 1$ for some $p > 1$, we would have an element $x \in \pi_1(\Sigma) \subset \pi_1(E_\varphi)$ such that $\varphi_*(x) \neq x$ but $\varphi_*^p(x) = x$. Suppose, without loss of generality, $x < \varphi_*(x)$ in the bi-ordering. Then $\varphi_*(x) < \varphi_*^2(x)$, and by induction and transitivity we conclude $x < \varphi_*^p(x) = x$, a contradiction.
3. Review of some basic facts on Lie algebras

Let $F$ be a free group. By ([Fox], section 4.5), $z \in F_n$ if and only if $z - 1 \in I^n$ where $I$ is the augmentation ideal of $\mathbb{Z}[F]$ ($I = \text{Ker } F \twoheadrightarrow \mathbb{Z}$). The map $\pi(z) = z - 1$ gives an injective homomorphism:

\begin{equation}
L_n(F) = F_n/F_{n+1} \xrightarrow{\pi} I^n/I^{n+1}.
\end{equation}

In fact if $x, y \in F_n$, then

\begin{align*}
\pi(xy) &= xy - y = (x - 1) + (y - 1) + (x - 1)(y - 1) \\&= (x - 1)(y - 1) \equiv (x - 1) + (y - 1) \mod I^{n+1}.
\end{align*}

**Lemma 3.1.** $\pi$ induces an injective homomorphism of Lie algebras:

\begin{equation}
\pi : L(F) = \bigoplus_{n=1}^{\infty} L_n(F) \longrightarrow I = \bigoplus_{n=1}^{\infty} I^n/I^{n+1}
\end{equation}

where the Lie product on $I$ is defined by $[\alpha, \beta] = \alpha \beta - \beta \alpha$.

**Proof** For $x \in F_n, y \in F_m$ we have

\begin{align*}
\pi[x, y] &= xyx^{-1}y^{-1} - 1 \\&= (xy - yx)x^{-1}y^{-1} \\&= (xy - yx) + (xy - yx)(x^{-1}y^{-1} - 1) \\&\equiv [(x - 1)(y - 1) - (y - 1)(x - 1)] \mod I^{n+m+1} \\&\equiv \pi(x)\pi(y) - \pi(y)\pi(x).
\end{align*}

\[\square\]

Let $H^{\otimes n} = H \otimes \cdots \otimes H$ ($n$ times) where $H = F_{ab}$ and let $\mathcal{H} = \bigoplus_{n=1}^{\infty} H^{\otimes n}$.

**Lemma 3.2.** a. For any positive integer $n$, the map

\begin{equation}
\psi_n : I^n/I^{n+1} \longrightarrow H^{\otimes n}
\end{equation}

given by $(x_i - 1) \cdots (x_i - 1) \mapsto a_i \otimes \cdots \otimes a_i$ is a homomorphism of abelian groups, where $x_i$ is a generator of $F$ and $a_i$ is the canonical image of $x_i$ in $H = F_{ab}$.

b. The map $\psi = \bigoplus \psi_n : I = \bigoplus I^n/I^{n+1} \longrightarrow \mathcal{H} = \bigoplus H^{\otimes n}$ is an isomorphism of Lie algebras, where the Lie structure of $\mathcal{H}$ is given by $[\alpha, \beta] = \alpha \otimes \beta - \beta \otimes \alpha$, for $\alpha \in H^{\otimes n}, \beta \in H^{\otimes m}$.

**Proof** The proof is routine (see [PR]).

\[\square\]

4. Review of some results of Labute

Let $R \in F = \langle x_1, \ldots, x_h \rangle$, $e = \sup \{n \in \mathbb{N} : R \in F_n \}$. We suppose $e > 1$ and $R$ primitive. Let $G = F/\langle \langle R \rangle \rangle$ and $\bar{R}$ be the class of $R$ in $F_{e}/F_{e+1} \subset gr(F)$. We of course have a natural map $gr(F) \longrightarrow gr(G)$. Let $I(\bar{R})$ be the ideal generated by $\bar{R}$ in the Lie algebra $gr(F) = \bigoplus F_{n}/F_{n+1}$.

That is, $I(\bar{R}) = \{ \lambda \cdot \bar{R} + n\bar{R} : \lambda \in gr(F), n \in \mathbb{Z} \}$ where $\lambda = \lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_p \oplus \cdots$, $\lambda_i \in L_i(F)$ and $\lambda \cdot \bar{R} = \bigoplus [\lambda_i, R]$. 

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**Theorem 4.1** (Labute [Lab]). With the above hypothesis on $R$,
1. $L_n(G) = G_n/G_{n+1}$ is a free $\mathbb{Z}$-module of finite rank, for any positive integer $n$.
2. $\text{gr}(G) \cong \text{gr}(F)/I(\hat{R})$.

**Remark:** Condition 2. means the following: $L_n(G)$ is the quotient of $L_n(F)$ by the equivalence relation $\sim_n$ defined as follows. Let $x, y \in F_n$, $\bar{x}, \bar{y}$ their classes in $F_n/F_{n+1}$. Then $\bar{x} \sim_n \bar{y}$ if and only if $\bar{x} \ast \bar{y}^{-1} \in I(\hat{R})$ (* is the abelian law in $F_n/F_{n+1}$). That is $\bar{x} \ast \bar{y}^{-1} = \lambda \cdot \bar{R} + p\bar{R}$, for some $\lambda \in \text{gr}(F)$, and $p \in \mathbb{Z}$. Since $\bar{x} \ast \bar{y}^{-1}$ has degree $n$ in $\text{gr}(F)$, and $\hat{R}$ has degree $e$ this means:
   (i) for $n < e$, $L_n(G) = L_n(F)$.
   (ii) for $n = e$, $L_n(G) = L_e(F)/(\hat{R})$, where $\langle \hat{R} \rangle$ is the subgroup generated by $\hat{R}$.
   (iii) for $n > e$, $\bar{x} \sim_n \bar{y}$ if and only if $\bar{x} \bar{y}^{-1} = [\lambda_{n-e}, \hat{R}]$ for some $\lambda_{n-e} \in L_{n-e}(F)$.

Recall the following homomorphisms:

$$L_n(F) \xrightarrow{\pi_n} I^n/I^{n+1} \xrightarrow{\psi_n} H^{\otimes n}$$

$$L(F) \xrightarrow{\pi} I = \oplus I^n/I^{n+1} \xrightarrow{\psi} H = \oplus H^{\otimes n}.$$ 

Denote by $\rho_0$ the image of $\hat{R}$ in $H^{\otimes e}$ by $\psi_e \circ \pi_e$.

Denote by $J(\rho_0)$ the image in $\mathcal{H}$ of the ideal $I(\hat{R})$ by $\psi \circ \pi$.

So $J(\rho_0) = \{ \lambda \cdot \rho_0 + n\rho_0 : \lambda \in \text{Im}(\psi \circ \pi), n \in \mathbb{Z} \}$ is an additive subgroup of $\mathcal{H}$, but no longer an ideal of $\mathcal{H}$ since $\psi \circ \pi$ is not surjective. The reason for considering $J(\rho_0)$ instead of the ideal of $\mathcal{H}$ generated by $\rho_0$ is that the induced map

$$L(F)/I(\hat{R}) \xrightarrow{\psi_{\pi_0}} \mathcal{H}/J(\rho_0)$$

continues to be injective. Of course $\mathcal{H}/J(\rho_0)$ is no longer a Lie algebra. It induces an injective homomorphism (of abelian groups)

$$L_n(G) = L_n(F)/\sim_n \hookrightarrow H^{\otimes n}/\sim_n$$

where $H^{\otimes n}/\sim_n$ has the following meaning. This is the quotient of $H^{\otimes n}$ by the relation:

for $x, y \in H^{\otimes n}$, $x \sim_n y$ if and only if $x - y = \lambda \cdot \rho_0 + p\rho_0 = \lambda \otimes \rho_0 - \rho_0 \otimes \lambda + p\rho_0$ for $\lambda \in \text{Im}(\psi \circ \pi)$, $p \in \mathbb{Z}$.

So if $n < e$, $H^{\otimes n}/\sim_n = H^{\otimes n}$,

If $n = e$, $H^{\otimes e}/\sim_e = H^{\otimes e}/\langle \rho_0 \rangle$,

If $n > e$, $x \sim_n y$ if and only if $x - y = \lambda \otimes \rho_0 - \rho_0 \otimes \lambda$ for some $\lambda \in \text{Im}(\psi_{n-e} \circ \pi_{n-e})$.

Now let $\varphi$ be an isomorphism of $G$ and $\varphi_{ab}$ the induced isomorphism of $\mathcal{H} = G_{ab} \simeq F_{ab}$.

Let $\tilde{\varphi} : F \to F$ be any homomorphism of the free group such that the following diagram commutes:

$$\begin{array}{ccc} 
F & \xrightarrow{\tilde{\varphi}} & F \\
\downarrow & & \downarrow \\
G & \xrightarrow{\varphi} & G 
\end{array}$$

Then $\varphi_{ab} = \varphi_{ab}$. Denote by $\tilde{\varphi}_n$ the induced homomorphism

$$\tilde{\varphi}_n : F_n/F_{n+1} \to F_n/F_{n+1}.$$ 

In [PR] we proved that the following diagram is commutative:
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\[ F_n / F_{n+1} \xrightarrow{\psi_n \circ \pi} H^\otimes_n \]

(7)

\[ \varphi_n \downarrow \quad \downarrow \varphi_n^\otimes_{ab} = \varphi_{ab} \otimes \cdots \otimes \varphi_{ab} \]

\[ F_n / F_{n+1} \xrightarrow{\psi_n \circ \pi} H^\otimes_n \]

Lemma 4.2. \( \varphi_{ab}^\otimes (\rho_0) = \pm \rho_0. \)

Proof. Because of the commutativity of diagram (6), \( \varphi(R) \in \langle \langle R \rangle \rangle, \) where \( \langle \langle R \rangle \rangle \) denotes the normal subgroup of \( F \) generated by \( R. \)

So \( \varphi(R) = \prod_{i=1}^{p} y_i R^{e_i} y_i^{-1} = \prod_{i=1}^{p} [y_i, R^{e_i}] R^{e_i} \equiv \prod_{i=1}^{p} R^{e_i} = R^{(\Sigma e_i)} \mod F_{c+1}. \) By commutativity of diagram (7) : \( \varphi_{ab}^\otimes (\rho_0) = \rho_0^{\Sigma e_i}. \) Since \( \varphi_{ab}^\otimes \) is an isomorphism of free abelian groups then \( \Sigma e_i = \pm 1. \)

Corollary 4.3. \( \varphi_{ab}^\otimes : H^\otimes_n \rightarrow H^\otimes_n \) induces a map on the quotient \( H^\otimes_n / \sim_n. \)

Proof. From the definition of \( \sim_n \) (see (5)) it is sufficient to prove that \( \varphi_{ab}^\otimes (\lambda_n - e \cdot \rho_0) \in J(\rho_0) \) for \( \lambda_n - e \in \text{Im}(\psi \circ \pi): \)

\[ \varphi_{ab}^\otimes (\lambda_n - e \cdot \rho_0) = \varphi_{ab}^\otimes (\lambda_n - e \otimes \rho_0 - \rho_0 \otimes \lambda_n - e) = \varphi_{ab}^\otimes (\lambda_n - e) \otimes (\pm \rho_0) - (\pm \rho_0) \otimes \varphi_{ab}^\otimes (\lambda_n - e). \]

By the commutativity of diagram (7), \( \varphi_{ab}^\otimes (\lambda_n - e) \in \text{Im} (\psi \circ \pi). \)

So diagram (7) gives rise to a commutative diagram:

\[ G_n / G_{n+1} \xrightarrow{\varphi_n} H^\otimes_n / \sim_n \]

(8)

\[ \varphi_n \downarrow \quad \downarrow \varphi_{ab}^\otimes \]

\[ G_n / G_{n+1} \xrightarrow{\varphi_n} H^\otimes_n / \sim_n \]

where the vertical arrows are isomorphisms and horizontal ones are injective.

Tensoring diagram (8) by \( \mathbb{R}, \) we get a commutative diagram

\[ (G_n / G_{n+1}) \otimes \mathbb{R} \xrightarrow{\varphi_n} (H^\otimes_n / \sim_n) \otimes \mathbb{R} \]

(9)

\[ \varphi_n \otimes \text{id}_\mathbb{R} \downarrow \quad \downarrow \varphi_{ab}^\otimes \otimes \text{id}_\mathbb{R} \]

\[ (G_n / G_{n+1}) \otimes \mathbb{R} \xrightarrow{\varphi_n} (H^\otimes_n / \sim_n) \otimes \mathbb{R} \]

where the horizontal arrows are still injective.

Denote by \( V_n \) (resp. \( \hat{\varphi}_n \)) the \( \mathbb{R} \)-vector space \( (H^\otimes_n / \sim_n) \otimes \mathbb{R} \) (resp. \( \varphi_{ab}^\otimes \otimes \text{id}_\mathbb{R} \)). Since \( G_n / G_{n+1} \) is torsion-free, by Labute’s theorem we get a commutative diagram, where the horizontal arrows are injective:

\[ G_n / G_{n+1} \rightarrow V_n \]

\[ \varphi_n \downarrow \quad \downarrow \hat{\varphi}_n \]

\[ G_n / G_{n+1} \rightarrow V_n \]

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**Lemma 4.4.** If $\varphi_{ab} : H \rightarrow H$ has only real positive eigenvalues, the same holds for $\varphi_n$ for any $n$.

**Proof** Tensoring the following diagram by $\mathbb{R}$:

\[
\begin{array}{ccc}
H^\otimes n & \longrightarrow & H^\otimes n / \sim_n \\
\downarrow \varphi_{ab}^\otimes & & \downarrow \varphi_{ab}^\otimes \\
H^\otimes n & \longrightarrow & H^\otimes n / \sim_n
\end{array}
\]

we get a commutative diagram of vector spaces with exact rows, where $E$ denotes the kernel, and the map $E \rightarrow E$ is the restriction of $\varphi_{ab}^\otimes \otimes \mathbb{R}$:

\[
\begin{array}{cccc}
0 & \longrightarrow & E & \longrightarrow & H^\otimes n \otimes \mathbb{R} & \longrightarrow & V_n & \longrightarrow & 0 \\
\downarrow & & \downarrow \varphi_{ab}^\otimes \otimes \mathbb{R} & & \downarrow \varphi_n & & & & \\
0 & \longrightarrow & E & \longrightarrow & H^\otimes n \otimes \mathbb{R} & \longrightarrow & V_n & \longrightarrow & 0
\end{array}
\]

The matrix of $\varphi_{ab}^\otimes \otimes \mathbb{R}$ has the following form in a suitable basis:

\[
\begin{pmatrix}
\alpha & ? \\
0 & \beta
\end{pmatrix}
\]

where $\alpha$ is the matrix of $\varphi_{ab}^\otimes \otimes \mathbb{R}$ restricted to $E$ and $\beta$ the matrix of $\varphi_n$. The eigenvalues of $\varphi_{ab}^\otimes \otimes \mathbb{R}$ are products of the eigenvalues of $\varphi_{ab} \otimes \mathbb{R}$ and therefore all real and positive. So $\beta$ has only real positive eigenvalues. \hfill $\Box$

5. **Proof of Theorem 2.1**

Assuming that $G$ and $\varphi : G \rightarrow G$ satisfy the hypotheses of Theorem 2.1, we need to construct an ordering of the elements of $G$ which is invariant under multiplication on both sides, and also invariant under the map $\varphi$. We may proceed exactly as in section 4 of [PR], which we outline here for the reader’s convenience.

Go back to diagram (10), where the vector space isomorphism $\varphi_n$ has all its eigenvalues real and positive. By standard linear algebra, there exists a basis $v_1, \ldots, v_k$ for $V_n$ with respect to which the matrix for $\varphi$ is upper triangular and has its (positive) eigenvalues $\lambda_1, \ldots, \lambda_k$ on the diagonal. We order the vectors in $V_n$ by reverse lexicographical ordering, using their coordinates relative to the basis $v_1, \ldots, v_k$. Thus if $x = x_1 v_1 + \cdots + x_k v_k$ and $y = y_1 v_1 + \cdots + y_k v_k$ are distinct vectors in $V_n$, we define $x < y$ if and only if $x_i < y_i$ (in the usual ordering of $\mathbb{R}$), at the last $i$ for which the coordinates differ. It is routine to check that this ordering of $V_n$ is invariant under vector addition and also that $x < y$ if and only if $\varphi(x) < \varphi(y)$. Restricting this ordering to the abelian group $G_n/G_{n+1}$ defines an ordering which is invariant under the isomorphism $\varphi_n$.

We now have $\varphi_n$-invariant orderings of the lower central quotients $G_n/G_{n+1}$, for each $n \in \mathbb{Z}$, and use this to define a $\varphi$-invariant bi-order on $G$ using a well-known technique for ordering groups which are residually nilpotent and have torsion-free lower central quotients. Namely, let $g, h \in G$ and consider $n = n(g, h) = n(h, g)$ to be the greatest integer such that $g^{-1}h$ belongs to $G_n$. Then define $g < h$ if the coset of $g^{-1}h$ is greater than the identity in the ordering of $G_n/G_{n+1}$ and $h < g$ otherwise. It is routine to check
that this defines a bi-ordering of $G$, and it is invariant under $\varphi$ because the orderings of $G_n/G_{n+1}$ are invariant under $\varphi_n$. □

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