

# Ordered Groups and Topology

Dale Rolfsen

University of British Columbia

Luminy, June 2001

---

Outline:

Lecture 1: Basics of ordered groups

Lecture 2: Topology and orderings

- $\pi_1$ , applications
- braid groups
- mapping class groups
- hyperplane arrangements
- surface braid groups

Lecture 3: Ordering 3-manifold groups

- Seifert fibrations
  - geometries
  - knot groups
  - foliations and  $\mathbb{R}$ -actions
- 

## Ordered groups

Definitions: Let  $G$  be a group, and  $<$  a strict total ordering of its elements. Then  $(G, <)$  is a *left-ordered* group (LO) if

$$g < h \Rightarrow fg < fh.$$

If the ordering is also right-invariant, we say that  $(G, <)$  is an *ordered* group (O), or for emphasis *bi-ordered*.

---

**Prop:**  $G$  is left-orderable if and only if there exists a subset  $\mathcal{P} \subset G$  such that:

$$\begin{aligned} \mathcal{P} \cdot \mathcal{P} &\subset \mathcal{P} \text{ (subsemigroup)} \\ G \setminus \{e\} &= \mathcal{P} \amalg \mathcal{P}^{-1} \end{aligned}$$

Given  $\mathcal{P}$  define  $<$  by:  $g < h$  iff  $g^{-1}h \in \mathcal{P}$ .

Given  $<$  take  $\mathcal{P} = \{g \in G : 1 < g\}$ .

---

The ordering is a bi-ordering iff also

$$g^{-1}\mathcal{P}g \subset \mathcal{P}, \forall g \in G.$$

Note: a group is right-orderable iff it is left-orderable.

---

Examples:

1.  $\mathbb{R}$ , the additive reals with the usual ordering.
2.  $\mathbb{R}^2$  with the lexicographical ordering.

3.  $\mathbb{Z}^2$  has uncountably many different orderings, one for each line through  $(0, 0)$  of irrational slope.

4.  $\mathbb{R} \setminus \{0\}$  under multiplication is not orderable, or even left-orderable. It has an element  $(-1)$  of order two.

We will see that there are surprisingly many nonabelian LO and O groups.

**Prop:** If  $G$  is left-orderable, then  $G$  is torsion-free.

**Prop:** If  $G$  is bi-orderable, then

- $G$  has no generalized torsion (product of conjugates of a nontrivial element being trivial).

- $G$  has unique roots:  $g^n = h^n \Rightarrow g = h$

- if  $[g^n, h] = 1$  in  $G$  then  $[g, h] = 1$

The class of LO groups is closed under: subgroups, direct products, free products, directed unions, extensions.

The class of O groups is closed under: subgroups, direct products, free products, directed unions, but not necessarily under extensions.

Both properties O and LO are local: a group has the property if and only if every finitely-generated subgroup has it.

**Prop:** (Extensions) Given an exact sequence

$$1 \longrightarrow F \xrightarrow{i} G \xrightarrow{p} H \longrightarrow 1$$

If  $F$  and  $H$  are left-orderable, then so is  $G$ , using the positive cone:

$$\mathcal{P}_G := i(\mathcal{P}_F) \cup p^{-1}(\mathcal{P}_H).$$

If  $F$  and  $H$  are bi-ordered, then this defines a bi-ordering of  $G$  if and only if

$$g^{-1}i(\mathcal{P}_F)g \subset i(\mathcal{P}_F), \quad \forall g \in G$$

Example: The Klein Bottle group:

$$\langle x, y : x^{-1}yx = y^{-1} \rangle$$

is LO, being an extension of  $\mathbb{Z}$  by  $\mathbb{Z}$ .

However, it is not bi-orderable:

$1 < y$  iff  $y^{-1} < 1$  holds in any LO-group.

However, in an O-group  $1 < y$  iff  $1 < x^{-1}yx$ . This would lead to a contradiction.

Warning:

Note that  $x < y$  and  $z < w$  imply  $xz < yw$  in an O-group, but not in an LO-group.

Example: in the Klein bottle group,  $y$  and  $x^2$  commute. So, although  $yx \neq xy$  we have

$$(yx)^2 = yx^2x^{-1}yx = x^2yy^{-1} = x^2.$$

So this group does not have unique roots.

---

**Exercise:** A left-ordered group  $(G, <)$  is bi-ordered iff

$$x < y \Leftrightarrow y^{-1} < x^{-1} \quad \forall x, y \in G.$$

**Thm:** (Rhemtulla) Suppose  $G$  is left-orderable. Then  $G$  is abelian iff every left-ordering is a bi-ordering.

---

**Def:** An ordering  $<$  of  $G$  is *Archimedean* if whenever  $1 < x < y$ , there exists a positive integer  $n$  such that  $y < x^n$ .

**Hölder's thm** (1902): Suppose  $(G, <)$  is an O-group which is Archimedean. Then  $G$  is isomorphic with a subgroup of the additive real numbers (and  $<$  corresponds to the natural ordering of  $\mathbb{R}$ ). In particular,  $G$  is abelian.

---

**Thm** (Conrad, 1959): If  $(G, <)$  is LO and Archimedean, then the ordering is actually a bi-ordering, so the conclusions of Hölder's theorem apply.

---

Why is orderability useful?

Group rings: For any group  $G$ , let  $\mathbb{Z}G$  denote the group ring of formal linear combinations  $n_1g_1 + \cdots + n_kg_k$ .

**Thm:** If  $G$  is LO, then  $\mathbb{Z}G$  has no zero divisors.

This is conjectured to be true for torsion-free groups.

**Thm:**(Malcev, Neumann) If  $G$  is an O-group, then  $\mathbb{Z}G$  embeds in a division ring.

**Thm:**(LaGrange, Rhemtulla) If  $G$  is LO and  $H$  is any group, then  $\mathbb{Z}G \cong \mathbb{Z}H$  implies  $G \cong H$ .

---

Group actions and orderability:

Say the group  $G$  acts on the set  $X$  via  $x \mapsto gx$  if  $(gh)x = g(hx)$ .  $G$  acts effectively if only  $1 \in G$  acts trivially on  $X$ .

**Thm:** A group  $G$  is LO if and only if there exists a totally ordered set  $X$  upon which  $G$  acts effectively by order-preserving bijections.

Example: The group  $Homeo_+(\mathbb{R})$  is LO.

**Thm:** A countable group  $G$  is LO if and only if it embeds in  $Homeo_+(\mathbb{R})$ .

If  $G$  acts on  $\mathbb{R}$  without fixed points, it is bi-orderable.

---

Another useful characterization:

**Thm:** (Burns-Hale) A group  $G$  is LO if and only if for every nontrivial finitely-generated subgroup  $H \subset G$ , there exists a left-ordered group  $L$  and a nontrivial homomorphism

$$H \rightarrow L.$$

**Def:** A group  $G$  is *locally indicable* if for every f.g. subgroup  $1 \neq H \subset G$ ,  $\exists$  nontrivial  $H \rightarrow \mathbb{Z}$ .

**Cor:** Locally indicable  $\Rightarrow$  left-orderable.

**Prop:** Bi-orderable  $\Rightarrow$  locally indicable.

---

The Klein bottle group is LI but not O.

Bergman:  $\langle x, y, z : x^2 = y^3 = z^7 = xyz \rangle$  is LO, but not LI. It's  $\pi_1$  of a homology 3-sphere.

Similarly,  $\widetilde{SL}_2(\mathbb{R})$  is LO but not LI.

---

**Thm:** Free groups are bi-orderable.

**Cor:** Excepting  $\mathbb{P}^2$ , surface groups are LO.

Proof: If  $M^2$  is noncompact or  $\partial M$  is nonempty,  $\pi_1(M)$  is free.

---

Proof that free groups are bi-orderable:

Let  $F = \langle x_1, x_2 \rangle$  denote the free group of rank two. We wish to construct an explicit bi-ordering on  $F$ .

The Magnus expansion: Consider the ring

$$\Lambda = \mathbb{Z}[[X_1, X_2]]$$

of formal power series in the non-commuting variables  $X_1$  and  $X_2$ . The Magnus map is the (multiplicative) homomorphism

$$\mu : F \rightarrow \Lambda$$

defined by:

$$\begin{aligned} x_i &\mapsto 1 + X_i \\ x_i^{-1} &\mapsto 1 - X_i + X_i^2 - X_i^3 + \dots \end{aligned}$$

---

**Lemma:**  $\mu$  is injective; its image lies in the group of units of  $\Lambda$  of the form  $1 + O(1)$ .

Define an ordering  $<$  on  $\Lambda$  by the following recipe: Write the elements of  $\Lambda$  in a standard form, with lower degree terms preceding higher degree terms, and within a given degree list the terms in sequence according to (say) the lexicographic ordering of the variables' subscripts.

Compare two elements of  $\Lambda$  by writing them both in standard form and ordering them according to the natural ordering of the coefficients at the *first* term at which they differ.

It defines an ordering  $<$  on  $\Lambda$  which is invariant under addition. Moreover, restricted to the group of units  $\{1 + O(1)\}$ , it is invariant under multiplication on both sides.

---

Since  $F$  is embedded in  $\{1 + O(1)\}$ , this defines a bi-invariant ordering for the free group.

Example:

$$\begin{aligned} \mu(x_1^{-1}x_2x_1) &= \\ &= (1 - X_1 + X_1^2 - \dots)(1 + X_2)(1 + X_1) \\ &= 1 + X_2 - X_1X_2 + X_2X_1 + O(3) \\ \mu(x_2) &= 1 + X_2 + 0X_1X_2 + \dots \end{aligned}$$

Therefore  $1 < x_1^{-1}x_2x_1 < x_2$ .

---

## Lecture 2: Topology and orderable groups

---

**Thm:** (Farrell) Suppose  $X$  is a paracompact Hausdorff space. Then  $\pi_1(X)$  is LO if and only if there is an embedding of the universal covering  $h: \tilde{X} \rightarrow X \times \mathbb{R}$  such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{h} & X \times \mathbb{R} \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

---

**Thm:** (Smythe) Consider a knot

$$K \subset M^2 \times \mathbb{R},$$

and a regular projection  $p(K)$  in  $M$ . Suppose  $K$  is homotopically trivial. Then there is a choice of over-under at the crossings of  $p(K)$  which creates a knot  $K'$  in  $M^2 \times \mathbb{R}$  with  $p(K') = p(K)$ , but  $K'$  is unknotted in  $M^2 \times \mathbb{R}$  (it bounds a disk).

---

## Ordering braid groups

$B_n$  has generators  $\sigma_1, \dots, \sigma_{n-1}$  and relations

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad |i - j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \end{aligned}$$

**Thm:** (Dehornoy)  $B_n$  is left-orderable.

We will outline three different proofs.

**Note:** for  $n > 2$ ,  $B_n$  cannot be bi-ordered.

Take  $x = \sigma_1 \sigma_2 \sigma_1$  and  $y = \sigma_1 \sigma_2^{-1}$ , and observe

$$x^{-1}yx = y^{-1}.$$

---

Proof 1 (Dehornoy): Define the positive cone  $\mathcal{P} \subset B_n$  by  $\beta \in \mathcal{P}$  iff there exists an expression

$$\beta = w_1 \sigma_i w_2 \sigma_i \cdots \sigma_i w_k$$

where each  $w_j \in \langle \sigma_{i+1} \cdots \sigma_{n-1} \rangle$ .

In other words the generator with the lowest subscript has only positive exponents.

It is easy to verify that  $\mathcal{P} \cdot \mathcal{P} \subset \mathcal{P}$ ;

To show that  $B_n \setminus \{1\} = \mathcal{P} \amalg \mathcal{P}^{-1}$ , it is not too hard to show that  $\mathcal{P}$  and  $\mathcal{P}^{-1}$  are disjoint.

The difficult part is to show that every nontrivial braid, or its inverse, can be expressed in the above form:  $B_n \setminus \{1\} = \mathcal{P} \cup \mathcal{P}^{-1}$

---

Proof 2 (Fenn, Green, Rolfsen, Rourke, Wiest): This uses the alternative view of  $B_n$  as the mapping class group of the disk with  $n$  punctures:

$$B_n \cong \mathcal{M}(D^2, n)$$

$D^2$  is pictured as a round disk in the complex plane enclosing the integers  $1, \dots, n$ . Given a mapping of the disk to itself, consider the image of the real line  $\mathbb{R}$ .

After an isotopy of the mapping  $f$ , one may assume that this image is “taut” in that the number of components of  $f(\mathbb{R}) \cap \mathbb{R}$  is minimized.

Then a braid  $\beta = [f]$  is considered positive iff the first departure of  $f(\mathbb{R})$  from  $\mathbb{R}$  goes into the upper half-plane.

Remarkably, this ordering is precisely the same as Dehornoy’s.

---

Proof 3 (Thurston, *a la* Short and Wiest): Again we consider  $B_n \cong \mathcal{M}(D^2, n)$ . The punctured disk  $D_n^2$  has universal covering  $\widetilde{D}_n$  embeddable in the hyperbolic plane  $\mathbb{H}$ . Choose a fixed basepoint  $*$  in one of the lifts of the boundary of  $D^2$ . Given  $\beta = [f]$ , with  $f : D_n^2 \rightarrow D_n^2$  let  $\tilde{f} : \widetilde{D}_n \rightarrow \widetilde{D}_n$  be the unique lifting of  $f$  which fixes  $*$ .

Note that  $\partial\widetilde{D}_n \cong S^1$ . Every such lift fixes an interval of  $S^1$  containing  $*$ , so we may consider  $\tilde{f}$  as an orientation-preserving mapping  $\mathbb{R} \rightarrow \mathbb{R}$ .

This action of  $B_n$  on  $\mathbb{R}$  shows  $B_n$  is LO.

---

Advantages of proof 3:

This approach defines infinitely many left-orderings of  $B_n$ , including Dehornoy’s. Some are order-dense, others (like D’s) are discrete.

Also it easily generalizes to other mapping class groups.

**Thm:** If  $M^2$  is a compact surface with nonempty boundary (with or without punctures), then  $\mathcal{M}(M^2)$  is LO.

---

The *pure* braid groups  $P_n$ :

**Thm:**(Kim-R.-Zhu)  $P_n$  is bi-orderable.

Proof: According to Artin’s combing technique,  $P_n$  is a semidirect product of free groups, which are bi-orderable. However, since bi-orderability is not necessarily preserved under semidirect products we need to exercise some care. We will proceed by induction: clearly  $P_1 = \{1\}$  and  $P_2 \cong \mathbb{Z}$  are biorderable. Suppose  $P_n$  is biordered. There is a standard inclusion

$$P_n \xhookrightarrow{i} P_{n+1}$$

and also a homomorphism

$$P_{n+1} \xrightarrow{r} P_n$$

which “forgets the last string” is a retraction of groups:  $r \circ i = id$ .

The kernel  $K = \ker(r)$  can be regarded as all  $(n+1)$ -string braids in which the first  $n$  strings are straight; so  $K$  can be also be regarded as the fundamental group of an  $n$ -punctured disk, a free group.

$$1 \rightarrow K \hookrightarrow P_{n+1} \xrightarrow{r} P_n \rightarrow 1$$

is exact.

**Lemma:** There is a bi-ordering on the free group  $K$  so that conjugation by any element of  $P_{n+1}$  is order-preserving.

Key fact: each such automorphism  $K \rightarrow K$  becomes the identity upon abelianization. The Magnus ordering is invariant under all such automorphisms.

This completes the proof that  $P_{n+1}$  is bi-orderable.

Properties: With appropriate choice of generators of  $K$  we have:

- This ordering of  $P_n$  is order-dense ( $n > 2$ ).
- It is compatible with the inclusions:

$$P_n \hookrightarrow P_{n+1},$$

and so bi-orders  $P_\infty$ .

- The semigroup  $P_n^+ = B_n^+ \cap P_n$  of Garside positive pure braids are all positive in the bi-ordering:

$$\beta \in P_n^+ \setminus \{1\} \Rightarrow 1 < \beta.$$

- $P_n^+$  is *well-ordered* by this bi-ordering.

Note:  $B_n^+$  is also well-ordered by the Dehornoy left-ordering of  $B_n$ . However, our ordering of  $P_n$  is very different from the restriction of any known left-ordering of  $B_n$ .

**Question:** Does there exist a bi-ordering  $<$  of  $P_n$ , which extends to a left-invariant ordering of  $B_n$ ?

**Answer: NO!!** Because of the following.

**Prop:** (Rhemtulla - R.) Suppose  $(G, <)$  is a left-ordered group. Suppose  $H \subset G$  is a finite-index subgroup such that  $(H, <)$  is a bi-ordered group. Then  $G$  is locally indicable.

**Prop:** (Gorin - Lin) For  $n > 4$ ,  $B_n$  is not locally-indicable. In fact  $[B_n, B_n]$  is finitely-generated and perfect.

Hyperplane arrangements:

Let  $\{H_\alpha\}$  be a finite family of complex hyperplanes in  $\mathbb{C}^n$ .

**Example:** For each  $i, j$  with  $1 < i < j < n$ , let  $H_{ij}$  be the hyperplane in  $\mathbb{C}^n$  defined by  $z_i = z_j$ . Then  $X = \mathbb{C}^n \setminus \cup H_{ij}$  is the set of distinct ordered  $n$ -tuples of complex numbers, and  $\pi_1(X) \cong P_n$ .

A class of hyperplane arrangements, generalizing this example, are those of “fibre type.” As noted by L. Paris, we have:

**Thm:** If  $\{H_\alpha\}$  is a hyperplane arrangement of fibre type, then  $\pi_1(\mathbb{C}^n \setminus \cup H_\alpha)$  is bi-orderable.

Proof: The group is a semidirect product of free groups, in which the actions on the free groups become the identity upon abelianization. The proof proceeds as in the pure braid groups.

---

Bi-ordering surface groups.

Let  $M^2$  be a connected surface.

If  $M = \mathbb{P}^2$ , then  $\pi_1(M) \cong \mathbb{Z}_2$  is not LO.

If  $M = \mathbb{P}^2 \# \mathbb{P}^2$ , the Klein bottle, then  $\pi_1(M)$  is LO, but not bi-orderable.

**Thm:** Except for the two examples above, every surface group  $\pi_1(M)$  is bi-orderable.

---

Surface braid groups  $B_n(M^2)$  and  $PB_n(M^2)$

**Thm:** (Gonzalez-Meneses) For  $M^2$  orientable,  $PB_n(M^2)$  is bi-orderable.

**Thm:** For  $M^2$  non-orientable,  $M \neq \mathbb{P}^2$ ,  $PB_n(M^2)$  is left-orderable, but not biorderable.

---

## Lecture 3: Three-manifold groups

Joint work with Steve Boyer and Bert Wiest, with help from Bouleau, Perron, Short, Sjerve.

We consider  $\pi_1(M^3)$ , where  $M^3$  is a *compact* 3-dimensional manifold.

$M^3$  may be nonorientable.

$\partial M^3$  may be empty or nonempty.

Everything is assumed PL (or  $C^\infty$ ).

---

**Prop:** Suppose  $G = G_1 * \cdots * G_k$ . Then  $G$  is LO if and only if every factor  $G_i$  is LO. Similarly for “O, virtually LO, or virtually O” replacing LO.

Therefore, we may assume w.l.o.g., that  $M^3$  is prime.

**Def:**  $M^3$  is irreducible if every  $S^2 \subset M^3$  bounds a 3-ball in  $M$ .

The only prime manifolds which are not irreducible are  $S^1 \times S^2$  and  $S^1 \tilde{\times} S^2$ ; both have fundamental group  $\mathbb{Z}$ , which is bi-orderable.

**Def:**  $M^2$  is  $\mathbb{P}^2$ -irreducible if irreducible and  $M$  contains no 2-sided projective plane. (relevant only for nonorientable  $M$ )

---

**Thm:** (Boileau, Howie, Short) Suppose  $M^3$  is compact, connected and  $\mathbb{P}^2$ -irreducible. Then  $\pi_1(M)$  is LO if and only if there exists a LO group  $L$  and a nontrivial homomorphism

$$\pi_1(M) \rightarrow L.$$

---

**Cor:** If  $M^3$  is compact, connected and  $\mathbb{P}^2$ -irreducible, and  $H_1(M)$  is infinite, then  $\pi_1(M)$  is LO.

In fact,  $b_1(M) > 0$  implies  $\pi_1(M)$  is locally indicable.

---

**Prop:** If  $M^3$  is an irreducible homology sphere and there is a nontrivial homomorphism

$$\pi_1(M) \rightarrow PLS_2(\mathbb{R}),$$

then  $\pi_1(M)$  is LO.

**Cor:** If  $M^3$  is a Seifert-fibred homology sphere, and not the Poincaré dodecahedral space, then  $\pi_1(M)$  is LO.

Recall that  $M^3$  is a Seifert-fibred space (SFS) if it is foliated by circles.

**Thm:** If  $M^3$  is a compact, connected SFS, then  $\pi_1(M)$  is LO iff

- (1)  $M \cong S^3$ , or
- (2)  $b_1(M) > 0$  and  $M \not\cong \mathbb{P}^2 \times S^1$ , or
- (3)  $M$  is orientable,  $\pi_1(M)$  is infinite, the base orbifold is  $S^2(\alpha_1, \dots, \alpha_k)$  and  $M$  admits a horizontal codimension 1 foliation.

The SFS with horizontal codimension 1 foliations are characterized by Eisenbud-Hirsch-Neumann, Jankins-Neumann, Naimi.

**Thm:** If  $M^3$  is a compact, connected SFS, then  $\pi_1(M)$  is bi-orderable iff

- (1)  $M \cong S^3$ ,  $S^1 \times S^2$ ,  $S^1 \times S^2$  or a solid Klein bottle, or
- (2)  $M$  is an honest circle bundle over a surface other than  $S^2$ ,  $\mathbb{P}^2$  or  $2\mathbb{P}^2$ .

**Cor:** If  $M$  is any compact SFS,  $\pi_1(M)$  is virtually bi-orderable.

**Question:** If  $M$  is an arbitrary 3-manifold, is  $\pi_1(M)$  virtually bi-orderable?

If so, this would answer affirmatively a conjecture of Waldhausen that  $M$  is virtually Haken.

**Question:** If  $M$  is an arbitrary 3-manifold, is  $\pi_1(M)$  virtually LO?

Geometries: recall the eight 3-dimensional geometries of Thurston.

Six are SFS geometries, also there's  $Sol$  and  $\mathbb{H}^3$ .

**Thm:** Let  $M$  be a closed, connected  $Sol$  manifold. Then

- (1)  $\pi_1(M)$  is left-orderable if and only if  $M$  is either non-orientable, or orientable and a torus bundle over the circle.
- (2)  $\pi_1(M)$  is bi-orderable if and only if  $M$  is a torus bundle over the circle whose monodromy in  $GL_2(\mathbb{Z})$  has at least one positive eigenvalue.
- (3)  $\pi_1(M)$  is virtually bi-orderable.

Hyperbolic 3-manifolds: many have  $\pi_1$  LO.

**Example:** (R. Roberts, J. Shareshian, M. Stein) There exist compact hyperbolic 3-manifolds  $M_{p,q,m}^3$  with  $\pi_1(M)$  not LO.

Here  $m < 0$ ,  $p > q > 1$  are relatively prime and  $M_{p,q,m}^3$  is a certain  $(p, q)$  Dehn filling of a fibre bundle over  $S^1$  with fibre a punctured torus and pseudo-Anosov

monodromy represented by the matrix  $\begin{pmatrix} m & 1 \\ -1 & 0 \end{pmatrix}$ .

$\pi_1(M_{p,q,m}^3)$  has generators  $t, a, b$  and relations:

$$t^{-1}at = aba^{m-1}$$

$$t^{-1}bt = a^{-1}$$

$$t^{-p} = (aba^{-1}b^{-1})^q$$

---

Orderability and geometry seem to be independent:

**Thm:** Each of the eight geometries models a 3-manifold with a LO  $\pi_1$  and also one whose  $\pi_1$  is not LO.

**Question:** If  $M$  is a hyperbolic 3-manifold, is  $\pi_1(M)$  virtually LO?

---

Knot groups. If  $K \subset S^3$  is a knot,  $\pi_1(S^3 \setminus K)$  is the group of  $K$ .

**Thm:** (Howie-Short) Every (classical) knot group is LO, in fact locally indicable. The same holds for any link group.

**Prop:** The group of a nontrivial torus knot is not bi-orderable.

**Cor:** Cabled knots and satellites of torus knots have non-biorderable groups. This includes the knots arising from singularities of complex curves in  $\mathbb{C}^2$ .

---

Fibred knots:  $S^3 \setminus K$  fibres over  $S^1$  with fibre a punctured surface  $F$ .

Torus knots are all fibred.

**Thm:** (Perron-R.) Suppose  $K$  is a fibred knot whose Alexander polynomial  $\Delta_K(t)$  has all roots real and positive. Then its group is bi-orderable.

---

**Examples:** The figure-eight knot  $4_1$  has

$$\Delta_K(t) = t^2 - 3t + 1.$$

Roots are  $(3 \pm \sqrt{5})/2$ .

Of 121 prime fibred prime knots of 10 or fewer crossings, only two others fulfill the condition:  $8_{12}$  and  $10_{137}$ .

---

**Question:** What are the orderability properties of higher-dimensional knot groups?

These can have torsion. In some cases one can use the following:

**Thm:** (Howie) If  $G$  is a one-relator torsion-free group, then  $G$  is locally-indicable (thus LO).

---