Geometric subgroups of mapping class groups

Luis Paris Dale Rolfsen

September 20, 2002

Abstract

This paper is a study of the subgroups of mapping class groups of Riemann surfaces, called "geometric" subgroups, corresponding to the inclusion of subsurfaces. Our analysis includes surfaces with boundary and with punctures. The centres of all the mapping class groups are calculated. We determine the kernel of inclusion-induced maps of the mapping class group of a subsurface, and give necessary and sufficient conditions for injectivity. In the injective case, we show that the commensurability class of a geometric subgroup completely determines up to isotopy the defining subsurface, and we characterize centralizers, normalizers, and commensurators of geometric subgroups.

Mathematics Subject Classification: Primary 57N05; Secondary 20F38

1 Introduction

Throughout the paper, M will denote a *compact, connected, oriented* surface. The boundary ∂M, if nonempty, is a finite collection of simple closed curves. Consider a finite subset $P = \{p_1, \dots, p_m\}$, of m distinct points (often called "punctures" or "marked points") in the interior of M. Define $\mathcal{H}(M, P)$ to be the group of orientation-preserving homeomorphisms $h : M \to M$ such that h is the identity on each boundary component of M and $h(P) = P$. Our main object of study is the *mapping class group* $\mathcal{M}(M, P) = \pi_0(\mathcal{H}(M, P))$, the set of isotopy classes of these mappings, with composition as the group operation. We emphasize that throughout an isotopy, the boundary, and also the points P remain fixed. It is clear that, up to isomorphism, these groups do not depend on the choice of P, but depend only on the cardinality $m = |P|$, so we may write (M, m) or $\mathcal{M}(M, m)$ in place of (M, P) or $\mathcal{M}(M, P)$, and simply $\mathcal{M}(M)$ for $\mathcal{M}(M, \emptyset)$. $\mathcal{M}(M, P)$ may equivalently be considered as the group of orientation-preserving *diffeomorphisms* of (M, P), up to smooth isotopy. We refer the reader to the survey articles [2], [14], [15], [30] and [31] and their bibliographies for more information.

Let $N \subset M$ be a *subsurface*, by which we mean a closed subset which is also a surface and for which we always assume the further properties: (1) every component of ∂N lies in the interior of M, (2) $P \cap \partial N = \emptyset$.

The inclusion $i : (N, N \cap P) \to (M, P)$ induces a natural mapping

$$i_* : \mathcal{M}(N, N \cap P) \to \mathcal{M}(M, P).$$

If $[h]$ is a class of a homeomorphism of N, then $i_*([h])$ is represented by extending h to M using the identity mapping on $M \setminus N$. The image $i_*(\mathcal{M}(N, N \cap P))$ will be called a *geometric subgroup* of $\mathcal{M}(M, P)$.

Our study of these subgroups depends on a careful analysis of curves in M and Dehn twists, which are the subject of Section 3. The mapping i_* is often, but not always, injective. We determine its kernel in Section 4. Section 5 is devoted to the centres of mapping class groups. These are certainly well-known to specialists and many of them can be found in the literature (see [15] and [17]). However, we need the general result for the remainder of the paper and the proofs are straightforward applications of Section 3. Our main result is that, assuming injectivity of the i_*, up to a finite number of exceptions, two geometric subgroups are commensurable if and only if they are equal if and only if their respective defining subsurfaces are isotopic (Theorem 6.5). Note that the assumption that ∂N lies entirely in the interior of M is necessary for the conclusion of Theorem 6.5; indeed, without this assumption, it is very easy to construct non-isotopic subsurfaces N and N' which define the same geometric subgroup, N satisfying $\partial N \cap \partial M = \emptyset$ and N' satisfying $\partial N' \cap \partial M \neq \emptyset$. From the main result, still assuming injectivity of the i_*, we characterize the commensurator, the normalizer and the centralizer of a geometric subgroup in $\mathcal{M}(M, P)$.

We close this introduction and illustrate the injectivity question by discussing some basic examples, which are well-known (c.f. [1] and [7]).

Examples: (1) $\mathcal{M}(D^2) \cong \{1\}$ and $\mathcal{M}(D^2, 1) \cong \{1\}$, where D^2 is a disk.

(2) Similarly, for the 2-sphere, $\mathcal{M}(S^2)$ and $\mathcal{M}(S^2, 1)$ are trivial.

The above examples are essentially the only surface mapping class groups which are trivial.

(3) $\mathcal{M}(S^1 \times I) \cong \mathbf{Z}$.

The mapping class group of the annulus $S^1 \times I$ is generated by a *Dehn twist*, described in Section 3.

(4) $\mathcal{M}(S^1 \times I, 1) \cong \mathbf{Z}^2$. Here, Dehn twists along the two boundary components are generators. They are not isotopic because of the puncture.

(5) As a family of examples, consider disks $D_1 \subset D_2 \subset \ldots$ in the complex plane, where the diameter of D_m is taken to be the real line interval $[1 - m/(m+1), m + 1/2]$. Take $P_m = D_m \cap \mathbf{Z} = \{1, \ldots, m\}$. Then it is also well-known that the mapping class groups $\mathcal{M}(D_m, P_m)$ are isomorphic with the classical braid groups B_m of Artin, which have generators $\sigma_1, \ldots, \sigma_{m-1}$ subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1, \qquad\qquad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

Under the isomorphism, $\sigma_j, \quad 0 < j < m$ corresponds with the (class of the) diffeomorphism consisting of a " half-twist" interchanging the integers j and $j+1$, and supported on a small neighborhood of the interval $[j, j+1] \subset \mathbf{C}$. See [1] and [7] for details, but beware some differences in choice of conventions. It is classical, but nontrivial, that for $n < m$ the homomorphism $B_n \to B_m$, taking $\sigma_j \in B_n$ to $\sigma_j \in B_m$ is injective, allowing us to write $B_n \subset B_m$. We chose notation so that under the isomorphisms, $B_n \to B_m$ corresponds to the inclusion-induced mapping $i_* : \mathcal{M}(D_n, P_n) \to \mathcal{M}(D_m, P_m)$. We conclude that in this case i_* is injective. Note that the closure of the complementary subsurface is an annulus with $m - n$ punctures. The commensurator, the normalizer and the centralizer of $\mathcal{M}(D_n, P_n)$ in $\mathcal{M}(D_m, P_m)$ are characterized in [8] and [35].

(6) The following is an example of an inclusion map which is not injective on mapping class groups. Take $M = S^2$ a 2-sphere with $P = 2$ points in S^2, and let D be a disk in S^2 which encloses the points P. We have the map $i_* : \mathcal{M}(D, P) \to \mathcal{M}(S^2, P)$. As already discussed, $\mathcal{M}(D, P)$ is the braid group B_2, which is infinite cyclic, generated by σ_1. However, σ_1^2 is isotopic with a Dehn twist along ∂D. In the larger surface $M = S^2$ this twist is isotopic with the identity (rel P). So the kernel of i_* in this case is the infinite cyclic subgroup of index 2 in $\mathcal{M}(D, P)$. $\mathcal{M}(S^2, 2)$ is cyclic of order 2.

(7) For the torus $T^2 = S^1 \times S^1$ with either zero or one puncture, the mapping class group is the modular group of invertible 2×2 matrices with integer entries: $\mathcal{M}(T^2) \cong \mathcal{M}(T^2, 1) \cong SL(2, \mathbf{Z})$. Dehn twists along the curves $S^1 \times *$ and $* \times S^1$ generate the mapping class group. Note that if A is an annulus neighborhood of one of these curves, and happens to enclose the puncture of $\mathcal{M}(T^2, 1)$, the map $i_* : \mathcal{M}(A, 1) \to \mathcal{M}(T^2, 1)$ fails to be injective.

2 Subgroups of mapping class groups

In this section we review some of the literature regarding subgroups of mapping class groups. Although there is little published on the geometric subgroups, which are the main concern of the present paper, certain other subgroups are quite well understood. First, we recall some general properties of the mapping class groups themselves. A more complete survey can be found in [30].

For the closed surface M_g of genus g, the mapping class groups $\mathcal{M}(M_g, m)$ are known to be finitely presented [9], [11], [23] [25], [26], [29], [36], [37]. The generators can be taken to be Dehn twists (discussed below) along curves and half-twists along arcs connecting the punctures.

According to Grossman [10] and Ivanov [15], $\mathcal{M}(M_g, m)$ is residually finite – for every nontrivial element, there is a homomorphism of the mapping class group onto a finite group which does not kill that element. This implies that $\mathcal{M}(M_g, m)$ is Hopfian ([24], Chapter IV) – every epimorphism $\mathcal{M}(M_g, m) \to \mathcal{M}(M_g, m)$ is an isomorphism. Conversely, in a recent paper [17] Ivanov and McCarthy proved that $\mathcal{M}(M_g, m)$ is co-Hopfian – every monomorphism $\mathcal{M}(M_g, m) \to \mathcal{M}(M_g, m)$ is an isomorphism.

Although, it contains torsion elements, $\mathcal{M}(M_g)$ has a finite index subgroup which is torsion free (see [30] for a sketch of a proof.) It has recently been shown by Mosher [32] that the mapping class groups are *automatic*. This implies that the word problem is solvable (in quadratic time) and many other consequences [6].

The outer automorphism group of $\mathcal{M}(M_g, m)$ has been determined by Ivanov [16] and McCarthy [28]. It is equal to $\mathbf{Z}/2\mathbf{Z}$ under the assumptions $g \neq 0$, $m \geq 3$ if $g = 1$, and $m \geq 1$ if $g = 2$.

The abelianization of $\mathcal{M}(M_g)$ is cyclic of order 12, when $g = 1$, cyclic of order 10 when $g = 2$, and trivial (that is, $\mathcal{M}(M_g)$ is perfect) for $g > 2$ [34].

Finite subgroups of $\mathcal{M}(M_g)$ have been extensively studied. The so-called Nielsen realization problem [38] was solved by Kerckhoff [22]. It asserts that for any finite subgroup F of $\mathcal{M}(M_g)$, there is a complex structure on M_g such that F is realized as a group of holomorphic automorphisms of M_g. According to a classical result of Hurwitz [12], the orders of finite subgroups are bounded: $|F| \leq 84(g-1)$, $g > 1$.

McCarthy [27] showed that subgroups of $\mathcal{M}(M_g)$ satisfy the Tits alternative: every subgroup either contains a free group on two generators, or a solvable subgroup of finite index. Birman, Lubotzky and McCarthy [3] proved that solvable subgroups of $\mathcal{M}(M_g)$ are virtually abelian, and gave upper bounds for the rank of free abelian subgroups. Ivanov [15] proved these two results for $\mathcal{M}(M_g, m)$ and showed that: if G is a subgroup of $\mathcal{M}(M_g, m)$ which is not virtually abelian, then

G contains an uncountably infinite number of maximal subgroups of infinite index. The *Frattini subgroup* $\phi(G)$ of a group G is the intersection of all its maximal subgroups. Ivanov also proved that the Frattini subgroup of a finitely generated subgroup of $\mathcal{M}(M_g, m)$ is nilpotent. Note that this property also holds for finitely generated linear groups [33].

The centres of the $\mathcal{M}(M_g, m)$ are well-known [15], [17]: cyclic of order two if $g = 1$ and $m \leq 2$, and if $g = 2$ and $m = 0$, and trivial otherwise. We determine the centres in the more general situation, with boundary, in the present paper.

The Torelli subgroup of the mapping class group consists of classes of mappings which induce the identity on the homology of the surface. These subgroups have been studied extensively in the series of papers [18], [19], [20], [21]. In particular, they describe a finite set of generators for the Torelli groups. The cohomological properties of mapping class groups and the Torelli groups have been studied intensively and are also well-described in [30] and [31]. We shall not use any of these deeper properties – indeed our methods are quite elementary and self-contained, requiring as background only some basic properties of curves on surfaces due to Epstein [5].

3 Curves and Dehn twists

Working within the context of a surface M with punctures P as described above, we shall consider a *simple closed curve* in $M \setminus P$ as an embedding $c : S^1 \to M \setminus P$ which does not intersect the boundary of M. Note that c has an orientation; the curve with the opposite orientation, but same image will be denoted c^{-1}. By abuse of notation, we also use the symbol c to denote the image of c. We will say that c is *essential* if it does not bound a disk in M disjoint from P, and that c is *generic* if it does not bound a disk in M containing 0 or 1 point of P.

Two simple closed curves a, b are *isotopic* if there exists a continuous family $h_t \in \mathcal{H}(M, P)$, $t \in [0, 1]$ of homeomorphisms such that h_0 is the identity and $h_1 \circ a = b$. Isotopy of curves is an equivalence relation which we denote by $a \simeq b$. Following [7] the *index of intersection* of two simple closed curves a and b is:

$$I(a, b) = \inf\{|a' \cap b'|; a' \simeq a, b' \simeq b\}$$

We note that:
1) $I(a, b) = \inf\{|a' \cap b|; a' \simeq a\}$;
2) If a is not generic, then $I(a, b) = 0$ for every simple closed curve b;
3) If $a \simeq b$ then $I(a, b) = 0$.

A *bigon* cobounded by two simple closed curves a and b in $M \setminus P$ is a disk $D \subset M \setminus P$ whose boundary is the union of an arc of a and an arc of b.

Proposition 3.1 (Epstein [5]) *Let $a, b : S^1 \to M \setminus P$ be two essential simple closed curves, and suppose a is isotopic to b.*

i) If $a \cap b = \emptyset$, then there exists an annulus in $M \setminus P$ whose boundary components are a and b.

ii) If $a \cap b \neq \emptyset$, and they intersect transversely, then a and b cobound a bigon. □

Proposition 3.2 *Let $a, b : S^1 \to M \setminus P$ be two essential simple closed curves, which intersect transversely. Then*

$$I(a, b) = |a \cap b|$$

if and only if a and b do not cobound a bigon.

Proof: It is clear that if a and b cobound a bigon, one can isotop one of the curves across the bigon and reduce the cardinality of the intersection by two. Now suppose they do not cobound a bigon, and choose a simple closed curve a' isotopic to a and transverse to a such that

$$|a' \cap b| = I(a, b).$$

We will argue by induction on $|a' \cap a|$ that

$$|a \cap b| = |a' \cap b| = I(a, b).$$

If $|a' \cap a| = 0$, then by Proposition 3.1, there is an annulus in $M \setminus P$ with boundary components a and a'. Each arc of intersection of b with the annulus must run from one boundary component to the other (see Figure 1), since neither a, b nor a', b cobound bigons. Therefore $|a \cap b| = |a' \cap b| = I(a, b)$.

Now suppose $|a' \cap a| > 0$. By Proposition 3.1, a and a' cobound a bigon. We may assume $a' \cap a \cap b$ is empty, so any arc of intersection of b with the bigon must have one endpoint in a and the other in a' (see Figure 1), again because neither a, b nor a', b cobound bigons. Therefore we may push a' across the bigon to obtain a new curve a'' isotopic to a and satisfying:

$$|a'' \cap a| = |a' \cap a| - 2 \quad \text{and} \quad |a'' \cap b| = |a' \cap b| = I(a, b).$$

By inductive hypothesis $|a \cap b| = |a'' \cap b| = I(a, b)$. □

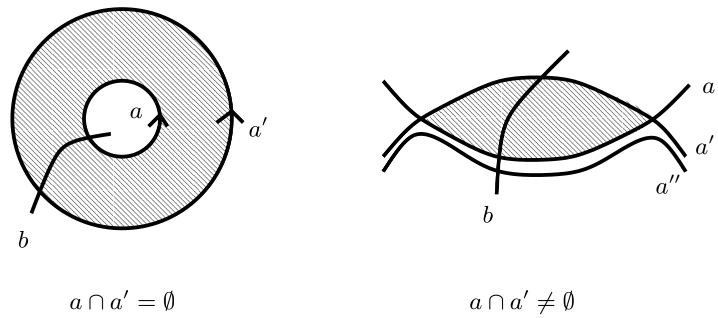

Figure 1: Curves cobounding an annulus and a bigon.

Definition: If we parametrize S^1 as the unit circle in the complex plane, and the interval $I = [0, 1]$, then the prototype *Dehn twist* $\tau : S^1 \times I \to S^1 \times I$ is given by

$$\tau(z, t) := (ze^{2\pi it}, t).$$

Note that τ is the identity on the boundary circles. More generally, let $a : S^1 \to M \setminus P$ be a simple closed curve, and let $N \subset M \setminus P$ be an annulus regular neighborhood of the image of a, parametrized by $\tilde{a} : S^1 \times I \to N$. Define the *Dehn twist along* a to be (the isotopy class of) the homeomorphism $A(x) = \tilde{a}\tau\tilde{a}^{-1}(x)$ for $x \in N$, $A(x) = x$ for x outside N.

We will use the convention throughout that a curve is denoted by a lower case letter and a Dehn twist along the curve is denoted by the corresponding capital letter. Note that, depending on parametrization chosen, there are two choices of Dehn twist along a, inverse to each other. Usually, the choice is immaterial, provided one is consistent throughout, but we will adopt the convention in illustrations that a curve crossing a will make a "right turn" at each encounter with a, after being acted on by A (see Figure 2). We also observe:

1) The Dehn twist along a^{-1} coincides with the Dehn twist along a.
2) The curve a is fixed by the Dehn twist A.
3) If two curves are isotopic, then so are their corresponding Dehn twists.
4) If h is a homeomorphism of M, the Dehn twist along $h(a)$ is hAh^{-1}.
5) If a is not generic, A is isotopic to the identity.

Proposition 3.3 *Let $a, b : S^1 \to M \setminus P$ be two simple closed curves, A the Dehn twist along a and n any integer. Then*

$$I(A^n(b), b) = |n| \cdot I(a, b)^2.$$

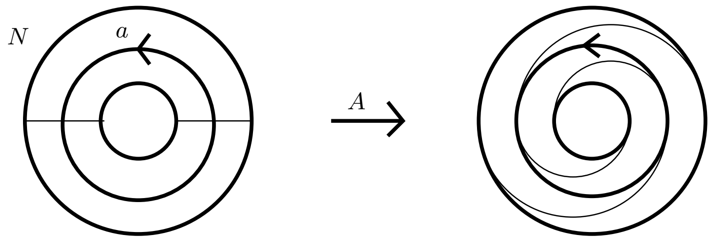

Figure 2: Dehn twist along curve a.

Proof: This is a special case of a formula in [7]. We outline a proof, leaving details to the reader. Assume $|a \cap b| = I(a,b)$. The cases $n = 0$ or $I(a,b) = 0$ being trivial, suppose they are nonzero. Construct the curve $A^n(b)$, which can be seen to cross b exactly $|n|I(a,b)$ times at each point of intersection of a with b (see Figure 3). The proof is completed by noting that this is the minimal intersection of $A^n(b)$ with b, up to isotopy. For otherwise, by Proposition 3.2 there would be a bigon cobounded by them. One can see this would imply that a and b also cobound a bigon, which is impossible, again by Proposition 3.2. $\square$

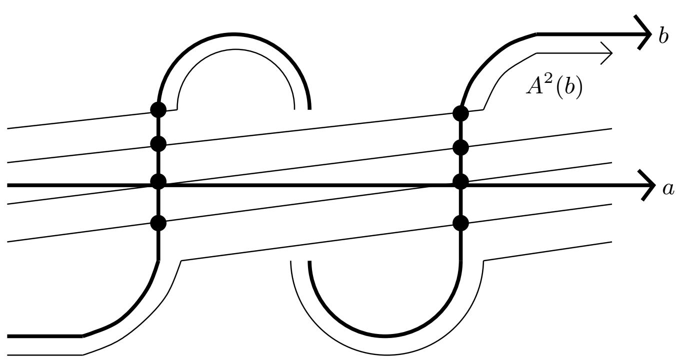

Figure 3: Intersection of b with $A^2(b)$.

Proposition 3.4 *Suppose $a_1, \dots, a_p : S^1 \to M \setminus P$ are generic simple closed curves such that:*

a) $a_i \cap a_j = \emptyset$ if $i \neq j$,

b) a_i is neither isotopic with a_j nor a_j^{-1}, if $i \neq j$,

c) none of the a_i is isotopic with a boundary component of M.

Then for each i, $1 \leq i \leq p$, there exists a simple closed curve $b : S^1 \to M \setminus P$ such that $a_j \cap b = \emptyset$ if $i \neq j$, and $|a_i \cap b| = I(a_i, b) > 0$.

Remark: The last condition implies that b must be generic in $M \setminus P$.

Proof: We "cut open" M along all the curves a_i to obtain the connected compact surfaces $N_1, \ldots, N_r$ with the property that the union of the interiors of the N_j is the interior of $M \setminus \bigcup_{i=1}^{p} a_i$ and each boundary component of N_j is either a boundary component of M or a copy of some a_i. There is a continuous projection of the disjoint union onto M:

$$\rho : \coprod_{j=1}^{r} N_j \to M,$$

which covers each curve a_i twice, and is injective on the union of the interiors of the N_j. Now fix $i \in \{1, \ldots, r\}$ and consider the curve $a_i = \rho(c_1) = \rho(c_2)$, where c_1 is a component of the boundary of some N_j and c_2 is a component of the boundary of N_k.

Case 1, $j = k$: Then c_1 and c_2 are different components of the boundary of the connected surface N_j. There is an arc $\tilde{b}$ in N_j with one endpoint in c_1, the other endpoint being the point of c_2 having same projection in a_i, and the interior of $\tilde{b}$ in the interior of N_j and avoiding the puncture set P (see Figure 4). Then take $b = \rho(\tilde{b})$. Clearly $a_j \cap b = \emptyset$ if $i \neq j$; moreover a_i and b are transverse and they cannot cobound a bigon, so $|a_i \cap b| = 1 = I(a_i, b)$.

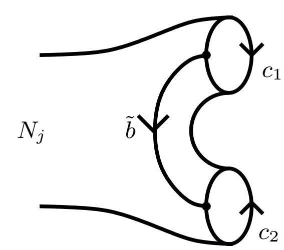

Figure 4: Constructing b, case 1.

Case 2, $j \neq k$: Then c_1 is a boundary component of N_j, and by hypothesis, if N_j is a disk, then $N_j \cap P$ contains at least two points, and if N_j is an annulus, $N_j \cap P$ contains at least one point. So in any case, there exists an arc b_1 in N_j with both endpoints in c_1, interior in the interior of N_j and disjoint from P, and such that b_1 and c_1 do not cobound a bigon in $N_j \setminus P \cap N_j$ (see Figure 5). In the same way we choose an arc b_2 in N_k, whose endpoints in c_2 project to $\rho(b_1 \cap c_1)$ in a_i, whose interior is interior to $N_k \setminus P \cap N_k$, and such that b_2 and c_2 do not cobound a bigon in $N_k \setminus P \cap N_k$. Define $b = \rho(b_1 \cup b_2)$. Then $a_j \cap b = \emptyset$ if $i \neq j$

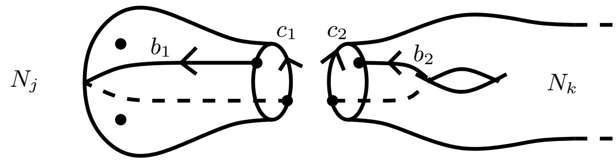

Figure 5: Constructing b, case 2.

and $|a_i \cap b| = I(a_i, b) = 2$, because we have arranged that a_i and b do not cobound a bigon in $M \setminus P$. □

Now consider a subsurface $N \subset M$; recall this includes the assumption that $P \cap \partial N = \emptyset$ and ∂N is interior to M. We will say that N is *essential* if each component of $\overline{M \setminus N}$ which is a disk has nonempty intersection with the puncture set P.

A component N' of $\overline{M \setminus N}$ will be called an *exterior cylinder* if N' is a cylinder (= annulus) disjoint from P, with both components of $\partial N'$ also being components of ∂N (see Figure 6).

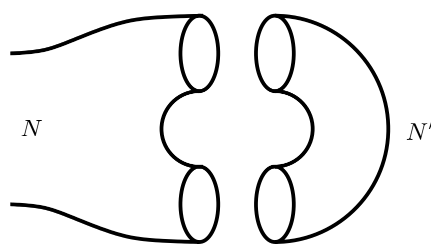

Figure 6: N' is an exterior cylinder for N.

Proposition 3.5 *Let $N \subset M$ be an essential subsurface and let $a, b : S^1 \to N \setminus N \cap P$ be essential simple closed curves. Assume that a is not isotopic in $N \setminus N \cap P$ to a boundary component of an exterior cylinder. Then a and b are isotopic in $M \setminus P$ if and only if they are isotopic in $N \setminus N \cap P$.*

Proof: The only nontrivial part is to show that if a and b are isotopic in $M \setminus P$, then they are isotopic in $N \setminus N \cap P$. We assume a and b intersect transversely and argue by induction on $|a \cap b|$.

If $|a \cap b| = 0$, then Proposition 3.1 implies there exists an annulus in $M \setminus P$ with (unoriented) boundary $a \cup b$. Since a is not isotopic to a boundary component of an exterior cylinder and N is essential, the annulus is disjoint from ∂N, and therefore it lies in $N \setminus N \cap P$. It follows that a and b are isotopic in $N \setminus N \cap P$.

If $|a \cap b| > 0$, then a and b cobound a bigon in $M \setminus P$, by Proposition 3.1. Because N is essential, the bigon is disjoint from ∂N, and therefore it lies in N. Pushing across this bigon defines an isotopy in $N \setminus P$ from b to a curve b' with $|b' \cap a| = |b \cap a| - 2$. By inductive hypothesis, b' is isotopic with a in $N \setminus N \cap P$, so the same is true of b. □

Proposition 3.6 *Consider two generic simple closed curves $a, b : S^1 \to M \setminus P$, and let A and B, respectively, denote Dehn twists along these curves. If j and k are integers, $j \neq 0$, such that $A^j = B^k$ in $\mathcal{M}(M,P)$, then a is isotopic to b or b^{-1} in $M \setminus P$.*

Remark: This proposition and the next one can be found in [17] for $\partial M = \emptyset$. The general case needs an extra argument to consider Dehn twists along boundary curves.

Proof: We will argue that if a is not isotopic with $b^{\pm 1}$ then $A^j \neq B^k$. It may be assumed that a and b meet transversely and that $|a \cap b| = I(a,b)$. First assume $I(a,b) > 0$. Then, by Proposition 3.3,

$$I(A^j(b), b) = |j| I(a,b)^2 > 0$$

$$I(B^k(b), b) = I(b,b) = 0$$

and we conclude $A^j \neq B^k$.

Now suppose that $I(a,b) = 0$. If M has nonempty boundary, consider the larger closed surface $\hat{M}$ obtained by gluing a torus minus a disk to each boundary component of M (if $\partial M = \emptyset$, let $\hat{M} = M$). By Proposition 3.5 a is not isotopic with $b^{\pm 1}$ in $\hat{M} \setminus P$. By Proposition 3.4 there is a simple closed curve c in $\hat{M} \setminus P$ such that $b \cap c = \emptyset$ and $|a \cap c| = I(a,c) > 0$. Then

$$I(A^j(c), c) = |j| I(a,c)^2 > 0$$

$$I(B^k(c), c) = I(c,c) = 0,$$

and therefore $A^j \neq B^k$ in $\mathcal{M}(\hat{M}, P)$; so $A^j \neq B^k$ in $\mathcal{M}(M,P)$. □

Proposition 3.7 *Consider two generic simple closed curves $a, b : S^1 \to M \setminus P$, and let A and B, respectively, denote Dehn twists along these curves. If j and k are integers, $j \neq 0$ and $k \neq 0$, such that A^j and B^k commute in $\mathcal{M}(M,P)$, then $I(a,b) = 0$.*

Proof: Assuming A^j and B^k commute, we have

$$A^j = B^k A^j B^{-k} = C^j,$$

where C is the Dehn twist along the curve $c = B^k(a)$. By Proposition 3.6 it follows that c is isotopic with $a^{\pm 1}$. Proposition 3.3 implies

$$0 = I(a^{\pm 1}, a) = I(c, a) = I(B^k(a), a) = |k| I(a, b)^2,$$

so $I(a, b) = 0$. □

Remark: An alternative proof of Proposition 3.7 can be deduced from [13], where it is shown that A and B commute if $I(a, b) = 0$, A and B satisfy the braid relation $ABA = BAB$ if $I(a, b) = 1$, and A and B generate a free group if $I(a, b) \geq 2$.

Proposition 3.8 *Suppose $a_1, \ldots, a_p : S^1 \to M \setminus P$ are generic simple closed curves which are pairwise disjoint, and no curve a_i is isotopic to a_j or a_j^{-1}, $i \neq j$. Consider the function*

$$h : \mathbf{Z}^p \to \mathcal{M}(M, P)$$

defined by

$$h(n_1, \ldots, n_p) = A_1^{n_1} \cdots A_p^{n_p},$$

where A_i is the Dehn twist about a_i. Then h is an injective homomorphism.

Proof: Because the curves are disjoint, the Dehn twists commute, and h is a homomorphism. To see it is injective, suppose $A_1^{n_1} \cdots A_p^{n_p}$ is the identity of $\mathcal{M}(M, P)$ for some $(n_1, \ldots, n_p)$. We again employ the trick of considering the closed surface $\hat{M}$, which is M plus a copy of a torus minus a disk glued to each boundary component. Clearly, each $a_i : S^1 \to \hat{M}$ is generic, and by Proposition 3.5 a_i is not isotopic to $a_j^{\pm 1}$ in $\hat{M} \setminus P$, when $i \neq j$. Now fix an index $i \in \{1, \ldots, p\}$. Proposition 3.4 supplies a simple closed curve b in $\hat{M} \setminus P$ disjoint from a_j, $\;i \neq j$, with

$$|a_i \cap b| = I(a_i, b) > 0.$$

We calculate, using commutativity of the twists, $b \cap a_j = \emptyset$, $\;i \neq j$, and Proposition 3.3:

$$\begin{aligned} 0 = I(b, b) &= I(A_1^{n_1} \cdots A_p^{n_p}(b), b) \\ &= I(A_i^{n_i}(b), b) = |n_i| I(a_i, b)^2. \end{aligned}$$

Therefore $n_i = 0$ and we have shown $i_* \circ h$ is injective, where $i : (M, P) \to (\hat{M}, P)$, so h is injective. □

Corollary 3.9 *If $a : S^1 \to M \setminus P$ is a generic simple closed curve, then the Dehn twist about a has infinite order in $\mathcal{M}(M, P)$.* □

Proposition 3.10 *Let $a_1, \ldots, a_p, b_1, \ldots, b_p : S^1 \to M \setminus P$ be essential simple closed curves satisfying:*

1) $a_i \cap a_j = \emptyset$ and $b_i \cap b_j = \emptyset$ if $i \neq j$;

2) a_i is not isotopic with $a_j^{\pm 1}$ and b_i is not isotopic with $b_j^{\pm 1}$ if $i \neq j$;

3) a_i is isotopic to b_i for each $i = 1, \ldots, p$.

Then there exists an isotopy $h_t \in \mathcal{H}(M, P)$ such that $h_0 = id$ and $h_1 \circ a_i = b_i$ for all $i = 1, \ldots, p$.

Proof: We will use a double induction. First, induction on p. The proposition is obvious if $p = 1$, so we assume it is true for $p-1$ pairs of curves. This means that, replacing each a_i by $h_1 \circ a_i$, we may assume that $a_i = b_i$ for $i = 1, \ldots, p-1$. Then we have a_p disjoint from $a_j = b_j, \ \ j < p$ and also b_p disjoint from $a_j = b_j, \ \ j < p$, and a_p isotopic in $M \setminus P$ to b_p. We will be done if we show that there is a further isotopy taking a_p to b_p which does not move the curves $a_j = b_j, \ \ j < p$. Taking a_p and b_p to be transverse, we will argue by induction on $|a_p \cap b_p|$.

If $|a_p \cap b_p| = 0$, then a_p and b_p cobound an annulus in $M \setminus P$, by Proposition 3.1. Any simple closed curve in this annulus must be either inessential or parallel to a boundary component, so our hypotheses guarantee that the annulus is disjoint from all the curves $a_j = b_j, \ \ j < p$. Then there is an isotopy across the annulus taking a_p to b_p; the isotopy may be taken to be the identity outside a small neighborhood of the annulus, so the other curves do not move. Suppose $|a_p \cap b_p| > 0$, then by Proposition 3.1, the curves cobound a bigon in $M \setminus P$, and we may argue as above that the bigon is disjoint from the other curves. An isotopy taking a_p across the bigon, fixed outside a neighborhood of the bigon, reduces the number $|a_p \cap b_p|$ and does not move the other curves. The inductive hypothesis now gives a final isotopy taking a_p to b_p. □

4 Subsurfaces and injectivity

Define the *pure* mapping class group of M relative to P to be the subgroup $\mathcal{PM}(M, P)$ of $\mathcal{M}(M, P)$ consisting of all (classes of) diffeomorphisms which fix P pointwise. Letting Σ_P denote the group of permutations of the set P, we have the exact sequence

$$1 \to \mathcal{PM}(M, P) \to \mathcal{M}(M, P) \to \Sigma_P \to 1.$$

Definitions: A pair equivalent to $(D^2, 2)$ is called a *pantalon of type I* (see Figure 7). We have already observed that $\mathcal{M}(D^2, 2)$ is infinite cyclic, generated by a "half-twist" σ which interchanges the two punctures. $\mathcal{PM}(D^2, 2)$ is the subgroup generated by σ^2, which is (up to isotopy) the same as a Dehn twist along the boundary.

A pair $(S^1 \times I, 1)$ is a *pantalon of type II* (see Figure 7). The mapping class groups $\mathcal{M}(S^1 \times I, 1)$ and $\mathcal{PM}(S^1 \times I, 1)$ coincide, and are isomorphic with $\mathbf{Z}^2$, generated by the Dehn twists along the two boundary components.

A *pantalon of type III* is a connected planar surface M with three boundary components, the puncture set P is taken to be empty (see Figure 7). Its mapping class group is $\mathcal{M}(M, 0) \cong \mathbf{Z}^3$, generated by Dehn twists along the three boundary curves.

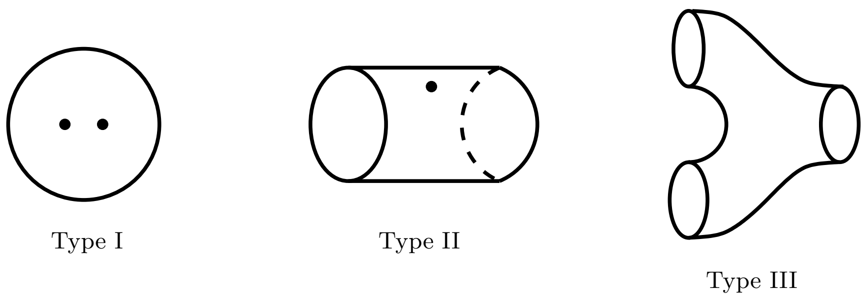

Figure 7: The three types of pantalons.

We now define one of our basic tools, a *pantalon decomposition* of a pointed surface (M, P) (see Figure 8). This consists of:

1) a collection $(M_i, P_i), i = 1, \cdots, r$, each pair being a pantalon of one of the three types, together with maps $\phi_i : (M_i, P_i) \to (M, P)$,

2) a collection $a_1, \cdots, a_p$ of simple closed curves in M, disjoint from each other, from P and from ∂M satisfying the following.

a) Each ϕ_i is injective on the interior of M_i.

b) $\phi_i(intM_i)$ and $\phi_j(intM_j)$ are disjoint if $i \neq j$.

c) ϕ_i takes each boundary component of M_i to one of the curves a_k or to a boundary component of M. Two boundary components of M_i are allowed to map to the same curve a_k.

d) $M = \cup_{i=1}^r \phi_i(M_i)$ and $P = \cup_{i=1}^r \phi_i(P_i)$.

Informally, we say that $a_1, \cdots, a_p$ determine a pantalon decomposition of (M, P), cutting M open into pantalons (M_i, P_i).

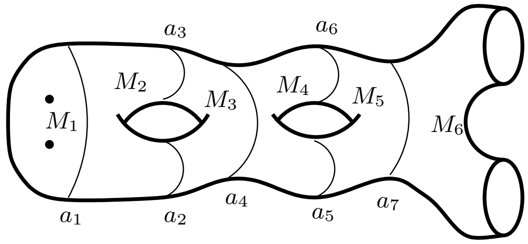

Figure 8: A pantalon decomposition.

It is straightforward to check that every connected compact orientable surface M with punctures P admits a pantalon decomposition, with the following exceptions:

a) $M = S^2$, a sphere, and $|P| \leq 3$;
b) $M = D^2$, a disk, $|P| \leq 1$;
c) $M = S^1 \times I$, an annulus, with $P = \emptyset$;
d) $M = T^2$ the torus, with $P = \emptyset$.

Recall that a subsurface $N \subset M$ is essential if no component of $\overline{M \setminus N}$ is a disk disjoint from P. If some component N' of $\overline{M \setminus N}$ is a disk with $|N' \cap P| = 1$, we call N' a *pointed disk* exterior to N.

Theorem 4.1 *Consider an essential subsurface $N \subset M$ with the associated homomorphism induced by inclusion:*

$$i_* : \mathcal{M}(N, N \cap P) \to \mathcal{M}(M, P)$$

i) If N is a disk and $|N \cap P| \leq 1$ then $\mathcal{M}(N, N \cap P)$ is trivial, and therefore i_* is injective.

ii) Suppose N is an annulus and $N \cap P = \emptyset$. If N has an exterior pointed disk, then the kernel of i_* is $\mathcal{M}(N, N \cap P)$; otherwise i_* is injective.

iii) Assuming that $(N, N \cap P)$ is not as in (i) or (ii), let $a_1, \ldots, a_r$ denote the boundary components of N which bound pointed disks exterior to N, and let b_j, b'_j, $j = 1, \ldots, s$ be the pairs of boundary components of N which cobound exterior cylinders (disjoint from P) (see Figure 9). Denote by A_i, B_j, B'_j the Dehn twists corresponding to the curves a_i, b_j, b'_j, respectively. Then the kernel of i_* is generated by

$$\{A_1, \ldots, A_r, B_1^{-1}B'_1, \ldots, B_s^{-1}B'_s\}$$

and is isomorphic to $\mathbf{Z}^{r+s}$.

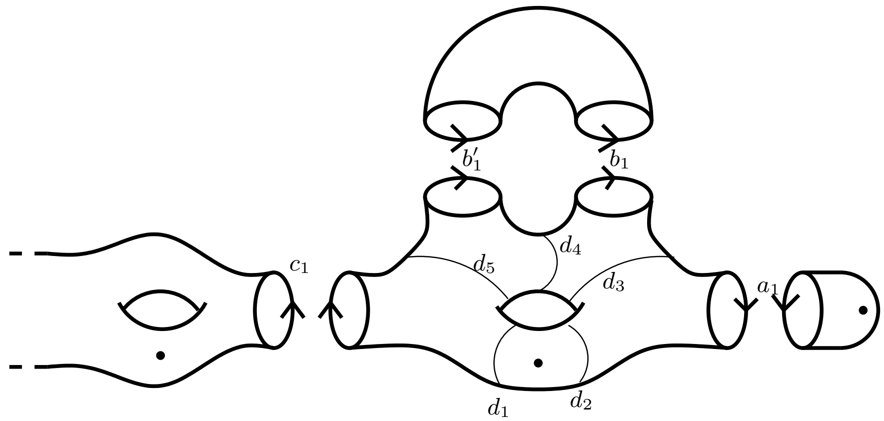

Figure 9: Subsurface N and pantalon decomposition.

Proof: Part (i) is obvious and (ii) is a direct consequence of Proposition 3.8. To prove (iii), let $[h] \in Ker(i_*)$, where $h : N \to N$. We have the following commutative diagram, with exact rows.

$$\begin{array}{ccccccccc} 1 & \to & \mathcal{PM}(N, P\cap N) & \to & \mathcal{M}(N, P\cap N) & \to & \Sigma_{P\cap N} & \to & 1 \\ & & \downarrow & & i_* \downarrow & & \downarrow & & \\ 1 & \to & \mathcal{PM}(M, P) & \to & \mathcal{M}(M, P) & \to & \Sigma_P & \to & 1 \end{array}$$

Since the homomorphism $\Sigma_{P\cap N} \to \Sigma_P$ is injective, $[h]$ is in $\mathcal{PM}(N, P\cap N)$.

Let $c_1, \ldots, c_t$ denote the components of ∂N different from the a_i and b_j, b_j'. In addition, let $d_1, \ldots, d_u : S^1 \to N \setminus P$ be simple closed curves which determine a pantalon decomposition of $(N, N \cap P)$ (see Figure 9). Note that all the curves we are considering are pairwise disjoint and non-isotopic. Since h is isotopic to the identity in $M \setminus P$, each $h \circ d_i$ is isotopic to d_i in $M \setminus P$. Proposition 3.5 implies that $h \circ d_i$ is isotopic to d_i in $N \setminus N \cap P$. By Proposition 3.10 we may suppose that $h \circ d_i = d_i$ for all $i = 1, \ldots, u$, and that h is the identity on the boundary of each pantalon. Using the structure of the pure mapping class groups of pantalons we conclude that

$$[h] = A_1^{\alpha_1} \cdots A_r^{\alpha_r} B_1^{\beta_1} B_1'^{\beta_1'} \cdots B_s^{\beta_s} B_s'^{\beta_s'} C_1^{\gamma_1} \cdots C_t^{\gamma_t} D_1^{\delta_1} \cdots D_u^{\delta_u}.$$

Therefore

$$1 = i_*[h] = B_1^{\beta_1+\beta_1'} \cdots B_s^{\beta_s+\beta_s'} C_1^{\gamma_1} \cdots C_t^{\gamma_t} D_1^{\delta_1} \cdots D_u^{\delta_u}.$$

By Proposition 3.8

$$\beta_1 + \beta_1' = \cdots = \beta_s + \beta_s' = \gamma_1 = \cdots = \gamma_t = \delta_1 = \cdots = \delta_u = 0.$$

Therefore

$$[h] = A_1^{\alpha_1} \cdots A_r^{\alpha_r} (B_1^{-1} B_1')^{\beta_1'} \cdots (B_s^{-1} B_s')^{\beta_s'}.$$

Conversely, it is clear that any $[h]$ of this form is in the kernel of i_*. Finally, Proposition 3.8 implies that $Ker(i_*)$ is isomorphic to $\mathbf{Z}^{r+s}$. □

Corollary 4.2 *Let $N \subset M$ be any subsurface (with $\partial N \cap P = \emptyset$) and let*

$$i_* : \mathcal{M}(N, N \cap P) \to \mathcal{M}(M, P)$$

the natural homomorphism.

i) If N is a disk and $|N \cap P| \leq 1$, then i_* is injective.

ii) If N is an annulus and $N \cap P = \emptyset$, then i_* is injective if and only if there is no boundary component of N which is the boundary of a disk intersecting P in less than two points.

iii) If $(N, N \cap P)$ is not as in (i) or (ii), then i_* is injective if and only if no component of $\overline{M \setminus N}$ is either an annulus disjoint from P whose boundary components are both boundary components of N, or a disk which contains less than two points of P.

5 Centres

We begin this section by considering a special mapping of the surface M of genus one, and with one boundary component; that is, M is a torus minus a disk, with empty puncture set P. We model M as a certain identification space of a planar surface, as follows (see Figure 10). Let

$$D = \{z \in \mathbf{C}; |z| \leq 4\}$$

$$D_1 = \{z \in \mathbf{C}; |z - 2| < 1\}$$

$$D_2 = \{z \in \mathbf{C}; |z + 2| < 1\}$$

Then $D \setminus (D_1 \cup D_2)$ is a pantalon (of type III) with boundary curves $a_1, a_2, c : S^1 \to \partial M$ which we parametrize as follows, $0 \leq \theta \leq 2\pi$:

$$c(e^{i\theta}) = 4e^{i\theta}$$

$$a_1(e^{i\theta}) = 2 + e^{i\theta}$$

$$a_2(e^{i\theta}) = -2 - e^{-i\theta}.$$

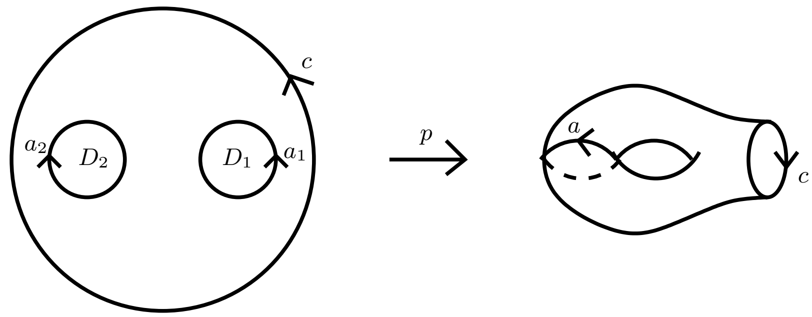

Figure 10: The projection map.

We consider $M = (D \setminus (D_1 \cup D_2))/\sim$ where we identify the points on the curves a_1 and a_2 by

$$a_1(e^{i\theta}) \sim a_2(e^{i\theta}).$$

Denote the natural projection by

$$p : D \setminus (D_1 \cup D_2) \to M.$$

The "meridian" curve $a : S^1 \to M$ is defined by

$$a = p \circ a_1 = p \circ a_2.$$

Now we define a homeomorphism $R : D \to D$ by the equation:

$$R(re^{i\theta}) = \begin{cases} re^{i(\theta-\pi)} & \text{if } \ 0 \leq r \leq 3 \\ re^{i(\theta-(r-2)\pi)} & \text{if } \ 3 \leq r \leq 4 \end{cases}$$

We have $R(D_1) = D_2$, $R(D_2) = D_1$, $R \circ a_1 = a_2^{-1}$ and $R \circ a_2 = a_1^{-1}$. Therefore, R induces a homeomorphism

$$\tilde{R} : M \to M$$

such that

$$\tilde{R} \circ a = a^{-1}.$$

Its class $\rho = [\tilde{R}] \in \mathcal{M}(M)$ will be called a *half-twist* of M along c relative to a. The Dehn twists A and C about the respective curves a and c relate to ρ as follows:

$$\rho^2 = C, \quad \rho A \rho^{-1} = A.$$

The latter equation follows since $\rho A \rho^{-1}$ is the Dehn twist about $\rho \circ a = a^{-1}$, and the Dehn twist about a^{-1} is the same as the Dehn twist about a for any simple closed curve a.

Proposition 5.1 *Let M, ρ, a and A be as in the above discussion. If G is the subgroup of $\mathcal{M}(M)$ consisting of classes of homomorphisms $h : M \to M$ such that $h \circ a$ is isotopic with a or a^{-1}, then G is generated by $\{A, \rho\}$ and is isomorphic with $\mathbf{Z}^2$.*

Proof: Let $[h] \in G$; we may assume $h \circ a = a$ or $h \circ a = a^{-1}$. Note that a determines a pantalon decomposition of M, with a single pantalon of type III, namely $D \setminus (D_1 \cup D_2)$.

If $h \circ a = a$, by the structure of the mapping class group of the pantalon, for some integers m, n:

$$[h] = A^n C^m = A^n \rho^{2m}.$$

If $h \circ a = a^{-1}$, then $\tilde{R} \circ h \circ a = a$ and $\rho[h]$ has the form

$$\rho[h] = A^n C^m = A^n \rho^{2m}.$$

Therefore

$$[h] = \rho^{-1} A^n \rho^{2m} = A^n \rho^{2m-1}.$$

It is clear that A and ρ are in G, so they generate G and we have already noted that they commute. If $A^n \rho^m = 1$, then

$$(A^n \rho^m)^2 = A^{2n} \rho^{2m} = A^{2n} C^m = 1,$$

and by Proposition 3.8, $2n = m = 0$. This shows $G \cong \mathbf{Z}^2$. □

We turn now to determining the centre $\mathcal{ZM}(M, P)$ of an arbitrary (compact orientable) surface M with puncture set P, that is, the subgroup of $\mathcal{M}(M, P)$ consisting of mapping classes which commute with all elements of $\mathcal{M}(M, P)$. First we record some simple cases: If M is a sphere or disk and $|P| \leq 1$, $\mathcal{ZM}(M, P) = \mathcal{M}(M, P) = \{1\}$. $\mathcal{ZM}(S^2, 2) = \mathcal{M}(S^2, 2) = \mathbf{Z}/2\mathbf{Z}$. $\mathcal{M}(S^2, 3) \cong \Sigma_3$ and therefore $\mathcal{ZM}(S^2, 3)$ is the trivial group. Each of the pantalons (type I, II or III) has an abelian mapping class group, so the centre is equal to the whole group. For the same reason $\mathcal{ZM}(S^1 \times I) = \mathcal{M}(S^1 \times I) \cong \mathbf{Z}$.

The case of the torus is somewhat more interesting. Since

$$\mathcal{M}(T^2, 0) \cong \mathcal{M}(T^2, 1) \cong SL(2, \mathbf{Z})$$

we see algebraically that the centre is the cyclic group of order two, consisting of the two diagonal matrices, $\pm I$, where I is the identity matrix. However, to warm up for the more complicated cases, whe will establish this fact geometrically. Consider the torus T^2 embedded in xyz-space as the set of points of distance 1 from the circle $x^2+y^2=4,\ z=0$ (see Figure 11). Let $s: T^2 \to T^2$ be the (orientation-preserving) involution $s(x,y,z)=(x,-y,-z)$. In the case of $\mathcal{M}(T^2,1)$, we suppose that $P=\{p\}$ is a point on the x-axis, so that $s(p)=p$. Let $\sigma=[s]\in\mathcal{M}(T^2)$ and $\sigma_1=[s]\in\mathcal{M}(T^2,\{p\})$.

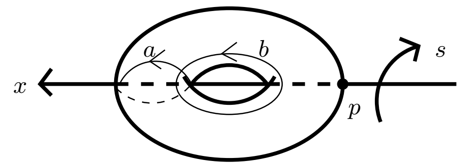

Figure 11: Involution generating the centre of $\mathcal{M}(T^2)$.

Proposition 5.2 *The centre of $\mathcal{M}(T^2)$ is the cyclic group of order 2 generated by σ; similarly $\mathcal{Z}\mathcal{M}(T^2,1)\cong\mathbf{Z}/2\mathbf{Z}$, generated by σ_1.*

Proof: We prove the first part; the case of $\mathcal{Z}\mathcal{M}(T^2,\{p\})$ being essentially the same. Since s has order 2, $\sigma^2=1$ in $\mathcal{M}(T^2)$. Let $a,b: S^1\to T^2$ be the circles parametrized by

$$a(e^{i\theta})=(2+\cos\theta,0,\sin\theta),\quad b(e^{i\theta})=(\cos\theta,\sin\theta,0).$$

Then

$$s\circ a=a^{-1}\ \text{ and }\ s\circ b=b^{-1}.$$

Since a is not isotopic to a^{-1} we conclude that $\sigma\neq 1$, and so σ has order 2. Letting A,B be the Dehn twists about a,b we have

$$\sigma A\sigma^{-1}=A\ \text{ and }\ \sigma B\sigma^{-1}=B,$$

and since A and B generate $\mathcal{M}(T^2)$ we conclude that $\sigma\in\mathcal{Z}\mathcal{M}(T^2)$.

Now suppose that $[h]\in\mathcal{Z}\mathcal{M}(T^2)$, where $h: T^2\to T^2$. Then $A=[h]A[h]^{-1}=C$ is a Dehn twist about the curve $c=h\circ a$. By Proposition 3.6, $c\simeq a^{\pm1}$, so we may assume that $h\circ a=a$ or $h\circ a=a^{-1}$. Case 1, $h\circ a=a$; we can cut T^2 open along a and conclude from the mapping class of the cylinder that $[h]=A^k$ for some integer k. Since $I(a,b)=1$ and A^k and B commute, Proposition 3.7 implies that $k=0$, and $[h]=1$ in this case. Case 2, $h\circ a=a^{-1}$. Then $s\circ h\circ a=a$ and we conclude from case 1 that $\sigma[h]=1$, so $[h]=\sigma^{-1}=\sigma$. □

Next we consider the torus with two marked points. Using the above model for T^2, let $P=\{(0,3,0),(0,-3,0)\}$, that it, a pair of points which are interchanged by the involution $s : T^2 \to T^2$ described above. Let $\sigma_2 \in \mathcal{M}(T^2, P) = \mathcal{M}(T^2, 2)$ be the class represented by s. The following is proved in the same manner as the above.

Proposition 5.3 *The centre of $\mathcal{M}(T^2, 2)$ is the cyclic group of order 2 generated by σ_2.* □

We now consider the closed surface M of genus 2, which we realize in $\mathbf{R}^3$ as the boundary of a uniform regular neighborhood of the union of the circles

$$(x+1)^2 + y^2 = 1,\ z = 0 \text{ and } (x-1)^2 + y^2 = 1,\ z = 0.$$

Let $s : \mathbf{R}^3 \to \mathbf{R}^3$ be the same involution as in the previous discussion, and let $\sigma \in \mathcal{M}(M)$ be represented by the restriction of s to M.

Proposition 5.4 *The centre of $\mathcal{M}(M)$, where M is a closed surface of genus 2, is the cyclic group of order 2 generated by σ.*

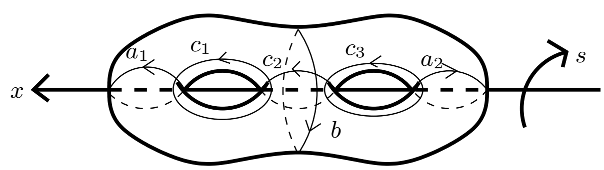

Figure 12: Generator of the centre of $\mathcal{M}(M_2)$.

Proof: Let $a_1, a_2, b, c_1, c_2, c_3$ be the simple closed curves as in Figure 12. In particular, b is the curve of intersection of M with the plane $x = 0$, c_1 and c_3 lie in the plane $z = 0$ and a_1, a_2, c_2 are in the plane $y = 0$. We have

$$s \circ a_i = a_i^{-1}, \quad s \circ b = b, \quad s \circ c_i = c_i^{-1}.$$

Since a_1 is not isotopic with a_1^{-1}, $\sigma \neq 1$ has order 2 in $\mathcal{M}(M)$. Let A_1,A_2,B, C_1, C_2, C_3 be the Dehn twists about the respective curves, which generate $\mathcal{M}(M)$, according to [23]. From the above equations we conclude that

$$\sigma A_i \sigma^{-1} = A_i, \quad \sigma B \sigma^{-1} = B, \quad \sigma C_i \sigma^{-1} = C_i,$$

so σ is indeed central in $\mathcal{M}(M)$.

Given $[h] \in \mathcal{Z}\mathcal{M}(M)$ we have $A_i = [h]A_i[h]^{-1}$, which is a Dehn twist about $h \circ a_i$. By Proposition 3.6 $h \circ a_i$ is isotopic with $a_i^{\pm 1}$. Similarly $h \circ b$ is isotopic with

$b^{\pm 1}$. By Proposition 3.10 we may assume that $h \circ a_i = a_i$ or a_i^{-1} and that $h \circ b = b$ or b^{-1}.

Let M_1 and M_2 be the closures of the two components of $M \setminus b$, with a_1 in M_1, a_2 in M_2. Now $h \circ b = b^{-1}$ can only happen if $h(M_1) = M_2$ and $h(M_2) = M_1$, which is impossible because $h \circ a_i = a_i^{\pm 1}$. So we must have $h \circ b = b$, $h(M_i) = M_i$. Letting ρ_i denote the half-twist of M_i along b relative to a_i, Proposition 5.1 implies that $[h]$ can be written in the form

$$[h] = \rho_1^{n_1} A_1^{m_1} \rho_2^{n_2} A_2^{m_2},$$

for some integers n_1, n_2, m_1, m_2. Consider

$$[h]^2 = A_1^{2m_1} A_2^{2m_2} B^{n_1+n_2}.$$

Since $[h]^2$ is central it commutes with C_1, as do A_2 and B. Therefore $A_1^{2m_1}$ also commutes with C_1. But $I(a_1, c_1) = 1$, so by Proposition 3.7, $2m_1 = 0$. Similarly $2m_2 = 0$ and $n_1 + n_2 = 0$. We note that either n_1 and n_2 are both odd or both even. If they are both even, then $h \circ a_1 = a_1$ and $h \circ a_2 = a_2$, if both odd, then $h \circ a_1 = a_1^{-1}$ and $h \circ a_2 = a_2^{-1}$.

Case 1, n_1 and n_2 are even: write $n_1 = 2k_1$ and $n_2 = 2k_2$. Then

$$[h] = \rho_1^{2k_1} \rho_2^{2k_2} = B^{k_1+k_2} = B^0 = 1,$$

because $k_1 + k_2 = (n_1 + n_2)/2 = 0$.

Case 2, n_1 and n_2 are odd. Then

$$s \circ h \circ a_1 = a_1 \quad \text{and} \quad s \circ h \circ a_2 = a_2$$

and by case 1, $\sigma[h] = 1$. Therefore $[h] = \sigma^{-1} = \sigma$. □

Proposition 5.5 *Let M denote the genus one oriented surface with one boundary component. Then the centre of $\mathcal{M}(M)$ is the infinite cyclic group generated by the half-twist ρ, defined in the discussion preceding Proposition 5.1.*

Proof: Let a be the curve on M, A the Dehn twist about a, $\tilde{R} : M \to M$, $\rho = [\tilde{R}]$, all as described in the discussion preceding Proposition 5.1. Let b be the curve on M which is the image, under the identification $D \setminus (D_1 \cup D_2) \to M$, of the interval [-1, 1], so that $I(a, b) = 1$. Let B be the Dehn twist about b; then $\{A, B\}$ generates $\mathcal{M}(M)$.

Noting that $\tilde{R} \circ a = a^{-1}$ and $\tilde{R} \circ b = b^{-1}$, we see that ρ commutes with A and B, and therefore ρ is in the centre of $\mathcal{M}(M)$. Since $\rho^2 = C$ and C has infinite order, ρ has also infinite order.

Now let ξ be in the centre of $\mathcal{M}(M)$, and $h : M \to M$ a homeomorphism with $[h] = \xi$. Then $A = \xi A \xi^{-1}$ is a Dehn twist about $h \circ a$, so by Proposition 3.6 $h \circ a$ is isotopic with a or a^{-1}. By Proposition 5.1 ξ is of the form

$$\xi = A^p \rho^q, \quad p, q \in \mathbf{Z}.$$

Because ξ and ρ commute with B, A^p commutes with B. But $I(a, b) = 1$, so Proposition 3.7 implies that $p = 0$ and therefore $\xi = \rho^q$. □

Remark: Let M be a genus one oriented surface with one boundary component as above. According to [36] $\mathcal{M}(M)$ is isomorphic to the Artin braid group $\langle A, B | ABA = BAB \rangle$, by [4] its center is the infinite cyclic group generated by $(ABA)^2$, and one can check directly that $\rho = (ABA)^2$.

Following is the main result regarding the centre of $\mathcal{M}(M, P)$, for a general Riemann surface. We have already noted that there are certain exceptional cases, so we list the hypotheses here for the generic result:

1) If M is a sphere, then assume $|P| \geq 4$,
2) If M is a disk, then assume $|P| \geq 3$,
3) If M is an annulus, then assume $|P| \geq 1$,
4) If M is a torus, then assume $|P| \geq 3$,
5) If M is a surface of genus one and one boundary component, then assume $|P| \geq 1$,
6) If M is a surface of genus 2, then assume $|P| \geq 1$.

Theorem 5.6 *Let M be a connected compact orientable surface with marked points $P \subset M$ and assume the hypotheses (1) - (6) above. Let $c_1, \cdots, c_q : S^1 \to \partial M$ be the boundary curves of M and $C_1, \cdots, C_q$ the Dehn twists about these curves. Then the centre $\mathcal{Z}\mathcal{M}(M, P)$ of $\mathcal{M}(M, P)$ is the subgroup generated by $\{C_1 \cdots, C_q\}$ and is isomorphic with $\mathbf{Z}^q$. In particular, if ∂M is empty, the centre of $\mathcal{M}(M, P)$ is trivial.*

Proof: It is clear that C_i is central in $\mathcal{M}(M, P)$; the fact that the subgroup generated by $C_1, \cdots, C_q$ is isomorphic with $\mathbf{Z}^q$ follows directly from Proposition 3.8. Now consider an element ξ in the centre of $\mathcal{M}(M, P)$, with representative homeomorphism $h : M \to M, \;\; [h] = \xi$.

Consider curves $a_1, \cdots, a_p$ which detemine a pantalon decomposition of (M, P). (All the cases which do not admit a pantalon decomposition have been excluded.) Since ξ is central, the Dehn twist A_i about the curves a_i satisfy

$$A_i = \xi A_i \xi^{-1},$$

and since $\xi A_i \xi^{-1}$ is a Dehn twist about $h \circ a_i$ we conclude from Proposition 3.6 that $h \circ a_i$ is isotopic with a_i or a_i^{-1}. By Proposition 3.10 we may suppose that

$$h \circ a_i = a_i^{\pm 1},$$

and consequently, h permutes the pantalons.

The pantalons (M_j, P_j) are mapped to M by maps ϕ_j. We call a curve a_i *separating* if it is in the image of ϕ_k and ϕ_l, for some $k \neq l$ (a separating curve in our sense need not separate the surface itself). First assume that a_i is separating and that $h \circ a_i = a_i^{-1}$. Then $h(\phi_k(M_k)) = \phi_l(M_l)$ and $h(\phi_l(M_l)) = \phi_k(M_k)$. We consider all the possibilities:

If (M_k, P_k) and (M_l, P_l) are pantalons of type I, then M is a 2-sphere and $|P| = 4$ (see Figure 13.i). Consider the exact sequence:

$$1 \to \mathcal{PM}(S^2, 4) \to \mathcal{M}(S^2, 4) \to \Sigma_4 \to 1.$$

Since the centre of Σ_4 is trivial, ξ must act trivially on P, whereas h interchanges the points $\phi_k(P_k)$ and $\phi_l(P_l)$, so this case cannot occur.

If (M_k, P_k) and (M_l, P_l) are pantalons of type II, then $\phi_k(M_k) \cap \phi_l(M_l) = a_i$ and there are other curves a_μ and a_ν which form the remaining boundary curves of these pantalons (see Figure 13.ii). Then, since h interchanges the pantalons, $h(a_\mu) = a_\nu^{\pm 1}$. On the other hand, $h(a_\mu) = a_\mu^{\pm 1}$, so we must have $a_\mu = a_\nu^{\pm 1}$ and conclude that M is a torus and $|P| = 2$. This case has been excluded.

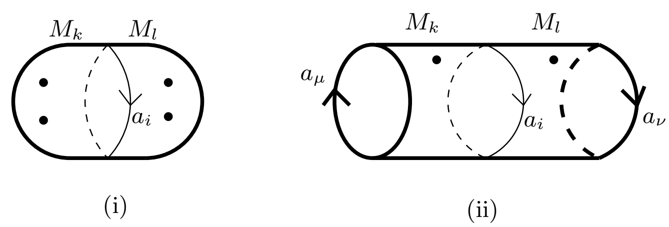

Figure 13: First cases, a_i separating.

If (M_k, P_k) and (M_l, P_l) are pantalons of type III with identifications, so that $\phi_k(M_k)$ and $\phi_l(M_l)$ are genus 1 surfaces with one boundary component a_i, then M is a closed surface of genus 2 and P is empty (see Figure 14.i). This case also has been excluded.

Finally suppose (M_k, P_k) and (M_l, P_l) are pantalons of type III, mapped homeomorphically by ϕ_k and ϕ_l (see Figure 14.ii). Then we argue as in the type II case that their other boundary components must be identified, and that M is closed, has genus 2, and P is empty, an excluded case.

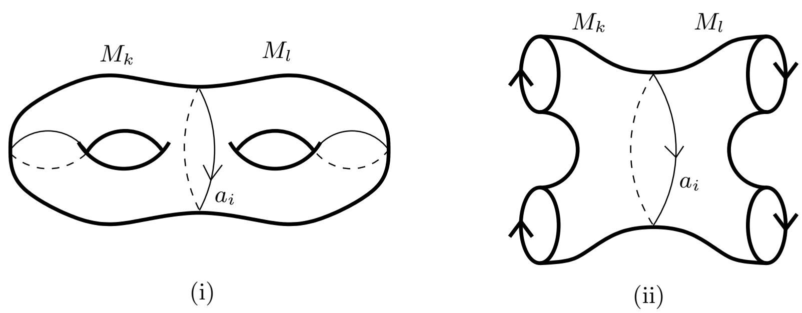

Figure 14: a_i separating type III pantalons.

Thus we have shown that if a_i is separating, then $h \circ a_i = a_i$. In addition we have seen that $h(\phi_k(M_k)) = \phi_k(M_k)$ for all k.

Recall the notation $m = |P|$ and q is the number of boundary components of M. We complete the proof by considering three cases:

Case 1: $m + q \geq 2$. Then we may assume all the a_i are separating (see Figure 15). If M is not a sphere or disk, then each pantalon can be taken to be of type II or III. If M is either a sphere or disk, the exact sequence

$$1 \to \mathcal{PM}(M, P) \to \mathcal{M}(M, P) \to \Sigma_P \to 1,$$

and the fact that Σ_P has trivial centre under the assumption $|P| \geq 3$, shows that $\xi \in \mathcal{PM}(M, P)$, or in other words, h fixes P pointwise. So we have, by the structure of the (pure) mapping class groups of the pantalons of type I, II and III:

$$\xi = A_1^{r_1} \cdots A_p^{r_p} C_1^{s_1} \cdots C_q^{s_q},$$

for some integers $r_1, \ldots, r_p, s_1, \ldots, s_q$. Now fix $i \in \{1, \ldots, p\}$. By Proposition 3.4, there exists a generic simple closed curve $b : S^1 \to M \setminus P$ such that $I(a_i, b) > 0$ but $a_j \cap b = \emptyset$ if $j \neq i$. If B is the Dehn twist about b, we see that B commutes with A_j, $j \neq i$. B also commutes with all the C_j and with ξ since they are central. It follows that B commutes with $A_i^{r_i}$. By Proposition 3.7 we conclude that $r_i = 0$. Since i was arbitrary,

$$\xi = C_1^{s_1} \cdots C_q^{s_q}.$$

Case 2: $m + q = 1$. We may now suppose that a_1 is not separating, but $a_2, \ldots, a_p$ are separating curves, and the pantalon (M_1, P_1) is of type III with two boundary curves identified to a_1, so that $\phi_1(M_1)$ is a surface of genus one with one boundary component, which we may take to be a_2 (see Figure 16). Moreover we

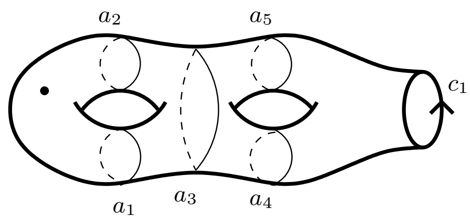

Figure 15: Pantalon decomposition, all a_i separating.

may assume $(M_2, P_2), \ldots, (M_r, P_r)$ are pantalons of type II or III, embedded in M. Let ρ denote the half-twist of $\phi_1(M_1)$ along a_2 relative to a_1. Then by Proposition 5.1 and the structure of the mapping class groups of type II and III pantalons, ξ can be written (assuming $q = 1, m = 0$ and noting that $\rho^2 = A_2$):

$$\xi = \rho^k A_1^{r_1} A_3^{r_3} \cdots A_p^{r_p} C_1^{s_1},$$

for integers $k, r_1, r_3, \ldots, r_p, s_1$. Then

$$\xi^2 = A_1^{2r_1} A_2^k A_3^{2r_3} \cdots A_p^{2r_p} C_1^{2s_1}.$$

Employing the same argument as in Case 1, we conclude that

$$2r_1 = k = 2r_3 = \cdots = 2r_p = 0,$$

and so $\xi = C_1^{s_1}$ if $q = 1$. If $q = 0$ and $m = 1$ we similarly conclude that $\xi = 1$.

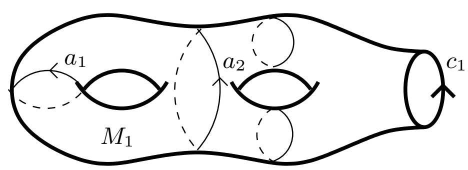

Figure 16: Pantalon decomposition with a_1 nonseparating.

Case 3: $m + q = 0$. In particular, $q = 0$. Then we need two singular pantalons in the decomposition of M, ξ has an expression as a product of two half-twists together with powers of the A_i and we conclude by examining ξ^2, exactly as in case 2 that the powers are all zero and therefore $\xi = 1$. □

6 Commensurability

If G is a group, then two subgroups $H, H' < G$ are said to be *commensurable* if $H \cap H'$ has finite index in both H and H'. Commensurability is an equivalence relation on the set of all subgroups of G, really of interest only for infinite groups. Following is an elementary property of commensurable subgroups, which we shall find useful.

Proposition 6.1 *Suppose H and H' are commensurable subgroups of G. Then for each $h \in H$, there exists a nonzero integer k such that $h^k \in H'$.*

Proof: If not, then $\{h^k\}, k \in \mathbf{Z}$ is an infinite set of elements, all in different cosets of H rel $H \cap H'$, contradicting finite index. □

Proposition 6.2 *Suppose $a_1, \ldots, a_p : S^1 \to M \setminus P$ are essential simple closed curves which are pairwise disjoint. Let $b : S^1 \to M \setminus P$ be an essential simple closed curve such that $I(a_i, b) = 0$ for all $i = 1, \ldots, p$. Then there exists a simple closed curve $c : S^1 \to M \setminus P$ isotopic to b and such that $a_i \cap c = \emptyset$ for all $i = 1, \ldots, p$.*

Proof: We may assume b transverse to all the a_i and argue by induction on the cardinality of $b \cap (a_1 \cup \cdots \cup a_p)$. If $b \cap (a_1 \cup \cdots \cup a_p)$ is empty, there is nothing to prove; suppose it is nonempty. Choose i so that $b \cap a_i \neq \emptyset$. Proposition 3.2 implies that b and a_i cobound a bigon D. There may be other intersections of a_j with D, but there is always an outermost bigon $D' \subset D$ cobounded by b and some a_k, and otherwise disjoint from $a_1 \cup \cdots \cup a_p$ (see Figure 17). Now we can push b across D' to obtain a curve b' isotopic with b and

$$|b' \cap (a_1 \cup \cdots \cup a_p)| = |b \cap (a_1 \cup \cdots \cup a_p)| - 2.$$

By inductive hypothesis, there exists an isotopy from b' to an essential simple closed curve c in $M \setminus P$ such that $a_i \cap c = \emptyset$ for all $i = 1, \ldots, p$. □

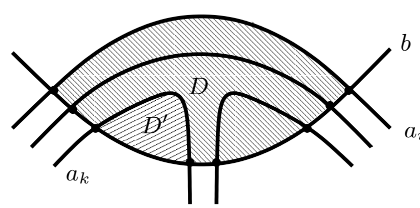

Figure 17: Bigon in proof of Proposition 6.2.

Definitions: Recall that a subsurface is *essential* if none of its exterior components is a disk with zero marked points. By a *marked subsurface* we mean a pair (N, Q) where N is a subsurface of M and $Q = P \cap N$. We will say that an essential marked subsurface (N, Q) of (M, P) is *injective* provided it satisfies:

a) If N is a disk, then $|Q| \geq 2$,

b) If N is an annulus with Q empty, then there is no pointed disk component of the exterior of N in M,

c) If N is not an annulus, or if $|Q| \geq 1$, then no component of $\overline{M \setminus N}$ is a disk with one marked point, or a cylinder with no marked points and both boundary components in ∂N.

By Corollary 4.2 these criteria assure the injectivity of

$$\mathcal{M}(N, Q) \to \mathcal{M}(M, P).$$

If (N, Q) and (N', Q') are marked subsurfaces of (M, P), we say they are *isotopic* provided there is a continuous family of homeomorphisms $h_t \in \mathcal{H}(M, P)$, $t \in [0, 1]$ such that $h_0 =$ identity, and $h_1(N, Q) = (N', Q')$. In particular, $Q = Q'$.

We would like to be able to say that geometric subgroups are commensurable if and only if they are equal, if and only if their defining subsurfaces are isotopic. However, just as with centres, there are some exceptions to the general principle. Our first family are the infinite cyclic geometric subgroups. The only marked surfaces with mapping class $\mathbf{Z}$ are $(S^1 \times I, 0)$ and $(D^2, 2)$; we note for future reference that these are the only mapping class groups which contain $\mathbf{Z}$ as a finite-index subgroup.

Proposition 6.3 *Suppose (N, Q) and (N', Q') are injective subsurfaces of (M, P), and suppose $N = S^1 \times I$ and Q is empty. Suppose $\mathcal{M}(N, Q)$ and $\mathcal{M}(N', Q')$ are commensurable subgroups of $\mathcal{M}(M, P)$. Then either (1) N' is a cylinder $S^1 \times I$ with Q' empty or (2) N' is a disk and $|Q'| = 2$.*

In case (1), N' is isotopic with N. In case (2), $(N', Q') = (D^2, 2)$, then one of the components (N_1, Q_1) of the exterior of (N, Q) in (M, P) is a $(D^2, 2)$, and (N_1, Q_1) is isotopic with (N', Q') (see Figure 18).

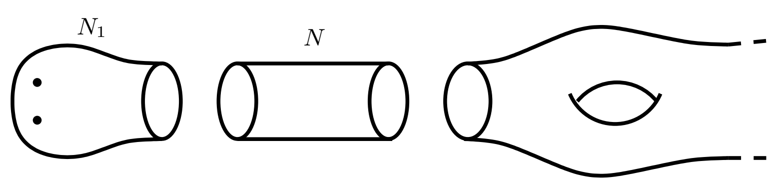

Figure 18: Injective subsurfaces in Proposition 6.3.

Proof: Since $\mathcal{M}(N, Q)$ is infinite cyclic, so is every nontrivial subgroup, including $\mathcal{M}(N, Q) \cap \mathcal{M}(N', Q')$, so (N', Q'), its mapping class group containing a finite index $\mathbf{Z}$, can only be $(S^1 \times I, \emptyset)$ or $(D^2, 2)$.

Case 1: (N', Q') is $(S^1 \times I, \emptyset)$. Let $a'(z) = (z, 1/2), \quad z \in S^1$ denote the central curve of N' and similarly label the central curve of N as $a : S^1 \to N$, and let A', A be Dehn twists of M about these curves. A and A' represent generators of $\mathcal{M}(N, Q)$ and $\mathcal{M}(N', Q')$, respectively. By commensurability their intersection has finite index, so there exist nonzero integers k, l such that

$$A^k = A'^l.$$

Proposition 3.6 implies that a' is isotopic with a or with a^{-1}. Since N and N' are regular neighborhoods, respectively, of a and a', it follows that N and N' are isotopic in this case.

Case 2: (N', Q') is $(D^2, 2)$. Choose notation as in Case 1, except that a' now denotes the boundary curve of N'. Again we have nonzero k and l so that $A^k = A'^l$, and we conclude that a' is isotopic with $a^{\pm 1}$. After an isotopy, we may assume a' equals a boundary component of N. It then follows that N' is a component of the exterior of N. □

Definition: A *doubled pantalon* is the marked surface obtained by pasting together two pantalons of the same type along their boundaries (see Figure 19). More specifically, if (M, P) is a marked surface and N a subsurface of M, the triple (M, N, P) is called a doubled pantalon in each of the cases:

Type I: $M \cong S^2, \quad N \cong D^2, \quad |P| = 4, \quad |P \cap N| = 2;$

Type II: $M \cong S^1 \times S^1 \supset S^1 \times I \cong N, \quad |P| = 2, \quad |P \cap N| = 1;$

Type III: $M =$ closed surface of genus two, $P =$ empty set, $N \cong \overline{M \setminus N} =$ type III pantalons.

One reason to be interested in doubled pantalons is that they provide examples of nonisotopic subsurfaces inducing commensurable geometric subgroups, which we will see to be an exceptional case.

Proposition 6.4 *Suppose that (M, N, P) is a doubled pantalon, and let $N' = \overline{M \setminus N}$. Then $\mathcal{M}(N, N \cap P)$ and $\mathcal{M}(N', N' \cap P)$ inject in $\mathcal{M}(M, P)$, and are commensurable subgroups. The subgroups are equal in the case of Types II or III. For Type I, the intersection $\mathcal{M}(N, N \cap P) \cap \mathcal{M}(N', N' \cap P)$ has index two in each of $\mathcal{M}(N, N \cap P)$ and $\mathcal{M}(N', N' \cap P)$. In each of the three types, the subsurfaces N and N' are non-isotopic in M (rel P).*

Proof: In the case of doubled pantalons of Type II or III, the two geometric subgroups both equal the free abelian group generated by twists along the common boundary of N and N'. This has rank 2 or 3, respectively. For Type I, the generator of $\mathcal{M}(N, N \cap P)$ is a half-twist interchanging the two points $N \cap P$, which clearly

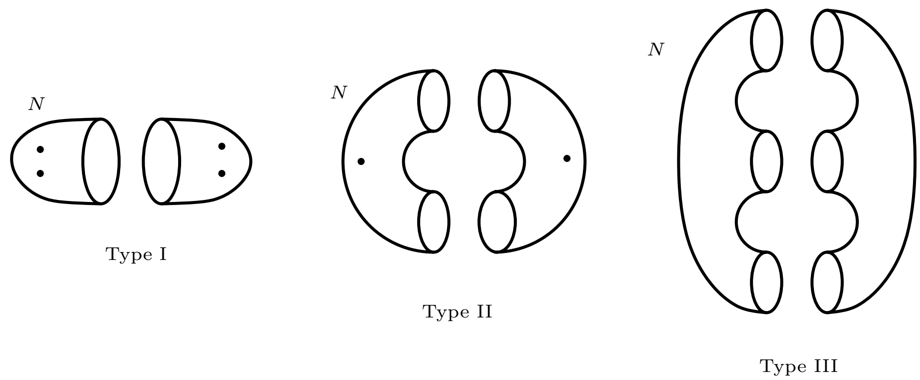

Figure 19: Doubled pantalons.

does not belong to $\mathcal{M}(N', N' \cap P)$, all of whose elements fix $N \cap P$ pointwise. Likewise the generator of $\mathcal{M}(N', N' \cap P)$ is not in $\mathcal{M}(N, N \cap P)$. But the squares of the generators, being a Dehn twist along the common boundary, coincide. The question of isotopy is clear for types I and II, because the surfaces N and N' enclose different marked points. In type III, the two pantalons, N and N' are nonisotopic, too. For an isotopy taking N to N' would take the boundary to itself, but with orientation reversed. But a simple homological calculation shows that if a, b, c are the (oriented) boundary curves of N, a cannot be isotopic with a^{-1}, b^{-1} or c^{-1}. □

In the following, consider two connected marked injective subsurfaces (N, Q), $(N', Q') \subset (M, P)$, it being understood that $Q = N \cap P$ and $Q' = N' \cap P$. We will refer to the following four statements:

a) $\mathcal{M}(N, Q)$ and $\mathcal{M}(N', Q')$ are commensurable subgroups of $\mathcal{M}(M, P)$;
b) $\mathcal{M}(N, Q) = \mathcal{M}(N', Q')$;
c) (N, Q) and (N', Q') are isotopic;
d) (N, Q) is isotopic with either (N', Q') or $(\overline{M \setminus N'}, P \setminus Q')$.

Theorem 6.5 *Suppose that N and N' are injective subsurfaces of M and that $(N, Q) \neq (S^1 \times I, 0) \neq (N', Q')$. Then:*
i) if (M, N, P) is not a doubled pantalon, $(a) \Leftrightarrow (b) \Leftrightarrow (c)$;
ii) if (M, N, P) is a doubled pantalon of Type II or III, $(a) \Leftrightarrow (b) \Leftrightarrow (d)$;
iii) if (M, N, P) is a doubled pantalon of Type I, $(b) \Leftrightarrow (c) \Rightarrow (a) \Leftrightarrow (d)$.

Proof: It is obvious that $(c) \Rightarrow (b) \Rightarrow (a)$ in all three cases. Before breaking into cases, let $d_1, \ldots, d_l : S^1 \to M \setminus P$ denote the boundary components of N and similarly $d'_1, \ldots, d'_{l'}$ the components of $\partial N'$. Let $\{a_i\}$ be curves in N defining a pantalon decomposition of N (see Figure 20). Let N_j denote the components of $\overline{M \setminus N}$. Choose curves b_{jk} giving a pantalon decomposition of $(N_j, N_j \cap P)$ for $(N_j, N_j \cap P) \neq (S^1 \times I, \emptyset)$.

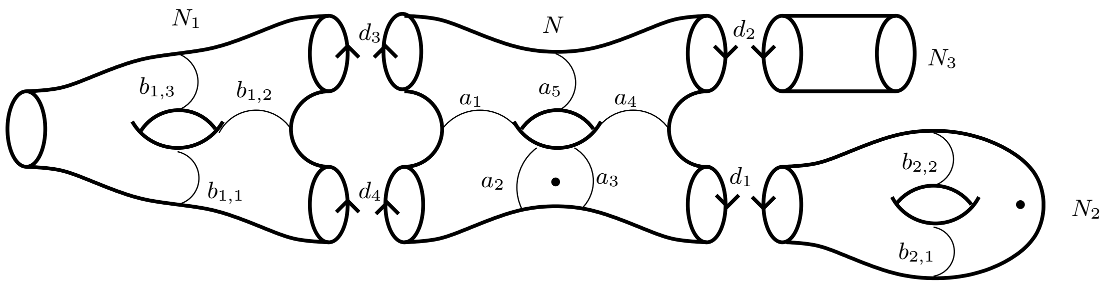

Figure 20: Pantalon decomposition of N and its complement.

We wish to show that in all cases $(a) \Rightarrow (d)$. So we assume commensurability of the geometric subgroups $\mathcal{M}(N, Q)$ and $\mathcal{M}(N', Q')$; the strategy is to show that after an isotopy, the boundaries of N and N' can be made to coincide. We break the argument into two steps.

Step 1: Each component d'_j of $\partial N'$ is isotopic (in M, rel P) with a component $(d_i)^{\pm 1}$ of ∂N, and vice-versa.

Proof of Step 1: First note that we may assume in this step that $\partial M = \emptyset$, by the trick used before: adjoin a genus one surface to each boundary component of M to obtain $\hat{M}$. $\mathcal{M}(M, P)$ injects in $\mathcal{M}(\hat{M}, P)$, so the subgroups $\mathcal{M}(N, Q)$ and $\mathcal{M}(N', Q')$ are unchanged, and the desired conclusion of step 1, if true in $\hat{M}$, will also hold in M by Proposition 3.5. In particular, since N is injective, none of the N_j is a cylinder with $N_j \cap P = \emptyset$, and the union of all the a_i, b_{jk} and d_i gives a pantalon decomposition of M. Let A_i, B_{jk}, and D_i denote the Dehn twists about the respective curves a_i, b_{jk}, d_i. Let $d' = d'_j$ be any boundary component of N', and D' the element of $\mathcal{M}(N', Q') \subset \mathcal{M}(M, P)$ represented by a Dehn twist along d'. We now calculate some intersection numbers:

$I(a_i, d') = 0$. Reason: Being a twist on a boundary curve, D' is central in $\mathcal{M}(N', Q')$, though perhaps not in $\mathcal{M}(M, P)$. By commensurability and Proposition 6.1 since $A_i \in \mathcal{M}(N, Q)$ there is an integer $s \neq 0$ such that $A_i^s \in \mathcal{M}(N', Q')$. It follows that A_i^s and D' commute. Then Proposition 3.7 implies that $I(a_i, d') = 0$.

$I(b_{jk}, d') = 0$. This is shown similarly, noting that B_{jk} commutes with all of $\mathcal{M}(N, Q)$, and by commensurability, a nonzero power of D' is in $\mathcal{M}(N, Q)$.

$I(d_i, d') = 0$ for all i. As in the previous case, a power of D' belongs to $\mathcal{M}(N, Q)$ and so commutes with D_i, a twist of ∂N.

Now Proposition 6.2 implies that d' may be assumed, after an isotopy, to be disjoint from the a_i, b_{jk}, d_i, i. e. d' lies entirely inside one of the pantalons in the decomposition of M corresponding to those curves. Being essential, d' is isotopic with one of the boundary curves of that pantalon (or its inverse). We have concluded that, up to isotopy of M rel P, we have one (and only one) of the following possibilities, for some i, j, k:

$$d' \simeq a_i^{\pm 1}, \quad d' \simeq b_{jk}^{\pm 1}, \text{ or } \quad d' \simeq d_i^{\pm 1}.$$

First assume $d' \simeq a_i^{\pm 1}$. Since a_i is not isotopic with a boundary component of N, there exists a generic simple closed curve e in $N \backslash Q$ such that $I(d', e) = I(a_i, e) > 0$. The Dehn twist E along e is an element of $\mathcal{M}(N, Q)$, a subgroup commensurable with $\mathcal{M}(N', Q')$, so there exists $t > 0$ such that $E^t \in \mathcal{M}(N', Q')$. Recalling that D' is central in $\mathcal{M}(N', Q')$, D' and E^t commute. By Proposition 3.7, $I(d', e) = 0$, a contradiction.

Next assume $d' \simeq b_{jk}^{\pm 1}$, one of the pantalon curves of the component N_j of the complement of N. As above, there is a simple closed curve e in $N_j \setminus (N_j \cap P)$ with $I(d', e) = I(b_{jk}, e) > 0$. Noting that the Dehn twist E commutes with all elements of $\mathcal{M}(N, N \cap P)$ and that a power of D' belongs to $\mathcal{M}(N, N \cap P)$, we obtain again the contradiction that $I(d', e) = 0$. Therefore $d' \simeq d_i^{\pm 1}$. So Step 1 is established, noting that by symmetry, any d_i is isotopic to some d'_k.

Step 2: Now, the boundary of M is not assumed to be necessarily empty. If, as in the hypothesis, $(N, Q) \neq (S^1 \times I, 0) \neq (N', Q')$, the curves $d_1, \dots, d_l$ are isotopically distinct, and likewise $d'_1, \dots, d'_{l'}$. We conclude that $l = l'$ and, after renumbering, each d'_i is isotopic in M rel P, with d_i or d_i^{-1}. By Proposition 3.10 there is an isotopy of $M \setminus P$ taking ∂N to $\partial N'$. It follows that $(a) \Rightarrow (d)$ in all cases.

We now show that the possibility $N = \overline{M \setminus N'}$ can occur only if (N, Q) is a pantalon in which case (M, N, P) is a doubled pantalon. Suppose that (N, Q) is not a pantalon (or $(S^1 \times I, 0), (D^2, 0), (D^2, 1)$ which are excluded by assumption). Then there exist curves $a, b : S^1 \to N \setminus Q$ with $I(a, b) > 0$. Let $A, B \in \mathcal{M}(N, Q)$ denote the corresponding Dehn twists. By commensurability, $A^k \in \mathcal{M}(N', Q')$ for some $k > 0$. But by disjointness, all elements of $\mathcal{M}(N', Q')$ commute with all elements of $\mathcal{M}(N, Q)$, so A^k and B commute. Proposition 3.7 implies $I(a, b) = 0$, a contradiction.

In summary, we have shown that in case (i), $(c) \Rightarrow (b) \Rightarrow (a) \Rightarrow (c)$. Cases (ii) and (iii) also follow from the above arguments and Proposition 6.4. □

7 Normalizers and Commensurators

The *commensurator* of a subgroup H of a group G is

$$Com_G(H) = \{g \in G \ : \ g^{-1}Hg \text{ and } H \text{ are commensurable}\},$$

the *normalizer* of H in G is

$$\mathcal{N}_G(H) = \{g \in G \ : \ g^{-1}Hg = H\},$$

and the *centralizer* of H in G is

$$\mathcal{Z}_G(H) = \{g \in G \ : \ gh = hg \text{ for all } h \in H\}.$$

In general we have

$$\mathcal{Z}_G(H) \subset \mathcal{N}_G(H) \subset Com_G(H). \quad \text{Also } H \subset \mathcal{N}_G(H).$$

For any subsurface N of M, we define the *stabilizer* $Stab(N)$ as

$$Stab(N) = \{[h] \in \mathcal{M}(M,P) : h(N) \text{ is isotopic to } N \text{ in } M \text{ rel } P \cup \partial M\}.$$

Noting that $h \simeq h'$ implies $h(N) \simeq h'(N)$, we see this is a well-defined subgroup of $\mathcal{M}(M,P)$.

As in the previous section, the doubled pantalons are exceptional cases, so we make the following constructions. Let (M,N,P) be a doubled pantalon of type I, II or III, embedded in the xyz-space symmetrically with respect to 180 degree rotation $s : M \to M$ about the x-axis, and so that $(N, N \cap P)$ is the intersection of $\underline{(M,P)}$ with the half-space $y \geq 0$ (see Figure 21). The map s interchanges N and $\overline{M \setminus N}$, and reverses the orientation of ∂N. We will call $\sigma = [s] : \mathcal{M}(M,P) \to \mathcal{M}(M,P)$, the *exchange* map, for each of the three types of doubled pantalon.

Theorem 7.1 *Let $N \subset M$ be an injective subsurface and denote $G = \mathcal{M}(M,P)$ and $H = \mathcal{M}(N, N \cap P)$.*

i) If (M,N,P) is not a doubled pantalon, then

$$Com_G(H) = \mathcal{N}_G(H) = Stab(N).$$

ii) If (M,N,P) is a doubled pantalon of type II or III, then

$$Com_G(H) = \mathcal{N}_G(H) = Stab(N) \times \langle\sigma\rangle.$$

iii) If (M,N,P) is a doubled pantalon of type I, then

$$Com_G(H) = Stab(N) \rtimes \langle\sigma\rangle \quad \text{and } \mathcal{N}_G(H) = Stab(N).$$

Here $\langle\sigma\rangle$ denotes the cyclic subgroup of order 2 generated by the exchange σ of (M,N,P) and $Stab(N)$ is normal in the semidirect product.

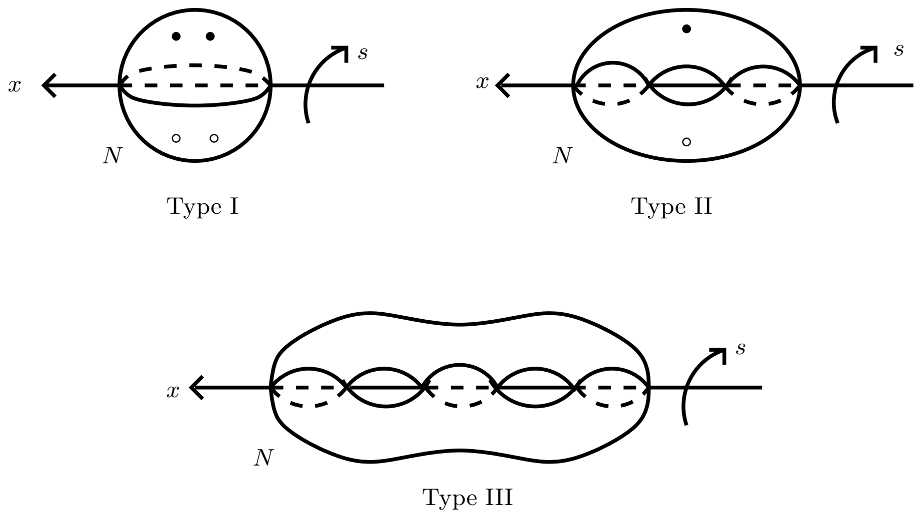

Figure 21: Exchange map for doubled pantalons.

Proof: Let $\xi \in Com_G(H)$ and let $h \in \mathcal{H}(M, P)$ represent ξ. Note that

$$\xi \mathcal{M}(N, P \cap N)\xi^{-1} = \mathcal{M}(h(N), P \cap h(N)).$$

First suppose (M, N, P) is not a doubled pantalon, and if N is a cylinder assume $P \cap N$ is nonempty. Since the groups $\mathcal{M}(N, P \cap N)$ and $\mathcal{M}(h(N), P \cap h(N))$ are commensurable, Theorem 6.5 implies that $\mathcal{M}(N, P \cap N) = \mathcal{M}(h(N), P \cap h(N))$ and that $h(N)$ is isotopic to N rel P. This shows that $Com_G(H) = \mathcal{N}_G(H) = Stab(N)$.

Next suppose that N is a cylinder and $P \cap N$ is empty. Then $h(N)$ is also a cylinder and by Proposition 6.3 $\mathcal{M}(N, P \cap N) = \mathcal{M}(h(N), P \cap h(N))$ and $h(N)$ is isotopic to N rel P. Again we conclude $Com_G(H) = \mathcal{N}_G(H) = Stab(N)$.

Now assume that (M, N, P) is a doubled pantalon of type II or III. By Theorem 6.5, $\mathcal{M}(N, P \cap N) = \mathcal{M}(h(N), P \cap h(N))$ and $h(N)$ is isotopic to N or $\overline{M \setminus N}$ rel P. Therefore $Com_G(H) = \mathcal{N}_G(H)$, but this group is bigger than $Stab(N)$; for example σ belongs to the normalizer, but does not stabilize N.

We can define an epimorphism $f : Com_G(H) \to \mathbf{Z}/2\mathbf{Z}$ by $f([h]) = 0$ if $h(N)$ is isotopic to N and $f([h]) = 1$ if $h(N)$ is isotopic to $\overline{M \setminus N}$. The kernel of f is $Stab(N)$. The homomorphism f has a section $s : \mathbf{Z}/2\mathbf{Z} \to Com_G(H)$ given by $s(0) = [id], s(1) = \sigma$, where σ is the exchange of (M, N, P). It follows that $Com_G(H) = Stab(N) \rtimes \langle\sigma\rangle$. By Propositions 5.3 and 5.5, σ is in the centre of $\mathcal{M}(M, P)$, so the product is, in fact, a direct product.

Finally, consider the case that (M, N, P) is a doubled pantalon of type I. By Theorem 6.5, $h(N)$ is isotopic with N or $\overline{M \setminus N}$. Moreover, if $\xi \in \mathcal{N}_G(H)$, then

$h(N)$ is isotopic with N, so we conclude that $\mathcal{N}_G(H) = Stab(N)$. By defining the homomorphism f and its section s exactly as above, we also conclude that $Com_G(H) = Stab(N) \rtimes \langle \sigma \rangle$. □

We proceed now to study the group $Stab(N)$ which, as seen before, determines the commensurator and the normalizer of $\mathcal{M}(N, N \cap P)$ in $\mathcal{M}(M, P)$.

We first state the following proposition which can be proved in the same manner as Theorem 4.1.

Proposition 7.2 *Let (N,Q) and (N',Q') be marked surfaces. Choose l boundary components $c_1, \ldots, c_l$ of N and l boundary components $c'_1, \ldots, c'_l$ of N', and denote by C_i and C'_i the Dehn twists corresponding to c_i and c'_i in $\mathcal{M}(N,Q)$ and $\mathcal{M}(N',Q')$, respectively. Let M be the surface obtained by pasting together N and N' along the curves c_i and $c_i'^{-1}$ for all $i = 1, \ldots, l$, and let $P = Q \cup Q' \subset M$ (see Figure 22). Assume that $|Q| \geq 2$ if N is a disk and $|Q| \geq 1$ if N is an annulus. Assume the same for N' and Q'. Consider the homomorphism $\phi : \mathcal{M}(N,Q) \times \mathcal{M}(N',Q') \to \mathcal{M}(M,P)$ induced by the inclusions of N and N' in M. Then the kernel of ϕ is generated by*

$$\{(C_1, C_1'^{-1}), \ldots, (C_l, C_l'^{-1})\},$$

and is isomorphic with $\mathbf{Z}^l$. □

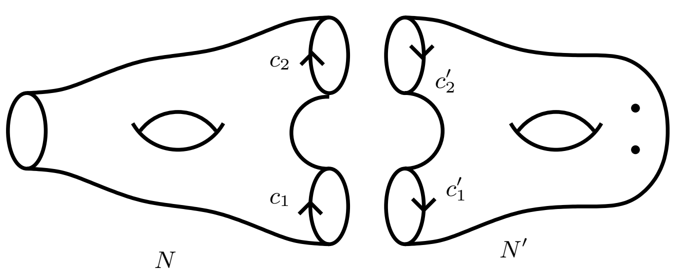

Figure 22: Marked surfaces whose union is M.

Let (M,P) be a marked surface, and let N be an injective subsurface of M. Let $c_1, \ldots, c_l$ denote the boundary components of N and Σ_l the symmetric group of $\{1, \ldots, l\}$. There is a natural homomorphism $\tau : Stab(N) \to \Sigma_l$ which associates to $[h] \in Stab(N)$ the unique $\sigma \in \Sigma_l$ such that $h(c_i)$ is isotopic to $c_{\sigma(i)}$. Here we assume that the curves c_i are provided with the orientation induced by the one of N. So, since $h(N)$ is isotopic to N, such a σ exists and is unique, even if N

is an annulus and $N \cap P = \emptyset$. The homomorphism τ is not surjective in general. However, one can explicitly describe its image which depends on the topology of the complement of $N \setminus N \cap P$ in $M \setminus P$. This image is long and tedious to describe and we let the reader off this description.

Let $N_1, \ldots, N_r$ denote the connected components of $\overline{M \setminus N}$. The inclusions of the N_i in M and of N in M induce a homomorphism

$$\phi : \mathcal{M}(N, N \cap P) \times \mathcal{M}(N_1, N_1 \cap P) \times \ldots \times \mathcal{M}(N_r, N_r \cap P) \to \mathcal{M}(M, P).$$

Assume that N is not an annulus with $N \cap P = \emptyset$. Let C_i denote the Dehn twist in $\mathcal{M}(N, N \cap P)$ corresponding to c_i, N_j the component of $\overline{M \setminus N}$ having c_i as boundary component, and C_i' the Dehn twist in $\mathcal{M}(N_j, N_j \cap P)$ corresponding to c_i. Then, by Proposition 7.2, the kernel of ϕ is generated by

$$\{C_1 C_1'^{-1}, \ldots, C_l C_l'^{-1}\}$$

and is a copy of $\mathbf{Z}^l$. The image of ϕ is obviously included in the kernel of τ and, by Proposition 3.10, any element of the kernel of τ belongs in the image of ϕ. So, we have proved:

Theorem 7.3 *Let $(N, N \cap P)$ be a marked injective subsurface of (M, P) different from $(S^1 \times I, \emptyset)$, and let τ and ϕ be the homomorphisms given above. Then we have the exact sequence*

$$1 \to \mathbf{Z}^l \to \mathcal{M}(N, N \cap P) \times \mathcal{M}(N_1, N_1 \cap P) \times \ldots \times \mathcal{M}(N_r, N_r \cap P) \xrightarrow{\phi} Stab(N) \xrightarrow{\tau} \Sigma_l.$$

□

Proposition 7.4 *Assume that N is an annulus, $N \cap P = \emptyset$, and there is no boundary component of N which is the boundary of a disk intersecting P in less than two points.*

i) Suppose M is a torus and P is empty. Then

$$Stab(N) = \mathcal{M}(N) \times \langle \sigma \rangle,$$

where σ is the element in $\mathcal{M}(M)$ of order two which generates the centre.

ii) Suppose $\overline{M \setminus N}$ has a unique connected component, N_1, and $(M, P) \neq (T^2, \emptyset)$ (see Figure 23.i). Then we have an exact sequence

$$1 \to \mathbf{Z} \to \mathcal{M}(N_1, P) \to Stab(N) \xrightarrow{\tau} \Sigma_2 \to 1.$$

iii) Suppose $\overline{M \setminus N}$ has two connected components, N_1 and N_2 (see Figure 23.ii). Then we have an exact sequence

$$1 \to \mathbf{Z} \to \mathcal{M}(N_1, N_1 \cap P) \times \mathcal{M}(N_2, N_2 \cap P) \to Stab(N) \xrightarrow{\tau} \Sigma_2.$$

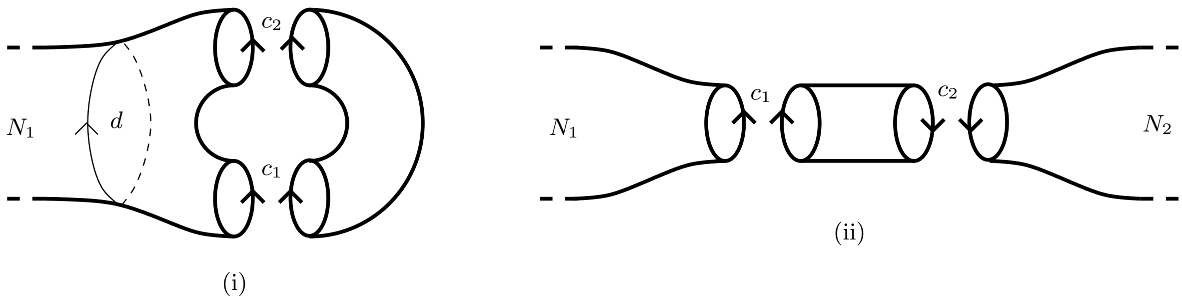

Figure 23: The annulus N of Proposition 7.4.

Proof: First, suppose that M is a torus and P is empty. By the structure of the mapping class group of an annulus, the kernel of τ is $\mathcal{M}(N) \simeq \mathbf{Z}$. Moreover, $\sigma \in Stab(N)$ and σ permutes (up to isotopy) the boundary components of N, thus τ is surjective and the map $(1,2) \mapsto \sigma$ gives a section of τ. Since σ is central in $\mathcal{M}(M)$, it follows that $Stab(N) = \mathcal{M}(N) \times \langle \sigma \rangle$.

Assume now that $\overline{M \setminus N}$ has a unique connected component, N_1, and $(M,P) \neq (T^2, \emptyset)$. Let c_1, c_2 denote the boundary components of N. We can cut N_1 along some closed essential curve d into two subsurfaces such that one of them is a pantalon of type III having d, c_1 and c_2 as boundary components (see Figure 23.i). Pasting this pantalon with N one obtains a genus one surface with one boundary component, d. Let ρ be the half-twist of this surface of genus one along d relative to c_1, as defined in Section 5. Then $\rho \in Stab(N)$ and $\tau(\rho) = (1,2)$. This shows that τ is surjective. Let C_i denote the Dehn twist along c_i in $\mathcal{M}(N_1, P)$, $\iota : \mathcal{M}(N_1, P) \to Stab(N)$ the homomorphism induced by the inclusion of N_1 in M, and $\theta : \mathbf{Z} \to \mathcal{M}(N_1, P)$ the homomorphism defined by $\theta(1) = C_1 C_2^{-1}$. Then the equality $\mathrm{Im}\iota = \mathrm{Ker}\tau$ follows from Proposition 3.10, and Theorem 4.1 implies $\mathrm{Ker}\theta = \{0\}$ and $\mathrm{Im}\theta = \mathrm{Ker}\iota$.

Assume now that $\overline{M \setminus N}$ has two connected components, N_1 and N_2. Let c_i denote the common boundary component of N and N_i, C_i the Dehn twist along c_i in $\mathcal{M}(N_i, N_i \cap P)$,

$$\iota : \mathcal{M}(N_1, N_1 \cap P) \times \mathcal{M}(N_2, N_2 \cap P) \to Stab(N)$$

the homomorphism induced by the inclusions of N_1 and N_2 in M, and

$$\theta : \mathbf{Z} \to \mathcal{M}(N_1, N_1 \cap P) \times \mathcal{M}(N_2, N_2 \cap P)$$

the homomorphism defined by $\theta(1) = (C_1, C_2^{-1})$. The equality $\mathrm{Im}\iota = \mathrm{Ker}\tau$ and the inclusion $\mathrm{Im}\theta \subset \mathrm{Ker}\iota$ are obvious, and the injectivity of θ follows from Corollary

3.9. So, it remains to prove the inclusion Ker$\iota \subset$ Imθ. If $(N_1, N_1 \cap P) = (S^1 \times I, \emptyset)$, then $\mathcal{M}(N_1, N_1 \cap P) = \mathcal{M}(N)$ is the infinite cyclic subgroup generated by $C_1 = C_2$, and therefore, since $\mathcal{M}(N_2, N_2 \cap P)$ injects in $\mathcal{M}(M, P)$ by Theorem 4.1, Ker$\iota \subset$ Imθ. If $(N_1, N_1 \cap P) \neq (S^1 \times I, \emptyset) \neq (N_2, N_2 \cap P)$, then the inclusion Ker$\iota \subset$ Imθ follows from Proposition 7.2. □

8 Centralizers

Theorem 8.1 *Let $N \subset M$ be an injective subsurface and denote $G = \mathcal{M}(M, P)$ and $H = \mathcal{M}(N, N \cap P)$.*

i) If N is an annulus and $N \cap P$ is empty, then

$$Com_G(H) = \mathcal{N}_G(H) = \mathcal{Z}_G(H) = Stab(N).$$

ii) If (M, N, P) is a doubled pantalon of type II or III, then

$$\mathcal{Z}_G(H) = \mathcal{M}(N, N \cap P) \times \langle \sigma \rangle,$$

where $\langle \sigma \rangle$ is the cyclic subgroup of order two generated by the exchange σ of (M, N, P).

iii) Suppose that $(N, N \cap P)$ is not as in (i) or (ii). Let $N_1, \dots, N_r$ denote the connected components of $\overline{M \setminus N}$. Then we have the exact sequence

$$1 \to \mathbf{Z}^l \to \mathcal{Z}\mathcal{M}(N, N \cap P) \times \mathcal{M}(N_1, N_1 \cap P) \times \dots \times \mathcal{M}(N_r, N_r \cap P) \xrightarrow{\phi} \mathcal{Z}_G(H) \to 1,$$

where l is the number of components of ∂N and ϕ is the homomorphism defined in Section 7. Moreover, assuming that $|N \cap P| \geq 3$ if N is a disk and $|N \cap P| \geq 1$ if N is a genus one surface with one boundary component, the restriction

$$\phi : \mathcal{M}(N_1, N_1 \cap P) \times \dots \times \mathcal{M}(N_r, N_r \cap P) \to \mathcal{Z}_G(H)$$

of ϕ to $\mathcal{M}(N_1, N_1 \cap P) \times \dots \times \mathcal{M}(N_r, N_r \cap P)$ is an isomorphism.

Proof: Suppose that N is an annulus and $N \cap P$ is empty. The equalities $Com_G(H) = \mathcal{N}_G(H) = Stab(N)$ are proved in Theorem 7.1 and the inclusion $\mathcal{Z}_G(H) \subset \mathcal{N}_G(H)$ is obvious. So, it remains to prove $Stab(N) \subset \mathcal{Z}_G(H)$. Let c_1, c_2 denote the boundary curves of N. The Dehn twists C_1 and C_2 along c_1 and c_2, respectively, coincide and generate $\mathcal{M}(N)$. Let $\xi \in Stab(N)$ and let $h \in \mathcal{H}(M, P)$ represent ξ. Then $h \circ c_1$ is isotopic with c_1 or c_2, thus $\xi C_1 \xi^{-1} = C_1$ or $C_2 = C_1$, therefore $\xi \in \mathcal{Z}_G(H)$.

Suppose now that $(N, N \cap P)$ is not as in (i) or (ii). We know by Theorem 7.1 that $\mathcal{Z}_G(H) \subset \mathcal{N}_G(H) = Stab(N)$. Consider the exact sequence of Theorem 7.3. Let $c_1, \dots, c_l$ denote the boundary components of N and C_i the Dehn twist in $\mathcal{M}(M, P)$ corresponding to c_i. Let $\xi \in \mathcal{Z}_G(H)$ and let $h \in \mathcal{H}(M, P)$ represent ξ. The transformation h cannot permute the c_i because $\xi C_i \xi^{-1}$ is the Dehn twist along $h \circ c_i$, the equality $\xi C_i \xi^{-1} = C_i$ implies by Proposition 3.6 that $h \circ c_i$ is isotopic with c_i or c_i^{-1}, and $(N, N \cap P) \neq (S^1 \times I, \emptyset)$ together with the injectivity of N imply that c_i^{-1} is not isotopic with some c_j, $j \neq i$. So, $\mathcal{Z}_G(H) \subset \mathrm{Ker}\tau = \mathrm{Im}\phi$. Then the exact sequence

$$1 \to \mathbf{Z}^l \to \mathcal{Z}\mathcal{M}(N, N \cap P) \times \mathcal{M}(N_1, N_1 \cap P) \times \ldots \times \mathcal{M}(N_r, N_r \cap P) \xrightarrow{\phi} \mathcal{Z}_G(H) \to 1$$

is a straightforward consequence of the exact sequence of Theorem 7.3.

Suppose now in addition that $|N \cap P| \geq 3$ if N is a disk and $|N \cap P| \geq 1$ if N is a genus one surface with one boundary component. Then, by Theorem 5.6, the centre of $\mathcal{M}(N, N \cap P)$ is the free abelian group of rank l generated by $\{C_1, \dots, C_l\}$. Thus it follows from the expression of the kernel of ϕ given in the proof of Theorem 7.3 that the restriction of ϕ to $\mathcal{M}(N_1, N_1 \cap P) \times \ldots \times \mathcal{M}(N_r, N_r \cap P)$ is an isomorphism.

Assume now that (M, N, P) is a doubled pantalon of type II or III. Let $\xi \in \mathcal{Z}_G(H)$ and let $h \in \mathcal{H}(M, P)$ represent ξ. We show as above that, if h stabilizes N (up to isotopy), then h fixes (up to isotopy) the boundary components of N, and therefore $h \in \mathcal{M}(N, N \cap P) = \mathcal{M}(\overline{M \setminus N}, \overline{M \setminus N} \cap P)$. So,

$$\mathcal{Z}_G(H) \cap Stab(N) = \mathcal{M}(N, N \cap P).$$

The exchange σ is an element of the centre of $\mathcal{M}(M, P)$ and thus an element of $\mathcal{Z}_G(H)$. Finally, from the equality $\mathcal{N}_G(H) = Stab(N) \times \langle \sigma \rangle$ of Theorem 7.1 follows the equality $\mathcal{Z}_G(H) = \mathcal{M}(N, N \cap P) \times \langle \sigma \rangle$. □

References

[1] J. Birman, "Braids, links, and mapping class groups", Annals of Mathematics Studies **82**, Princeton University Press and University of Tokyo Press, Princeton, 1975.

[2] J. Birman, *Mapping class groups of surfaces*, Braids, AMS-IMS-SIAM Jt. Summer Res. Conf., Santa Cruz/Calif. 1986, Contemp. Math. **78**, 1988, pp. 13–43.

[3] J. Birman, A. Lubotzky, J. McCarthy, *Abelian and solvable subgroups of the mapping class group*, Duke Math. J. **50** (1983), 1107-1120.

[4] W. Chow, *On the algebraical braid group*, Ann. Math. **49** (1948), 654–658.

[5] D.B.A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. **115** (1966), 83–107.

[6] D.B.A. Epstein, J. W. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston, "Word processing in groups," Jones and Bartlett, Boston, 1992.

[7] A. Fathi, F. Laudenbach, and V. Poénaru, "Travaux de Thurston sur les surfaces, Séminaire Orsay", Astérisque **66-67**, 1979.

[8] R. Fenn, D. Rolfsen, and J. Zhu, *Centralisers in the braid group and singular braid monoid*, Enseign. Math. **42** (1996), 75–96.

[9] S. Gervais, *A finite presentation of the mapping class group of an oriented surface*, preprint, 1998.

[10] E. Grossman, *On the residual finiteness of certain mapping class groups*, J. London Math. Soc. **9** (1974), 160–164.

[11] A. Hatcher, W. Thurston, *A presentation for the mapping class group of a closed orientable surface*, Topology **19** (1980), 221–237.

[12] A. Hurwitz, *Über algebraische Gebilde mit eindeutige Transformationen in sich*, Math. Ann. **41** (1893), 403–442.

[13] A. Ishida, *The structure of subgroups of mapping class groups generated by two Dehn twists*, Proc. Japan Acad. Ser. A **72** (1996), 240–241.

[14] N.V. Ivanov, *Complexes of curves and the Teichmüller modular group*, Russ. Math. Surv. **42** (1987), 55-107.

[15] N.V. Ivanov, "Subgroups of Teichmüller modular groups", Translation of Mathematical Monographs **115**, Amer. Math. Soc., Providence, 1992.

[16] N.V. Ivanov, *Automorphisms of Teichmüller modular groups*, Topology and geometry, Rohlin Semin. 1984–1986, Lect. Notes Math. **1346**, Springer-Verlag, Berlin-Heidelberg-New York, 1988, pp. 199–270.

[17] N.V. Ivanov, J.D. McCarthy, *On injective homomorphisms between Teichmüller modular groups I*, Invent. Math. **135** (1999), 425–486.

[18] D. Johnson *A survey of the Torelli group*, Low-dimensional topology (San Francisco, Calif., 1981), Contemp. Math., 20, Amer. Math. Soc., Providence, R.I., 1983, 165–179.

[19] D. Johnson, *The structure of the Torelli group. I. A finite set of generators for* $\mathcal{I}$, Ann. of Math. (2) **118** (1983), no. 3, 423–442.

[20] D. Johnson, *The structure of the Torelli group. II. A characterization of the group generated by twists on bounding curves*, Topology **24** (1985), no. 2, 113–126.

[21] D. Johnson, *The structure of the Torelli group. III. The abelianization of $\mathcal{I}$*, Topology **24** (1985), no. 2, 127–144.

[22] S. Kerckhoff, *The Nielsen realization problem*, Ann. Math. **117** (1983), 235–265.

[23] W.B.R. Lickorish, *A finite set of generators for the homeotopy group of a 2-manifold*, Proc. Camb. Philos. Soc. **60** (1964), 769–778.

[24] R.C. Lyndon, P.E. Schupp, "Combinatorial group theory", Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[25] E. Looijenga, *Affine Artin groups and the fundamental groups of some moduli spaces*, preprint, 1998.

[26] M. Matsumoto, *A presentation of mapping class groups in terms of Artin groups and geometric monodormy of singularities*, preprint, 1998.

[27] J.D. McCarthy, *A Tits alternative for subgroups of surface mapping class groups*, Trans. Amer. Math. Soc. **291** (1985), 583–612.

[28] J.D. McCarthy, *Automorphisms of surface mapping class groups. A recent theorem of N. Ivanov*, Invent. Math. **84** (1986), 49–71.

[29] J. McCool, *Some finitely presented subgroups of the automorphism group of a free group*, J. Algebra **35** (1975), 205–213.

[30] G. Mislin, *Mapping class groups, characteristic classes, and Bernouilli numbers*, CRM Proc. Lect. Notes **6** (1994), 103–131.

[31] S. Morita, *Mapping class groups of surfaces and three-dimensional manifolds*, Proc. Int. Congr. Math., Kyoto/Japan 1990, Vol. I, 1991, pp. 665–674.

[32] L. Mosher, *Mapping class groups are automatic*, Ann. Math. **142** (1995), 303–384.

[33] V.P. Platonov, *Frattini subgroups of linear groups and finitary approximability*, Dokl. Akad. Nauk SSSR **171** (1966), 798–801; English transl., Soviet Math. Dokl. **7** (1966), 1557–1560.

[34] J. Powell, *Two theorems on the mapping class group of a surface*, Proc. Amer. Math. Soc. **68** (1978), 347–350.

[35] D. Rolfsen, *Braid subgroup normalisers, commensurators and induced representations*, Invent. Math. **130** (1997), 575–587.

[36] B. Wajnryb, *A simple presentation for the mapping class group of an orientable surface*, Isr. J. Math. **45** (1983), 157–174.

[37] B. Wajnryb, *Mapping class group of a surface is generated by two elements*, Topology **35** (1996), 377–383.

[38] H. Zieschang, *Finite groups of mapping classes of surfaces*, Lect. Notes Math. **875**, Springer-Verlag, Berlin-Heidelberg-New York, 1981.

Luis PARIS
Laboratoire de Topologie
Université de Bourgogne
UMR 5584 du CNRS
BP 47870
21078 Dijon Cedex
FRANCE
lparis@u-bourgogne.fr

Dale ROLFSEN
Department of Mathematics
University of British Columbia
Vancouver
British Columbia V6T 1Z2
CANADA
rolfsen@math.ubc.ca