The Poincaré conjecture and its cousins

Dale Rolfsen

University of British Columbia

By the dawn of the 20th century the classification of surfaces, or 2-manifolds, was well-understood.

In particular, it was known that a 2-manifold which is closed (compact, connected, empty boundary) and simply-connected must be homeomorphic to the 2-sphere, S^2.

In 1904, Henri Poincaré asked if the analogous assertion is true for dimension three.

Poincaré conjecture: If M^3 is a closed 3-manifold which is simply-connected, then M^3 is homeomorphic with S^3, the standard 3-sphere.

An equivalent form is the following: If Q^3 is a compact, contractible 3-manifold, then Q is homeomorphic with the standard 3-ball.

The PC has been the "holy grail" for low-dimensional topologists for many years, and several notorious false proofs have been put forward.

Almost exactly a century after it was proposed, the PC is now considered solved in the affirmative by Grigori Perelman.

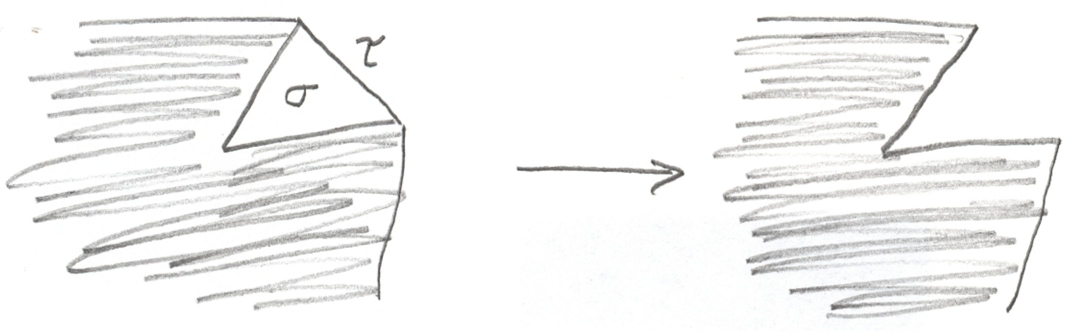
This talk is a discussion of some other conjectures in group theory and low-dimensional topology which are closely related, or even equivalent to, the PC.

Collapsing and simple-homotopy:

Suppose the finite polyhedron K has a simplex σ^n which has a free face τ^{n-1} (meaning $int(\tau)$ does not intersect any other part of K). Then the transition:

$$K \longrightarrow K \setminus \{int(\sigma) \cup int(\tau)\}$$

is called an elementary collapse. The inverse of this operation is an elementary expansion.



J. H. C. Whitehead defined "simple homotopy" to be the equivalence relation among polyhedra which is generated by elementary collapse and expansion. Subdivision is also allowed.

If two polyhedra have the same simple homotopy type, then they are homotopy equivalent, but the converse is not true. Whitehead torsion is an obstruction to going in the other direction.

A sequence of expansions and collapses involving simplices of dimension at most n is called an n-*deformation*.

Theorem (Whitehead-Wall): ($n \neq 2$) If polyhedra K^n and L^n are simple-homotopy equivalent, then there exists an $n+1$-deformation from K to L.

Generalized geometric AC conjecture: same for $n = 2$.

Geometric AC conjecture: K^2 contractible $\Rightarrow K$ 3-deforms to a point.

With the proof of the PC, we now know that the geometric ACC is true for 2-complexes K^2 which happen to embed in a 3-manifold. Call such a complex a *spine*. There is an algorithm, due to Neuwirth, to decide if a given 2-complex is a spine.

Theorem: The AC conjecture is true for spines.

proof: Let N^3 be a regular neighbourhood in a manifold containing the contractible K^2, so that N collapses to K. The PC implies N^3 is homeomorphic with the standard 3-ball, and hence collapsible to a point. This gives the 3-deformation asserted by the ACC:

$$K^2 \swarrow N^3 \searrow pt$$

Zeeman conjecture: K^2 contractible $\Rightarrow K \times I$ collapses to a point.

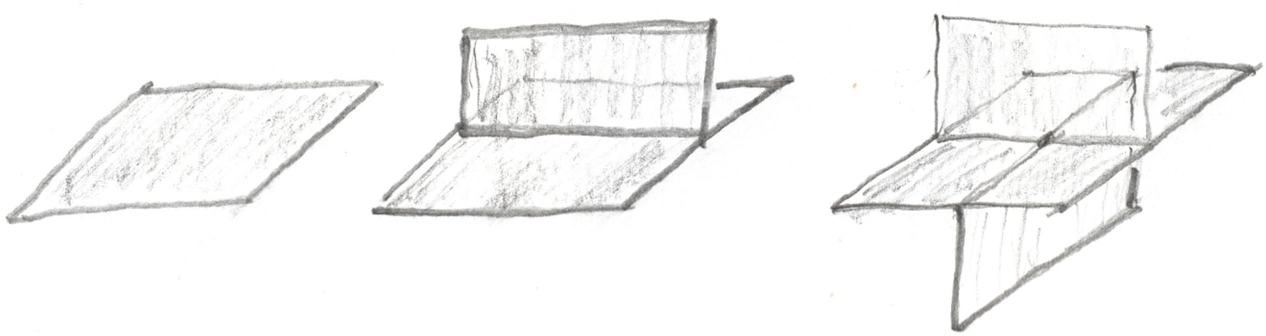
Clearly the ZC implies the ACC, because the transition $K \swarrow K \times I \searrow pt$ gives a 3-deformation.

The ZC also implies the PC, by the following argument: Suppose that Q^3 is a compact, contractible manifold. Q collapses to a "spine" K^2, also contractible. By ZC, $K \times I$ collapses to a point. Therefore $Q \times I$ collapses to a point, and (being a collapsible 4-manifold) it must be a 4-ball. Now $Q \subset \partial(Q \times I) \cong S^3$ and so Q is a 3-ball.

A converse....

A 2-complex is *standard* if it is modeled on the cone upon Δ_1^3, the 1-skeleton of a 3-simplex.

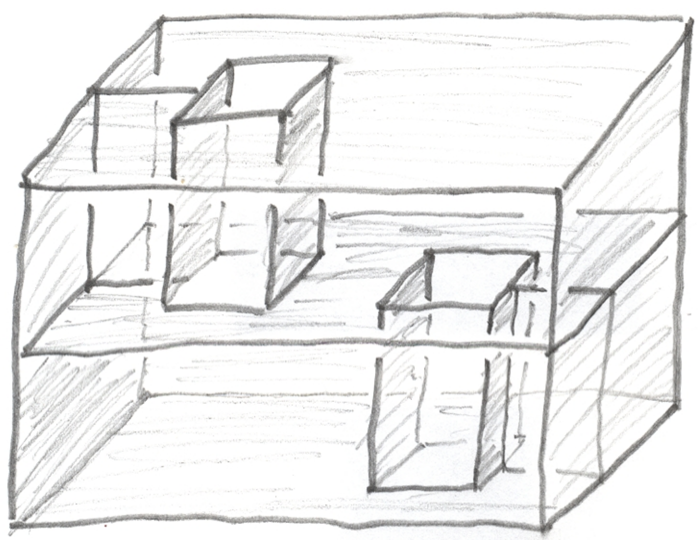
Every 3-manifold with nonempty boundary collapses to a standard spine and is determined by such a spine.



Local structure of a standard complex

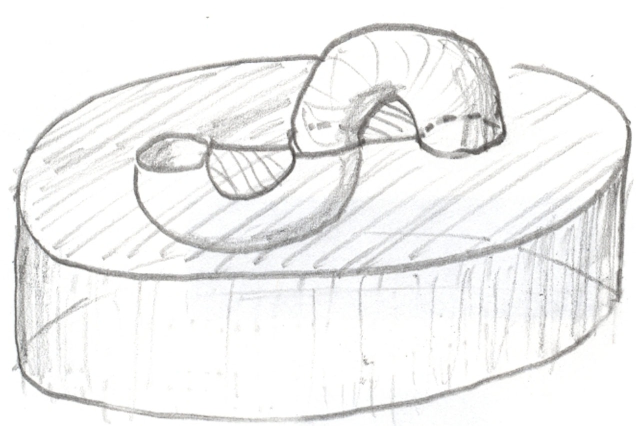
Bing's house with two rooms

A standard spine of the cube



It is contractible, but not collapsible

The igloo



Another contractible, non-collapsible 2-polyhedron

Theorem: (Gillman - R.) The ZC, restricted to standard spines, is equivalent to the PC.

Key idea of the proof: If K^2 is a spine of M^3 and has trivial homology groups, then (by an explicit construction) $K \times I$ collapses to a subset homeomorphic to M. If K is contractible, so is M, and assuming PC, M is a 3-ball, and so $K \times I \to M \to pt$ verifies the ZC for K.

Corollary: The ZC and ACC are true for standard spines.

Another well-known problem concerning 2-D polyhedra, but which seems less connected to the PC.

Whitehead conjecture: If K^2 is a polyhedron which is aspherical ($\pi_n(K) = 0, \forall n \geq 2$), and L^2 is a subpolyhedron, then L is aspherical.

Equivalently, if $L^2 \subset K^2$ and their universal covers are $\tilde{L}$ and $\tilde{K}$, then

$\tilde{K}$ contractible $\Rightarrow$ $\tilde{L}$ contractible.

Group theoretic cousins of the PC ....

First Stallings conjecture: Let $\Sigma_g =$ closed orientable surface of genus $g > 1$, F_1 and F_2 free groups of rank g,

$$\eta : \pi_1(\Sigma_g) \to F_1 \times F_2$$

a *surjective* homomorphism. Then there is a simple closed curve in Σ_g representing a non-trivial element of $\ker(h)$.

Second Stallings conjecture: Let $g > 1$,

$$G = \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1, y_1] \cdots [x_g, y_g] = 1 \rangle$$

F_1 and F_2 free, rank g and

$$\eta : G \to F_1 \times F_2$$

surjective homomorphism. Then η factors through an essential map $G \to G_1 \star G_2$, a free product.

Here, *essential* means that the image of the map is not conjugate to one of the factors G_i.

Note the FSC is a mixture of algebra and topology, whereas the SSC is purely group-theoretic.

Theorem: (Stallings, Jaco) The FSC and SSC are each equivalent to the PC.

Corollary: The two Stallings conjectures are true.

The connection between the group theory and the 3-manifolds is via Heegaard splittings. Every closed oriented 3-manifold is the union of two handlebodies, whose intersection is their common boundary, Σ_g. The map η is the product of the inclusion-induced maps of the surface into the two handlebodies, at the fundamental group level.

Hempel has formulated this in a somewhat different way. Call two group homomorphisms $h_1, h_2 : G \to H$ *equivalent* if there is an automorphism $\alpha : G \to G$ with $h_1 \circ \alpha = h_2$.

Let

$$G = \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1, y_1] \cdots [x_g, y_g] = 1 \rangle$$

and F_1 and F_2 free groups of rank g as above. There is an obvious surjective homomorphism

$$\phi : G \to F_1 \times F_2$$

which takes the x_i to the generators of F_1 and the y_i to the generators of F_2.

Theorem: (Hempel) The PC is true if and only if ϕ is the only surjection of G to $F_1 \times F_2$, up to equivalence.

Corollary: Up to equivalence, ϕ is the unique surjection $G \to F_1 \times F_2$.

Back to Andrews-Curtis, group theoretic version:

Suppose $\langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_n \rangle$ is a "balanced" group presentation. (A relation $u = v$ may represent the relator uv^{-1}.)

Examples: $\langle x, y |\, x, y \rangle$

$$\langle x, y |\, x^p y^q, x^r y^s \rangle, \qquad ps - rq = \pm 1$$

$$\langle x, y |\, x^{-1} y^2 x = y^3, y^{-1} x^2 y = x^3 \rangle$$

$$\langle x, y |\, x^4 y^3 = y^2 x^2, x^6 y^4 = y^3 x^3 \rangle$$

$$\langle x, y, z |\, y^{-1} x y = x^2, z^{-1} y z = y^2, x^{-1} z x = z^2 \rangle$$

all present the trivial group.

Consider the operations, which do not change the group presented:

(1) replace r_i by its inverse r_i^{-1},

(2) replace r_i by $r_i r_j$, $\quad i \neq j$,

(3) replace r_i by $g r_i g^{-1}$, where $g \in F(x_1, \ldots, x_n)$.

Balanced Andrews-Curtis conjecture: If the group presented is the trivial group, then the set $r_1, \ldots, r_n$ may be transformed to $x_1, \ldots, x_n$ by a finite sequence of these three operations (and free reduction of the relators).

If true, the BACC implies that any regular neighbourhood of a contractible 2-dimensional polyhedron in $\mathbb{R}^5$ is a 5-ball.

Consider also the (possibly) weaker:

Andrews-Curtis Conjecture: A balanced presentation of the trivial group can be reduced to the empty presentation by (1)-(3) above, and operation (4) and its inverse:

(4) introduce a new generator x_{n+1} and relator r_{n+1} which coincides with x_{n+1}.

This conjecture is equivalent to the geometric ACC:

K^2 contractible $\Rightarrow K$ 3-deforms to a point.

The connection here between the group theory and polyhedra is through the basic construction of a 2-complex from a group presentation

$$G = \langle x_1, \ldots, x_m \,|\, r_1, \ldots r_n \rangle$$

Begin with a bouquet of m circles, one for each generator.

Sew n disks to this bouquet, the i^{th} disk attached to the bouquet along its boundary circle by "reading off" the relator r_i.

The Klein bottle, the polyhedron corresponding to

$$\langle x, y \,|\, xyx^{-1}y \rangle$$

The dunce hat, corresponding to

$$\langle x \,|\, x^2x^{-1} \rangle$$

It is contractible, but not collapsible.

The fundamental group of the resulting polyhedron is the group G. If the presentation is balanced ($m = n$), and the group G is trivial, then the polyhedron is contractible.

The Andrews-Curtis operations (1) - (4) on a group presentation correspond to 3-deformations of the corresponding polyhedra.

More possible counter-examples to the ACC:

$$\langle x, y, \mid x^n = y^{n+1}, xyx = yxy \rangle \text{ Akbulut} - \text{Kirby}$$

$$\langle x, y, \mid x = [x^p, y^q], y = [x^r, y^s] \rangle \text{ Gordon}$$

Miller and Schupp: If $w = w(x, y)$ is a word with zero exponent sum in x:

$$\langle x, y, \mid x^{-1} y^n x = y^{n+1}, x = w \rangle$$

The Grigurchuk-Kurchanov conjecture

Consider the free group $F_{2n} = \langle a_1, \ldots, a_n, b_1, \ldots b_n \rangle$ and Let

$$\beta : F_{2n} \longrightarrow F_n \times F_n$$

be the homomorphism which takes $a_1, \ldots a_n$ to the generators of the first F_n and $b_1, \ldots, b_n$ to the generators of the second free group in the product.

GK conj: Any surjective homomorphism

$$h : F_{2n} \longrightarrow F_n \times F_n$$

is equivalent to β, that is $h \circ \alpha = \beta$ for some automorphism α of F_{2n}.

Theorem: The GK conjecture implies the AC conjecture.

Note the similarity with Hempel's group theoretic analog of the PC, now known to be true. It is the same as the GK conjecture, with the genus n surface group replacing F_{2n}.

Another open question of combinatorial group theory is the following

Consider a group G with a presentation $\langle X|R\rangle$. Let y be a new generator and r a single new relator, a word in $X \cup \{y\}$.

Kervaire conjecture: If the group $\langle X \cup \{y\}|R \cup \{r\}\rangle$ is trivial, then G must have been the trivial group.

In summary, we have discussed conjectures of geometric topology . . .

- Geometric AC conjecture – still unsolved, but true for spines of 3-manifolds

- Zeeman's conjecture – also open in general, implies both ACC and PC, true for standard spines

- Whitehead's conjecture – still unsolved and group-theoretical conjectures . . .

• Andrews-Curtis conjecture – still open

• Stallings' two conjectures – equivalent to the PC, and hence "solved"

• Kervaire's conjecture – known true for torsion-free groups

• Uniqueness of surjection $\phi : \pi_1(\Sigma_g) \to F_g \times F_g$ – equivalent to the PC

• Uniqueness of surjection $\beta : F_{2n} \to F_n \times F_n$ – which implies the ACC

Stallings' "proof" strategy for the PC involved reducing it to equivalent group-theoretical problems. Although they did not serve to prove the PC, it eventually worked in reverse. Perelman's proof of the PC verified Stallings' conjectures!

In "How not to prove the Poincaré conjecture" (1966), John Stallings concluded with these words of advice:

"I have committed the sin of falsely proving the Poincaré conjecture. ...I was unable to find flaws in my "proof" for quite a while, even though the error is very obvious. It was a psychological problem, a blindness, an excitement, an inhibition of reasoning by an underlying fear of being wrong. Techniques leading to the abandonment of such inhibitions should be cultivated by every honest mathematician."