Ordered Groups and Topology

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Outline:
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Lecture 2: Topology and orderings
- π_1, applications
- braid groups
- mapping class groups
- hyperplane arrangements
- surface braid groups

Lecture 3: Ordering 3-manifold groups
- Seifert fibrations
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Ordered groups

Definitions: Let G be a group, and $<$ a strict total ordering if its elements. Then $(G, <)$ is a *left-ordered* group (LO) if

$$g < h \Rightarrow fg < fh.$$

If the ordering is also right-invariant, we say that $(G, <)$ is an *ordered* group (O), or for emphasis *bi-ordered.*

Prop: G is left-orderable if and only if there exists a subset $\mathcal{P} \subset G$ such that:

$$\mathcal{P} \cdot \mathcal{P} \subset \mathcal{P} \text{ (subsemigroup)}$$
$$G \setminus \{e\} = \mathcal{P} \coprod \mathcal{P}^{-1}$$

Given $\mathcal{P}$ define $<$ by: $g < h$ iff $g^{-1}h \in \mathcal{P}$.
Given $<$ take $\mathcal{P} = \{g \in G : 1 < g\}$.

The ordering is a bi-ordering iff also

$$g^{-1}\mathcal{P}g \subset \mathcal{P},\ \forall g \in G.$$

Note: a group is right-orderable iff it is left-orderable.

Examples:
1. $\mathbb{R}$, the additive reals with the usual ordering.
2. $\mathbb{R}^2$ with the lexicographical ordering.

3. $\mathbb{Z}^2$ has uncountably many different orderings, one for each line through $(0,0)$ of irrational slope.
4. $\mathbb{R} \setminus \{0\}$ under multiplication is not orderable, or even left-orderable. It has an element (-1) of order two.
We will see that there are surprisingly many nonabelian LO and O groups.

Prop: If G is left-orderable, then G is torsion-free.
Prop: If G is bi-orderable, then
• G has no generalized torsion (product of conjugates of a nontrivial element being trivial).
• G has unique roots: $g^n = h^n \Rightarrow g = h$
• if $[g^n, h] = 1$ in G then $[g, h] = 1$

The class of LO groups is closed under: subgroups, direct products, free products, directed unions, extensions.
The class of O groups is closed under: subgroups, direct products, free products, directed unions, but not necessarily under extensions.
Both properties O and LO are local: a group has the property if and only if every finitely-generated subgroup has it.

Prop: (Extensions) Given an exact sequence

$$1 \longrightarrow F \xrightarrow{i} G \xrightarrow{p} H \longrightarrow 1$$

If F and H are left-orderable, then so is G, using the positive cone:

$$\mathcal{P}_G := i(\mathcal{P}_F) \cup p^{-1}(\mathcal{P}_H).$$

If F and H are bi-ordered, then this defines a bi-ordering of G if and only if

$$g^{-1} i(\mathcal{P}_F) g \subset i(\mathcal{P}_F), \ \ \forall g \in G$$

Example: The Klein Bottle group:

$$\langle x, y : x^{-1} y x = y^{-1} \rangle$$

is LO, being an extension of $\mathbb{Z}$ by $\mathbb{Z}$.
However, it is not bi-orderable:
$1 < y$ iff $y^{-1} < 1$ holds in any LO-group.
However, in an O-group $1 < y$ iff $1 < x^{-1} y x$. This would lead to a contradiction.

Warning:
Note that $x < y$ and $z < w$ imply $xz < yw$ in an O-group, but not in an LO-group.
Example: in the Klein bottle group, y and x^2 commute. So, although $yx \neq x$ we have

$$(yx)^2 = yx^2 x^{-1} y x = x^2 y y^{-1} = x^2.$$

So this group does not have unique roots.

Exercise: A left-ordered group $(G, <)$ is bi-ordered iff

$$x < y \Leftrightarrow y^{-1} < x^{-1} \quad \forall x, y \in G.$$

Thm: (Rhemtulla) Suppose G is left-orderable. Then G is abelian iff every left-ordering is a bi-ordering.

Def: An ordering $<$ of G is *Archimedian* if whenever $1 < x < y$, there exists a positive integer n such that $y < x^n$.
Hölder's thm (1902): Suppose $(G, <)$ is an O-group which is Archimedian. Then G is isomorphic with a subgroup of the additive real numbers (and $<$ corresponds to the natural ordering of $\mathbb{R}$). In particular, G is abelian.

Thm (Conrad, 1959): If $(G, <)$ is LO and Archimedian, then the ordering is actually a bi-ordering, so the conclusions of Hölder's theorem apply.

Why is orderability useful?
Group rings: For any group G, let $\mathbb{Z}G$ denote the group ring of formal linear combinations $n_1 g_1 + \cdots + n_k g_k$.
Thm: If G is LO, then $\mathbb{Z}G$ has no zero divisors.
This is conjectured to be true for torsion-free groups.
Thm:(Malcev, Neumann) If G is an O-group, then $\mathbb{Z}G$ embeds in a division ring.
Thm:(LaGrange, Rhemtulla) If G is LO and H is any group, then $\mathbb{Z}G \cong \mathbb{Z}H$ implies $G \cong H$.

Group actions and orderability:
Say the group G acts on the set X via $x \mapsto gx$ if $(gh)x = g(hx)$. G acts effectively if only $1 \in G$ acts trivially on X.
Thm: A group G is LO if and only if there exists a totally ordered set X upon which G acts effectively by order-preserving bijections.
Example: The group $Homeo_+(\mathbb{R})$ is LO.
Thm: A countable group G is LO if and only if it embeds in $Homeo_+(\mathbb{R})$.
If G acts on R without fixed points, it is bi-orderable.

Another useful characterization:
Thm: (Burns-Hale) A group G is LO if and only if for every nontrivial finitely-generated subgroup $H \subset G$, there exists a left-ordered group L and a nontrivial homomorphism

$$H \to L.$$

Def: A group G is *locally indicable* if for every f.g. subgroup $1 \neq H \subset G$, $\exists$ nontrivial $H \to \mathbb{Z}$.
Cor: Locally indicable $\Rightarrow$ left-orderable.
Prop: Bi-orderable $\Rightarrow$ locally indicable.

The Klein bottle group is LI but not O.
Bergman: $\langle x, y, z : x^2 = y^3 = z^7 = xyz \rangle$ is LO, but not LI. It's π_1 of a homology 3-sphere.
Similarly, $\widetilde{SL}_2(\mathbb{R})$ is LO but not LI.

Thm: Free groups are bi-orderable.
Cor: Excepting $\mathbb{P}^2$, surface groups are LO.
Proof: If M^2 is noncompact or ∂M is nonempty, $\pi_1(M)$ is free.

Proof that free groups are bi-orderable:
Let $F = \langle x_1, x_2 \rangle$ denote the free group of rank two. We wish to construct an explicit bi-ordering on F.
The Magnus expansion: Consider the ring

$$\Lambda = \mathbb{Z}[[X_1, X_2]]$$

of formal power series in the non-commuting variables X_1 and X_2. The Magnus map is the (multiplicative) homomorphism

$$\mu : F \to \Lambda$$

defined by:

$$\begin{aligned} x_i &\mapsto 1 + X_i \\ x_i^{-1} &\mapsto 1 - X_i + X_i^2 - X_i^3 + \cdots \end{aligned}$$

Lemma: μ is injective; its image lies in the group of units of Λ of the form 1 + O(1).
Define an ordering $<$ on Λ by the following recipe: Write the elements of Λ in a standard form, with lower degree terms preceding higher degree terms, and within a given degree list the terms in sequence according to (say) the lexicographic ordering of the variables' subscripts.
Compare two elements of Λ by writing them both in standard form and ordering them according to the natural ordering of the coefficients at the *first* term at which they differ.
It defines an ordering $<$ on Λ which is invariant under addition. Moreover, restricted to the group of units $\{1 + O(1)\}$, it is invariant under multiplication on both sides.

Since F is embedded in $\{1 + O(1)\}$, this defines a bi-invariant ordering for the free group.
Example:
$\mu(x_1^{-1} x_2 x_1) =$
$= (1 - X_1 + X_1^2 - \cdots)(1 + X_2)(1 + X_1)$
$= 1 + X_2 - X_1 X_2 + X_2 X_1 + O(3)$
$\mu(x_2) = 1 + X_2 + 0 X_1 X_2 + \cdots$

Therefore $1 < x_1^{-1} x_2 x_1 < x_2$.

Lecture 2: Topology and orderable groups

Thm: (Farrell) Suppose X is a paracompact Hausdorff space. Then $\pi_1(X)$ is LO if and only if there is an embedding of the universal covering $h : \tilde{X} \to X \times \mathbb{R}$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{h} & X \times \mathbb{R} \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

Thm: (Smythe) Consider a knot

$$K \subset M^2 \times \mathbb{R},$$

and a regular projection $p(K)$ in M. Suppose K is homotopically trivial.
Then there is a choice of over-under at the crossings of $p(K)$ which creates a knot K' in $M^2 \times \mathbb{R}$ with $p(K') = p(K)$, but K' is unknotted in $M^2 \times \mathbb{R}$ (it bounds a disk).

Ordering braid groups

B_n has generators $\sigma_1, \ldots, \sigma_{n-1}$
and relations

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i \,, \;\; |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \end{aligned}$$

Thm: (Dehornoy) B_n is left-orderable.
We will outline three different proofs.
Note: for $n > 2$, B_n cannot be bi-ordered.
Take $x = \sigma_1 \sigma_2 \sigma_1$ and $y = \sigma_1 \sigma_2^{-1}$, and observe

$$x^{-1} y x = y^{-1}.$$

Proof 1 (Dehornoy): Define the positive cone $\mathcal{P} \subset B_n$ by
$\beta \in \mathcal{P}$ iff there exists an expression

$$\beta = w_1 \sigma_i \, w_2 \sigma_i \cdots \sigma_i \, w_k$$

where each $w_j \in \langle \sigma_{i+1} \cdots \sigma_{n-1} \rangle$.
In other words the generator with the lowest subscript has only positive exponents.
It is easy to verify that $\mathcal{P} \cdot \mathcal{P} \subset \mathcal{P}$;
To show that $B_n \setminus \{1\} = \mathcal{P} \coprod \mathcal{P}^{-1}$, it is not too hard to show that $\mathcal{P}$ and $\mathcal{P}^{-1}$ are disjoint.

The difficult part is to show that every nontrivial braid, or its inverse, can be expressed in the above form: $B_n \setminus \{1\} = \mathcal{P} \cup \mathcal{P}^{-1}$

Proof 2 (Fenn, Green, Rolfsen, Rourke, Wiest): This uses the alternative view of B_n as the mapping class group of the disk with n punctures:

$$B_n \cong \mathcal{M}(D^2, n)$$

D^2 is pictured as a round disk in the complex plane enclosing the integers $1, \ldots, n$. Given a mapping of the disk to itself, consider the image of the real line $\mathbb{R}$.
After an isotopy of the mapping f, one may assume that this image is "taut" in that the number of components of $f(\mathbb{R}) \cap \mathbb{R}$ is minimized.
Then a braid $\beta = [f]$ is considered positive iff the first departure of $f(\mathbb{R})$ from $\mathbb{R}$ goes into the upper half-plane.
Remarkably, this ordering is precisely the same as Dehornoy's.

Proof 3 (Thurston, *a la* Short and Wiest): Again we consider $B_n \cong \mathcal{M}(D^2, n)$. The punctured disk D_n^2 has universal covering $\widetilde{D_n}$ embeddable in the hyperbolic plane $\mathbb{H}$. Choose a fixed basepoint $*$ in one of the lifts of the boundary of D^2. Given $\beta = [f]$, with $f : D_n^2 \to D_n^2$ let $\tilde{f} : \widetilde{D_n} \to \widetilde{D_n}$ be the unique lifting of f which fixes $*$.
Note that $\partial\widetilde{D_n} \cong S^1$. Every such lift fixes an interval of S^1 containing $*$, so we may consider $\tilde{f}$ as an orientation-preserving mapping $\mathbb{R} \to \mathbb{R}$.
This action of B_n on $\mathbb{R}$ shows B_n is LO.

Advantages of proof 3:
This approach defines infinitely many left-orderings of B_n, including Dehornoy's.
Some are order-dense, others (like D's) are discrete.
Also it easily generalizes to other mapping class groups.
Thm: If M^2 is a compact surface with nonempty boundary (with or without punctures), then $\mathcal{M}(M^2)$ is LO.

The *pure* braid groups P_n:
Thm:(Kim-R.-Zhu)) P_n is bi-orderable.
Proof: According to Artin's combing technique, P_n is a semidirect product of free groups, which are bi-orderable. However, since bi-orderability is not necessarily preserved under semidirect products we need to exercise some care. We will proceed by induction: clearly $P_1 = \{1\}$ and $P_2 \cong \mathbb{Z}$ are biorderable. Suppose P_n is biordered. There is a standard inclusion

$$P_n \overset{i}{\hookrightarrow} P_{n+1}$$

and also a homomorphism

$$P_{n+1} \overset{r}{\longrightarrow} P_n$$

which "forgets the last string" is a retraction of groups: $r \circ i = id$.

The kernel $K = \ker(r)$ can be regarded as all $(n+1)$ -string braids in which the first n strings are straight; so K can be also be regarded as the fundamental group of an n-punctured disk, a free group.

$$1 \to K \hookrightarrow P_{n+1} \xrightarrow{r} P_n \to 1$$

is exact.
Lemma: There is a bi-ordering on the free group K so that conjugation by any element of P_{n+1} is order-preserving.
Key fact: each such automorphism $K \to K$ becomes the identity upon abelianization. The Magnus ordering is invariant under all such automorphisms.
This completes the proof that P_{n+1} is bi-orderable.

Properties: With appropriate choice of generators of K we have:

- This ordering of P_n is order-dense ($n > 2$).
- It is compatible with the inclusions:

  $$P_n \hookrightarrow P_{n+1},$$

  and so bi-orders P_∞.
- The semigroup $P_n^+ = B_n^+ \cap P_n$ of Garside positive pure braids are all positive in the bi-ordering:

  $$\beta \in P_n^+ \setminus \{1\} \Rightarrow 1 < \beta.$$

- P_n^+ is *well-ordered* by this bi-ordering.

Note: B_n^+ is also well-ordered by the Dehornoy left-ordering of B_n. However, our ordering of P_n is very different from the restriction of any known left-ordering of B_n.
Question: Does there exist a bi-ordering $<$ of P_n, which extends to a left-invariant ordering of B_n?
Answer: NO!! Because of the following.
Prop: (Rhemtulla - R.) Suppose $(G, <)$ is a left-ordered group. Suppose $H \subset G$ is a finite-index subgroup such that $(H, <)$ is a bi-ordered group. Then G is locally indicable.
Prop: (Gorin - Lin) For $n > 4$, B_n is not locally-indicable. In fact $[B_n, B_n]$ is finitely-generated and perfect.

Hyperplane arrangements:
Let $\{H_\alpha\}$ be a finite family of complex hyperplanes in $\mathbb{C}^n$.
Example: For each i, j with $1 < i < j < n$, let H_{ij} be the hyperplane in $\mathbb{C}^n$ defined by $z_i = z_j$. Then $X = \mathbb{C}^n \setminus \cup H_{ij}$ is the set of distinct ordered n-tuples of complex numbers, and $\pi_1(X) \cong P_n$.

A class of hyperplane arrangements, generalizing this example, are those of "fibre type." As noted by L. Paris, we have:
Thm: If $\{H_\alpha\}$ is a hyperplane arrangement of fibre type, then $\pi_1(\mathbb{C}^n \setminus \cup H_\alpha)$ is bi-orderable.
Proof: The group is a semidirect product of free groups, in which the actions on the free groups become the identity upon abelianization. The proof proceeds as in the pure braid groups.

Bi-ordering surface groups.
Let M^2 be a connected surface.
If $M = \mathbb{P}^2$, then $\pi_1(M) \cong \mathbb{Z}_2$ is not LO.
If $M = \mathbb{P}^2 \# \mathbb{P}^2$, the Klein bottle, then $\pi_1(M)$ is LO, but not bi-orderable.
Thm: Except for the two examples above, every surface group $\pi_1(M)$ is bi-orderable.

Surface braid groups $B_n(M^2)$ and $PB_n(M^2)$
Thm: (Gonzalez-Meneses) For M^2 orientable, $PB_n(M^2)$ is bi-orderable.
Thm: For M^2 non-orientable, $M \neq \mathbb{P}^2$, $PB_n(M^2)$ is left-orderable, but not biorderable.

Lecture 3: Three-manifold groups

Joint work with Steve Boyer and Bert Wiest, with help from Bouleau, Perron, Short, Sjerve.
We consider $\pi_1(M^3)$, where M^3 is a *compact* 3-dimensional manifold.
M^3 may be nonorientable.
∂M^3 may be empty or nonempty.
Everything is assumed PL (or C^∞).

Prop: Suppose $G = G_1 * \cdots * G_k$. Then G is LO if and only if every factor G_i is LO. Similarly for "O, virtually LO, or virtually O" replacing LO.
Therefore, we may assume w.l.o.g., that M^3 is prime.
Def: M^3 is irreducible if every $S^2 \subset M^3$ bounds a 3-ball in M.
The only prime manifolds which are not irreducible are $S^1 \times S^2$ and $S^1 \tilde{\times} S^2$; both have fundamental group $\mathbb{Z}$, which is bi-orderable.
Def: M^2 is $\mathbb{P}^2$-irreducible if irreducible and M contains no 2-sided projective plane. (relevant only for nonorientable M)

Thm: (Boileau, Howie, Short) Suppose M^3 is compact, connected and $\mathbb{P}^2$-irreducible. Then $\pi_1(M)$ is LO if and only if there exists a LO group L and a nontrivial homomorphism

$$\pi_1(M) \to L.$$

Cor: If M^3 is compact, connected and $\mathbb{P}^2$-irreducible, and $H_1(M)$ is infinite, then $\pi_1(M)$ is LO.
In fact, $b_1(M) > 0$ implies $\pi_1(M)$ is locally indicable.

Prop: If M^3 is an irreducible homology sphere and there is a nontrivial homomorphism

$$\pi_1(M) \to PLS_2(\mathbb{R}),$$

then $\pi_1(M)$ is LO.
Cor: If M^3 is a Seifert-fibred homology sphere, and not the Poincaré dodecahedral space, then $\pi_1(M)$ is LO.
Recall that M^3 is a Seifert-fibred space (SFS) if it is foliated by circles.

Thm: If M^3 is a compact, connected SFS, then $\pi_1(M)$ is LO iff
(1) $M \cong S^3$, or
(2) $b_1(M) > 0$ and $M \ncong \mathbb{P}^2 \times S^1$, or
(3) M is orientable, $\pi_1(M)$ is infinite, the base orbifold is $S^2(\alpha_1, \ldots, \alpha_k)$ and M admits a horizontal codimension 1 foliation.
The SFS with horizontal codimension 1 foliations are characterized by Eisenbud-Hirsch-Neumann, Jankins-Neumann, Naimi.

Thm: If M^3 is a compact, connected SFS, then $\pi_1(M)$ is bi-orderable iff
(1) $M \cong S^3$, $S^1 \times S^2$, $S^1 \times S^2$ or a solid Klein bottle, or
(2) M is an honest circle bundle over a surface other than S^2, $\mathbb{P}^2$ or $2\mathbb{P}^2$.
Cor: If M is any compact SFS, $\pi_1(M)$ is virtually bi-orderable.

Question: If M is an arbitrary 3-manifold, is $\pi_1(M)$ virtually bi-orderable?
If so, this would answer affirmatively a conjecture of Waldhausen that M is virtually Haken.
Question: If M is an arbitrary 3-manifold, is $\pi_1(M)$ virtually LO?

Geometries: recall the eight 3-dimensional geometries of Thurston.
Six are SFS geometries, also there's *Sol* and $\mathbb{H}^3$.
Thm: Let M be a closed, connected *Sol* manifold. Then
(1) $\pi_1(M)$ is left-orderable if and only if M is either non-orientable, or orientable and a torus bundle over the circle.
(2) $\pi_1(M)$ is bi-orderable if and only if M is a torus bundle over the circle whose monodromy in $GL_2(\mathbb{Z})$ has at least one positive eigenvalue.
(3) $\pi_1(M)$ is virtually bi-orderable.

Hyperbolic 3-manifolds: many have π_1 LO.
Example: (R. Roberts, J. Shareshian, M. Stein) There exist compact hyperbolic 3-manifolds $M^3_{p,q,m}$ with $\pi_1(M)$ not LO.
Here $m < 0$, $p > q > 1$ are relatively prime and $M^3_{p,q,m}$ is a certain (p, q) Dehn filling of a fibre bundle over S^1 with fibre a punctured torus and pseudo-Anosov monodromy represented by the matrix $\begin{pmatrix} m & 1 \\ -1 & 0 \end{pmatrix}$.
$\pi(M^3_{p,q,m})$ has generators t, a, b and relations:

$$t^{-1}at = aba^{m-1}$$

$$t^{-1}bt = a^{-1}$$

$$t^{-p} = (aba^{-1}b^{-1})^q$$

Orderability and geometry seem to be independent:
Thm: Each of the eight geometries models a 3-manifold with a LO π_1 and also one whose π_1 is not LO.
Question: If M is a hyperbolic 3-manifold, is $\pi_1(M)$ virtually LO?

Knot groups. If $K \subset S^3$ is a knot, $\pi_1(S^3 \setminus K)$ is the group of K.
Thm: (Howie-Short) Every (classical) knot group is LO, in fact locally indicable. The same holds for any link group.
Prop: The group of a nontrivial torus knot is not bi-orderable.
Cor: Cabled knots and satellites of torus knots have non-biorderable groups. This includes the knots arising from singularities of complex curves in $\mathbb{C}^2$.

Fibred knots: $S^3 \setminus K$ fibres over S^1 with fibre a punctured surface F.
Torus knots are all fibred.
Thm: (Perron-R.) Suppose K is a fibred knot whose Alexander polynomial $\Delta_K(t)$ has all roots real and positive. Then its group is bi-orderable.

Examples: The figure-eight knot 4_1 has

$$\Delta_K(t) = t^2 - 3t + 1.$$

Roots are $(3 \pm \sqrt{5})/2$.
Of 121 prime fibred prime knots of 10 or fewer crossings, only two others fulfill the condition: 8_{12} and 10_{137}.

Question: What are the orderability properties of higher-dimensional knot groups?
These can have torsion. In some cases one can use the following:
Thm: (Howie) If G is a one-relator torsion-free group, then G is locally-indicable (thus LO).
