Invariant ordering of surface groups and 3-manifolds which fibre over S^1

By BERNARD PERRON

Laboratoire de Topologie,
Université de Bourgogne, BP 47870
21078 - *Dijon Cedex, France.*
e-mail: perron@topolog.u-bourgogne.fr

and DALE ROLFSEN

Pacific Institute for the Mathematical Sciences and
Department of Mathematics,
University of British Columbia,
Vancouver, BC, Canada V6T 1Z2.
e-mail: rolfsen@math.ubc.ca

(*Received*)

Abstract

It is shown that, if Σ is a closed orientable surface and $\varphi : \Sigma \to \Sigma$ a homeomorphism, then one can find an ordering of $\pi_1(\Sigma)$ which is invariant under left- and right-multiplication, as well as under $\varphi_*: \pi_1(\Sigma) \to \pi_1(\Sigma)$, provided all the eigenvalues of the map induced by φ on the integral first homology groups of Σ are real and positive. As an application, if M^3 is a closed orientable 3-manifold which fibres over the circle, then its fundamental group is bi-orderable if the associated homology monodromy has all eigenvalues real and positive. This holds, in particular, if the monodromy is in the Torelli subgroup of the mapping class group of Σ.

1. *Introduction*

It is well-known that the fundamental group $\pi_1(\Sigma)$ of a closed orientable surface is *bi-orderable*, that is, the elements of the group may be given a total linear ordering which is invariant under multiplication on both sides. If $\varphi: \Sigma \to \Sigma$ is an automorphism of the surface, we show that $\pi_1(\Sigma)$ can be given a bi-ordering which is invariant under $\varphi_*: \pi_1(\Sigma) \to \pi_1(\Sigma)$ provided all the eigenvalues of the homology map induced by φ are real and positive. This generalizes a similar result of [**PR**] for free groups to the somewhat more complicated case of surface groups, or more generally to certain one-relator groups. The proof depends crucially on a theorem of Labute [**Lab**].

We apply this result to 3-manifolds M^3 which fibre over the circle as follows. Suppose $M^3 \to S^1$ is a fibration, with fibre a closed oriented surface Σ, and monodromy $\varphi: \Sigma \to \Sigma$. M^3 may be regarded as the mapping torus E_φ of φ. From the homotopy exact sequence of the fibration,

$$1 \longrightarrow \pi_1(\Sigma) \longrightarrow \pi_1(M^3) \longrightarrow \pi_1(S^1) \longrightarrow 1$$

and the orderability of $\pi_1(\Sigma)$ and $\pi_1(S^1) = \mathbb{Z}$, one can conclude (for any φ) that $\pi_1(M^3)$ is left-orderable (i.e. has an ordering invariant under left-multiplication). The fundamental group of M^3 is an HNN extension of $\pi_1(\Sigma)$, in other words, it is isomorphic to the group $\pi_1(\Sigma)$, with an extra generator t, subject to the relations $t^{-1}\ xt = \varphi_*(x)$, for all generators x of $\pi_1(\Sigma)$. To construct a bi-ordering for $\pi_1(M^3)$, one needs a bi-ordering of $\pi_1(\Sigma)$ which is invariant under φ_*. Thus $\pi_1(M^3)$ is bi-orderable if all the eigenvalues of the homology map induced by φ are real and positive.

2. *The main result*

Let G be a group. Define the descending central series of G by

$$G_1 = G, \quad G_n = [G, G_{n-1}]$$

where $[G, G_{n-1}]$ is the group generated by commutators $[g, h] = ghg^{-1}h^{-1},\ g \in G,\ h \in G_{n-1}$. We set $L_n(G) = G_n/G_{n+1}$ and $gr(G) = \oplus_{n=1}^{\infty} L_n(G)$.

Then $L_n(G)$ are abelian groups and $gr(F)$ has a Lie algebra structure, by defining the Lie product $(u, v) \mapsto [u, v] = uvu^{-1}v^{-1} \in L_{n+m}$, for $u \in L_n(G)$, $v \in L_m(G)$.

Let F be a free group generated by $x_1, \cdots, x_h$ and $R \in F$. Let $e(R) = \sup\{n\ ;\ R \in F_n\}$. We will assume the following condition:

$(*)$ $\quad e(R) > 1$ *and* R *is primitive, i.e.* R *is not a power modulo* $F_{e(R)+1}$.

Suppose $G = F/\langle\langle R\rangle\rangle$ is the corresponding single relator group. We make the following additional hypothesis:

$$(**) \qquad \bigcap_{n=1}^{\infty} G_n = \{1\}.$$

Let G^{ab} be the abelianization of G. By $(*)$, G^{ab} is free abelian of rank h. More precisely the canonical map $F^{ab} \longrightarrow G^{ab}$ is an isomorphism.

Now let φ be an isomorphism of G, φ_{ab} be the induced isomorphism on G^{ab}. We consider the hypothesis.

$(***)$ $\quad \varphi_{ab}$ *has all its eigenvalues real and positive (possibly with multiplicity).*

THEOREM 2·1. *Let* G *be the single relator group* $F/\langle\langle R\rangle\rangle$ *satisfying hypothesis* $(*)$ *and* $(**)$*, and suppose* φ *is an isomorphism of* G *satisfying* $(***)$*. Then there is a bi-ordering of* G *which is invariant under* φ*.*

This will be proved in Section 5.

COROLLARY 2·2. *Assuming the hypotheses of Theorem* 2·1*, the HNN extension of* G *by* $\mathbb{Z}$ *defined by* φ *is bi-orderable.*

Proof If $1 \to A \to B \to C \to 1$ is an exact sequence of groups, with A and C bi-orderable, then B is biorderable provided conjugation of B upon A preserves a bi-ordering of A. The ordering is defined by taking $b_1 < b_2$ in B if either $b_1^{-1}b_2$ lies in A and is greater than the identity there, or else its image is greater than the identity in C. □

Remark : Hypotheses $(*)$ and $(**)$ are verified for G the fundamental group of a *closed* orientable surface of genus g. Here $h = 2g$, $F = \langle x_1, \cdots, x_g, y_1, \cdots, y_g\rangle$ and $R = [x_1, y_1] \cdots [x_g, y_g]$.

COROLLARY 2·3. *Let Σ_g be a closed oriented surface of genus g, φ a homeomorphism of Σ_g such that the induced isomorphism on $H_1(\Sigma_g;\mathbb{Z})$ has all eigenvalues real and positive. Let E_φ be the mapping torus of Σ_g associated to φ (this is a 3-manifold fibering over S^1). Then the fundamental group of E_φ is bi-orderable. This is true in particular if φ belongs to the Torelli subgroup of the mapping class group of Σ_g (that is, $\varphi_* = \mathrm{id}$ at the homological level).* □

COROLLARY 2·4. *If M is a 3-manifold which fibres over the circle, with fibre a torus (possibly with punctures), then $\pi_1(M)$ is virtually bi-orderable. In fact, it has a bi-orderable subgroup of index at most six.*

Proof The monodromy matrix A is a 2×2 matrix with determinant 1 (if the fibre is a punctured torus, the monodromy is the block sum of A with a number of identity matrices). By considering the characteristic polynomial $\chi_A(t) = t^2 - trace(A)t + 1$, we see that the eigenvalues of A are real if $|trace(A)| > 2$, and otherwise are roots of unity of order 2, 3, 4 or 6. Accordingly the matrix A^p, with $p = 1, 2, 3, 4$ or 6, will have real positive eigenvalues. This is the monodromy matrix of a p-fold cover of M. □

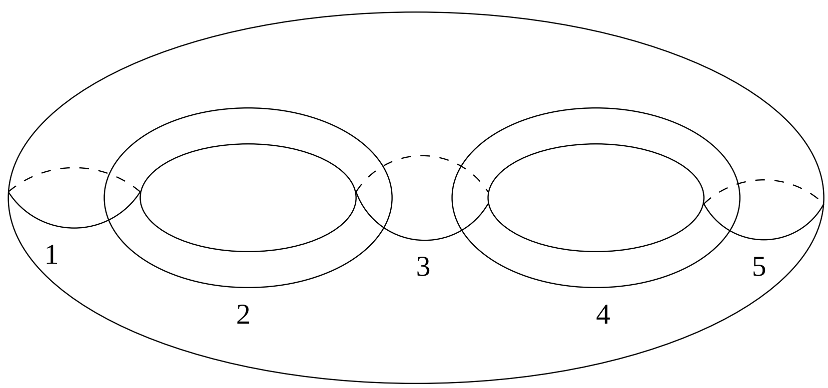

Fig. 1. Curves on a genus 2 surface

Example: Let $T_1, \ldots, T_5$ denote the Dehn twists along the curves labelled $1, \ldots, 5$ on the genus two surface pictured in Figure 1. Define $\varphi = T_1T_3(T_5)^2T_2^{-1}T_4^{-1}$. According to [**CB**], p.79, the characteristic polynomial of φ_* is $t^4 - 9t^3 + 21t^2 - 9t + 1$. It is irreducible over $\mathbb{Z}$ and has all it roots real and positive, so φ_* satisfies $(***)$ and the corresponding 3-manifold E_φ has bi-orderable fundamental group. Moreover, φ is pseudo-Anosov and therefore E_φ is hyperbolic.

Remark: It was mentioned in the introduction that for any homeomorphism $\varphi : \Sigma \to \Sigma$, the fibred manifold E_φ has left-orderable fundamental group. We note that if φ is periodic, even at the fundamental group level, then $\pi_1(E_\varphi)$ cannot be bi-orderable. If there were a bi-ordering on $\pi_1(E_\varphi)$, which is the HNN extension determined by $\varphi_* : \pi_1(\Sigma) \to \pi_1(\Sigma)$, then the ordering would be invariant under conjugation and therefore φ_*-invariant. However, if $\varphi_* \neq 1$ but $\varphi_*^p = 1$ for some $p > 1$, we would have an element $x \in \pi_1(\Sigma) \subset \pi_1(E_\varphi)$ such that $\varphi_*(x) \neq x$ but $\varphi_*^p(x) = x$. Suppose, without loss of generality, $x < \varphi_*(x)$ in the bi-ordering. Then $\varphi_*(x) < \varphi_*^2(x)$, and by induction and transitivity we conclude $x < \varphi_*^p(x) = x$, a contradiction.

3. *Review of some basic facts on Lie algebras*

Let F be a free group. By ([**Fox**], section 4.5), $z \in F_n$ if and only if $z-1 \in I^n$ where I is the augmentation ideal of $\mathbb{Z}[F]$ ($I = \text{Ker } \mathbb{Z}F \xrightarrow{\varepsilon} \mathbb{Z}$). The map $\pi(z) = z-1$ gives an injective homomorphism:

$$L_n(F) = F_n/F_{n+1} \xrightarrow{\pi} I^n/I^{n+1}. \tag{1}$$

In fact if $x,\ y \in F_n$, then

$$\pi(xy) = xy - 1 = (x-1) + (y-1) + (x-1)(y-1) \equiv (x-1) + (y-1) \bmod I^{n+1}.$$

LEMMA 3·1. *π induces an injective homomorphism of Lie algebras:*

$$\pi : L(F) = \oplus_{n=1}^{\infty} L_n(F) \quad \longrightarrow \quad \mathcal{I} = \oplus_{n=1}^{\infty} I^n/I^{n+1} \tag{2}$$

where the Lie product on $\mathcal{I}$ is defined by $[\alpha, \beta] = \alpha\beta - \beta\alpha$.

Proof For $x \in F_n,\ y \in F_m$ we have

$$\begin{aligned}
\pi[x,y] &= xyx^{-1}y^{-1} - 1 \\
&= (xy - yx)x^{-1}y^{-1} \\
&= (xy - yx) + (xy - yx)(x^{-1}y^{-1} - 1) \\
&\equiv [(x-1)(y-1) - (y-1)(x-1)] \quad \bmod I^{n+m+1} \\
&\equiv \pi(x)\pi(y) - \pi(y)\pi(x).
\end{aligned}$$

□

Let $H^{\otimes n} = H \otimes \cdots \otimes H$ (n times) where $H = F_{ab}$ and let $\mathcal{H} = \oplus_{n=1}^{\infty} H^{\otimes n}$.

LEMMA 3·2. *a. For any positive integer n, the map*

$$\psi_n : I^n/I^{n+1} \longrightarrow H^{\otimes n} \tag{3}$$

given by $(x_{i_1} - 1) \cdots (x_{i_n} - 1) \longrightarrow a_{i_1} \otimes \cdots \otimes a_{i_n}$ is a homomorphism of abelian groups, where x_i is a generator of F and a_i is the canonical image of x_i in $H = F_{ab}$.

b. The map $\psi = \oplus \psi_n : \mathcal{I} = \oplus I^n/I^{n+1} \longrightarrow \mathcal{H} = \oplus H^{\otimes n}$ is an isomorphism of Lie algebras, where the Lie structure of $\mathcal{H}$ is given by $[\alpha, \beta] = \alpha \otimes \beta - \beta \otimes \alpha$, for $\alpha \in H^{\otimes n}$, $\beta \in H^{\otimes m}$.

Proof The proof is routine (see [**PR**]). □

4. *Review of some results of Labute*

Let $R \in F = \langle x_1, \cdots, x_h \rangle,\ e = \sup\{n \in N\ ;\ R \in F_n\}$. We suppose $e > 1$ and R primitive. Let $G = F/\langle\langle R \rangle\rangle$ and $\bar{R}$ be the class of R in $F_e/F_{e+1} \subset gr(F)$. We of course have a natural map $gr(F) \longrightarrow gr(G)$. Let $I(\bar{R})$ be the ideal generated by $\bar{R}$ in the Lie algebra $gr(F) = \oplus F_n/F_{n+1}$.

That is, $I(\bar{R}) = \{\lambda \cdot \bar{R} + n\bar{R}\ ;\ \lambda \in gr(F),\ n \in \mathbb{Z}\}$ where $\lambda = \lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_p \oplus \cdots$, $\lambda_i \in L_i(F)$ and $\lambda \cdot \bar{R} = \oplus_i [\lambda_i, R]$.

THEOREM 4·1 (Labute [**Lab**]). *With the above hypothesis on R,*
1. $L_n(G) = G_n/G_{n+1}$ is a free $\mathbb{Z}$-module of finite rank, for any positive integer n.
2. $gr(G) \cong gr(F)/I(\bar{R})$. □

Remark : Condition 2. means the following: $L_n(G)$ is the quotient of $L_n(F)$ by the equivalence relation $\sim_n$ defined as follows. Let $x,\ y \in F_n,\ \bar{x},\ \bar{y}$ their classes in F_n/F_{n+1}. Then $\bar{x} \sim_n \bar{y}$ if and only if $\bar{x} * \bar{y}^{-1} \in I(\bar{R})$ ($*$ is the abelian law in F_n/F_{n+1}). That is $\bar{x} * \bar{y}^{-1} = \lambda \cdot \bar{R} + p\bar{R}$, for some $\lambda \in gr(F)$, and $p \in \mathbb{Z}$. Since $\bar{x} * \bar{y}^{-1}$ has degree n in $gr(F)$, and $\bar{R}$ has degree e this means:
(i) for $n < e$, $L_n(G) = L_n(F)$.
(ii) for $n = e$, $L_e(G) = L_e(F)/\langle\bar{R}\rangle$, where $\langle\bar{R}\rangle$ is the subgroup generated by $\bar{R}$.
(iii) for $n > e$, $\bar{x} \sim \bar{y}$ if and only if $\bar{x}\bar{y}^{-1} = [\lambda_{n-e}, \bar{R}]$ for some $\lambda_{n-e} \in L_{n-e}(F)$.

Recall the following homomorphisms:

$$L_n(F) \xrightarrow{\pi_n} I^n/I^{n+1} \xrightarrow{\psi_n} H^{\otimes n}$$

$$L(F) \xrightarrow{\pi} \mathcal{I} = \oplus I^n/I^{n+1} \xrightarrow{\psi} \mathcal{H} = \oplus H^{\otimes n}.$$

Denote by ρ_0 the image of $\bar{R}$ in $H^{\otimes e}$ by $\psi_e \circ \pi_e$.
Denote by $J(\rho_0)$ the image in $\mathcal{H}$ of the ideal $I(\bar{R})$ by $\psi \circ \pi$.
So $J(\rho_0) = \{\lambda \cdot \rho_0 + n\rho_0\ ;\ \lambda \in Im(\psi \circ \pi),\ n \in \mathbb{Z}\}$ is an additive subgroup of $\mathcal{H}$, but no longer an ideal of $\mathcal{H}$ since $\psi \circ \pi$ is not surjective. The reason for considering $J(\rho_0)$ instead of the ideal of $\mathcal{H}$ generated by ρ_0 is that the induced map

$$L(F)/I(\bar{R}) \xrightarrow{\psi\circ\pi} \mathcal{H}/J(\rho_0) \tag{4}$$

continues to be injective. Of course $\mathcal{H}/J(\rho_0)$ is no longer a Lie algebra. It induces an injective homomorphism (of abelian groups)

$$L_n(G) = L_n(F)/\sim_n \hookrightarrow H^{\otimes n}/\sim_n \tag{5}$$

where $H^{\otimes n}/\sim_n$ has the following meaning. This is the quotient of $H^{\otimes n}$ by the relation: for $x,\ y \in H^{\otimes n},\ x \sim_n y$ if and only if $x - y = \lambda \cdot \rho_0 + p\rho_0 = \lambda \otimes \rho_0 - \rho_0 \otimes \lambda + p\rho_0$ for $\lambda \in \mathrm{Im}(\psi \circ \pi),\ p \in \mathbb{Z}$.
So if $n < e,\ H^{\otimes n}/\sim_n = H^{\otimes n}$,
If $n = e,\ H^{\otimes e}/\sim_e = H^{\otimes e}/\langle\rho_0\rangle$,
If $n > e,\ x \sim_n y$ if and only if $x - y = \lambda \otimes \rho_0 - \rho_0 \otimes \lambda$ for some $\lambda \in \mathrm{Im}(\psi_{n-e} \circ \pi_{n-e})$.
Now let φ be an isomorphism of G and φ_{ab} the induced isomorphism of $H = G_{ab} \simeq F_{ab}$. Let $\tilde{\varphi} : F \longrightarrow F$ be any homomorphism of the free group such that the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\tilde{\varphi}} & F \\ \downarrow & & \downarrow \\ G & \xrightarrow{\varphi} & G \end{array} \tag{6}$$

Then $\widetilde{\varphi}_{ab} = \varphi_{ab}$. Denote by $\widetilde{\varphi_n}$ the induced homomorphism

$$\widetilde{\varphi_n} : F_n/F_{n+1} \longrightarrow F_n/F_{n+1}.$$

In [**PR**] we proved that the following diagram is commutative:

(7)
$$\begin{array}{ccc} F_n/F_{n+1} & \overset{\psi_n\circ\pi_n}{\hookrightarrow} & H^{\otimes n} \\ \widetilde{\varphi_n} \ \downarrow & & \downarrow \widetilde{\varphi}_{ab}^{\otimes n} = \varphi_{ab} \otimes \cdots \otimes \varphi_{ab} \\ F_n/F_{n+1} & \overset{\psi_n\circ\pi_n}{\hookrightarrow} & H^{\otimes n} \end{array}$$

LEMMA 4·2. $\varphi_{ab}^{\otimes e}\ (\rho_0) = \pm\rho_0$.

Proof Because of the commutativity of diagram (6), $\widetilde{\varphi}(R) \in \langle\langle R\rangle\rangle$, where $\langle\langle R\rangle\rangle$ denotes the normal subgroup of F generated by R.

So $\widetilde{\varphi}(R) = \prod_{i=1}^{p} y_i R^{\varepsilon_i} y_i^{-1} = \prod_{i=1}^{p} [y_i, R^{\varepsilon_i}]R^{\varepsilon_i} \equiv \prod_{i=1}^{p} R^{\varepsilon_i} = R^{(\Sigma\varepsilon_i)} \mod F_{e+1}$. By commutativity of diagram (7) : $\varphi_{ab}^{\otimes e}\ (\rho_0) = \rho_0^{\Sigma\varepsilon_i}$. Since $\varphi_{ab}^{\otimes e}$ is an isomorphism of free abelian groups then $\Sigma\varepsilon_i = \pm 1$. □

COROLLARY 4·3. $\varphi_{ab}^{\otimes n} : H^{\otimes n} \longmapsto H^{\otimes n}$ *induces a map on the quotient* $H^{\otimes n}/\sim_n$.

Proof From the definition of $\sim_n$ (see (5)) it is sufficient to prove that $\varphi_{ab}^{\otimes n}\ (\lambda_{n-e}\cdot\rho_0) \in J(\rho_0)$ for $\lambda_{n-e} \in \mathrm{Im}(\psi\circ\pi)$:

$$\begin{aligned} \varphi_{ab}^{\otimes n}\ (\lambda_{n-e}\cdot\rho_0) &= \varphi_{ab}^{\otimes}(\lambda_{n-e}\otimes\rho_0 - \rho_0\otimes\lambda_{n-e}) \\ &= \varphi_{ab}^{\otimes}\ (\lambda_{n-e})\otimes(\pm\rho_0) - (\pm\rho_0)\otimes\varphi_{ab}^{\otimes}(\lambda_{n-e}). \end{aligned}$$

By the commutativity of diagram (7), $\varphi_{ab}^{\otimes}(\lambda_{n-e}) \in \mathrm{Im}\ (\psi\circ\pi)$.

So diagram (7) gives rise to a commutative diagram:

(8)
$$\begin{array}{ccc} G_n/G_{n+1} & \hookrightarrow & H^{\otimes n}/\sim_n \\ \varphi_n \downarrow & & \downarrow \varphi_{ab}^{\otimes n} \\ G_n/G_{n+1} & \hookrightarrow & H^{\otimes n}/\sim_n \end{array}$$

where the vertical arrows are isomorphisms and horizontal ones are injective.

Tensoring diagram (8) by $\mathbb{R}$, we get a commutative diagram

(9)
$$\begin{array}{ccc} (G_n/G_{n+1})\otimes\mathbb{R} & \hookrightarrow & (H^{\otimes n}/\sim_n)\otimes\mathbb{R} \\ \varphi_n\otimes id_{\mathbb{R}} \downarrow & & \downarrow \varphi_{ab}^{\otimes n}\otimes id_{\mathbb{R}} \\ (G_n/G_{n+1})\otimes\mathbb{R} & \hookrightarrow & (H^{\otimes n}/\sim_n)\otimes\mathbb{R} \end{array}$$

where the horizontal arrows are still injective.

Denote by V_n (resp. $\hat{\varphi}_n$) the $\mathbb{R}$-vector space $(H^{\otimes n}/\sim_n)\otimes\mathbb{R}$ (resp. $\varphi_{ab}^{\otimes n}\otimes id_{\mathbb{R}}$). Since G_n/G_{n+1} is torsion-free, by Labute's theorem we get a commutative diagram, where the horizontal arrows are injective:

(10)
$$\begin{array}{ccc} G_n/G_{n+1} & \hookrightarrow & V_n \\ \varphi_n \downarrow & & \downarrow \hat{\varphi}_n \\ G_n/G_{n+1} & \hookrightarrow & V_n \end{array}$$

LEMMA 4·4. *If $\varphi_{ab} : H \longrightarrow H$ has only real positive eigenvalues, the same holds for $\hat{\varphi}_n$ for any n.*

Proof Tensoring the following diagram by $\mathbb{R}$:

$$\begin{array}{ccc} H^{\otimes n} & \longrightarrow & H^{\otimes n}/\sim_n \\ \downarrow \varphi_{ab}^{\otimes n} & & \downarrow \varphi_{ab}^{\otimes n} \\ H^{\otimes n} & \longrightarrow & H^{\otimes n}/\sim_n \end{array}$$

we get a commutative diagram of vector spaces with exact rows, where E denotes the kernel, and the map $E \to E$ is the restriction of $\varphi_{ab}^{\otimes n} \otimes \mathbb{R}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & H^{\otimes n} \otimes \mathbb{R} & \longrightarrow & V_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \varphi_{ab}^{\otimes n} \otimes \mathbb{R} & & \downarrow \hat{\varphi}_n & & \\ 0 & \longrightarrow & E & \longrightarrow & H^{\otimes n} \otimes \mathbb{R} & \longrightarrow & V_n & \longrightarrow & 0 \end{array}$$

The matrix of $\varphi_{ab}^{\otimes n} \otimes \mathbb{R}$ has the following form in a suitable basis :

$$\left(\begin{array}{c|c} \alpha & ? \\ \hline 0 & \beta \end{array} \right)$$

where α is the matrix of $\varphi_{ab}^{\otimes n} \otimes \mathbb{R}$ restricted to E and β the matrix of $\hat{\varphi}_n$. The eigenvalues of $\varphi_{ab}^{\otimes n} \otimes \mathbb{R}$ are products of the eigenvalues of $\varphi_{ab} \otimes \mathbb{R}$ and therefore all real and positive. So β has only real positive eigenvalues. □

5. *Proof of Theorem* 2·1

Assuming that G and $\varphi: G \to G$ satisfy the hypotheses of Theorem 2·1, we need to construct an ordering of the elements of G which is invariant under multiplication on both sides, and also invariant under the map φ. We may proceed exactly as in section 4 of [**PR**], which we outline here for the reader's convenience.

Go back to diagram (10), where the vector space isomorphism $\hat{\varphi}_n$ has all its eigenvalues real and positive. By standard linear algebra, there exists a basis $v_1, \cdots, v_k$ for V_n with respect to which the matrix for $\hat{\varphi}$ is upper triangular and has its (positive) eigenvalues $\lambda_1, \cdots, \lambda_k$ on the diagonal. We order the vectors in V_n by reverse lexicographical ordering, using their coordinates relative to the basis $v_1, \cdots, v_k$. Thus if $x = x_1 v_1 + \cdots + x_k v_k$ and $y = y_1 v_1 + \cdots + y_k v_k$ are distinct vectors in V_n, we define $x < y$ if and only $x_i < y_i$ (in the usual ordering of $\mathbb{R}$), at the last i for which the coordinates differ. It is routine to check that this ordering of V_n is invariant under vector addition and also that $x < y$ if and only if $\hat{\varphi}(x) < \hat{\varphi}(y)$. Restricting this ordering to the abelian group G_n/G_{n+1} defines an ordering which is invariant under the isomorphism φ_n.

We now have φ_n-invariant orderings of the lower central quotients G_n/G_{n+1}, for each $n \in \mathbb{Z}$, and use this to define a φ-invariant bi-order on G using a well-known technique for ordering groups which are residually nilpotent and have torsion-free lower central quotients. Namely, let $g,\ h \in G$ and consider $n = n(g,h) = n(h,g)$ to be the greatest integer such that $g^{-1}h$ belongs to G_n. Then define $g < h$ if the coset of $g^{-1}h$ is greater than the identity in the ordering of G_n/G_{n+1} and $h < g$ otherwise. It is routine to check that this defines a bi-ordering of G, and it is invariant under φ because the orderings of G_n/G_{n+1} are invariant under φ_n. □

REFERENCES

[**CB**] A. Casson, S. Bleiler : Automorphisms of surfaces after Nielsen and Thurston. *London Math. Soc. Student Texts* 9 (1988).

[**Fox**] R. H. Fox, Free differential calculus I. *Annals of Math* 57 (1953), 547–560.

[**Lab**] J.-P. Labute, On the descending central series of groups with a single defining relation. *Journal of Algebra* 14 (1970), 16–20.

[**PR**] B. Perron, D. Rolfsen, On orderability of fibred knot groups, *Math. Proc. Camb. Phil. Soc.* 135(2003), 135–147.