Orderable groups with applications to topology

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A group G is *left-orderable* (LO) if its elements can be given a (strict) total ordering $<$ which is left invariant:

$$g < h \Rightarrow fg < fh \quad \text{if} \quad f, g, h \in G.$$

Alternative viewpoint:

Let $P = \{g \in G \mid g > 1\}$ be the positive cone in a LO group G. Then:

(1) P is closed under multiplication, and

(2) if $g \in G \setminus \{1\}$ exactly one of g, g^{-1} is in P.

Conversely, if a group G has a subset P satisfying (1) and (2), then G is left-orderable, defining $g < h \Leftrightarrow g^{-1}h \in P$.

Proposition: LO groups are torsion-free, i. e. no elements of finite order.

Reason: If $g \neq 1$, say $g > 1$. Then $g^2 > g$, by left-invariance. So $g^2 > 1$, by transitivity. Inductively, $g^n > 1$ for all $n > 0$, so $g^n \neq 1$. Similarly if $g < 1$.

Proposition: If G is LO and R is a ring without zero divisors, then the group algebra RG has no zero-divisors.

(This is conjectured to be true for torsion-free groups in general.)

Examples of LO groups:

- $(\mathbb{R}, +)$ (2-sided invariant)

(but not the multiplicative group $(\mathbb{R} \setminus \{0\}, \cdot)$)

- Free groups and torsion-free abelian groups. (these have two-sided invariant orders)

- Braid groups (P. Dehornoy) (but NOT 2-sided invariant)

- $Homeo_{+}(\mathbb{R})$ = the group of order preserving homeomorphisms of the real line.

How to left-order $Homeo_+(\mathbb{R})$:

Let $x_1, x_2, \ldots$ be a countable dense subset of $\mathbb{R}$. If f, g are order-preserving homeomorphisms $\mathbb{R} \to \mathbb{R}$ and $f \neq g$, let $n = n(f,g)$ be the first n such that $f(x_n) \neq g(x_n)$. Then define

$$f < g \Leftrightarrow f(x_n) < g(x_n)$$

.

Fact: Every countable LO group is isomorphic with a subgroup of $Homeo_+(\mathbb{R})$

The family of LO groups is closed under:

- subgroups
- direct products (use lexicographic order)
- free products
- quotients by convex normal subgroups
- extensions: if $G \to H$ is a surjective homomorphism with kernel K, and both K and H are LO, then G is LO.

Application to topology: One of the principal connections between topology and group theory is through the fundamental group $\pi_1(X)$.

Surface groups: Let Σ_g denote the connected, compact, orientable surface of genus g. (The torus $S^1 \times S^1$ has genus 1.) Then $\pi_1(\Sigma_g)$ has a presentation with $2g$ generators $a_1, b_1, \ldots, a_g, b_g$ subject to the single relation:

$$[a_1, b_1] \cdot [a_2, b_2] \cdots [a_g, b_g] = 1.$$

Here $[a, b] = aba^{-1}b^{-1}$ denotes the commutator.

The Klein bottle K^2:

This nonorientable surface may be considered as the union of two Möbius bands, attached to each other along their boundaries. Its fundamental group has presentation:

$$\pi_1(K^2) \cong \langle a, b \mid a^2 = b^2 \rangle.$$

Alternatively, one may consider K^2 as the orbit space of $\mathbb{R}^2$ under the action of the (discrete) group $G \subset Isom(\mathbb{R}^2)$ generated by:

$$X : (x, y) \to (1 + x, -y) \quad \text{and} \quad Y : (x, y) \to (x, 1 + y).$$

In other words, $\mathbb{R}^2 \to K^2$ is a covering space, and the fundamental group of K can be identified with G, which has the presentation:

$$\pi_1(K) \cong G \cong \langle X, Y \mid XYX^{-1} = Y^{-1} \rangle$$

One can also verify this isomorphism by the substitutions

$$a = X,\ b = XY^{-1}.$$

Proposition: The fundamental group of K^2 is left-orderable.

Proof: Identify this with the group G of isometries of $\mathbb{R}^2$, as above. If $g \in G$, consider $g(0,0) = (x_0, y_0)$. Define g to be positive if and only if

either $x_0 > 0$ or $x_0 = 0$ and $y_0 > 0$.

More generally, we have:

Theorem: The fundamental group of every surface except $\mathbb{R}P^2$ is left-orderable. Moreover, all (possibly nonorientable and non-compact) surface groups have 2-sided invariant orderings, except for $\mathbb{R}P^2$ and K^2.

Left-orderability is very common among fundamental groups of 3-manifolds, too. For example:

Theorem: (Short - Howie) Suppose M^3 is a connected compact orientable 3-manifold which is irreducible. Then $\pi_1(M^3)$ is left-orderable if and only if it has a homomorphic image which is left-orderable.

Cor: If M^3 is as above, and the abelianization $H_1(M^3)$ of $\pi_1(M^3)$ is infinite, then $\pi_1(M^3)$ is LO.

Cor: If K is a knot in $\mathbb{R}^3$ or S^3, then the fundamental group of its complement is left-orderable. That is, "knot groups" are LO.

An application: an obstruction to the existence of mappings of nonzero degree.

Suppose M and N are closed orientable 3-manifolds. Is there a continuous function $M \to N$ of nonzero degree?

Theorem: If $\pi_1(N)$ is left-orderable, $\pi_1(M)$ is not left-orderable and M is irreducible. Then then the answer is NO!

There are many 3-manifolds whose groups are torsion-free, yet not left-orderable.

Example: The Weeks manifold W^3 is the closed hyperbolic 3-mainifold of minimal volume. Calegari-Dunfield: $\pi_1(W^3)$ is not left-orderable.

A surgery description of the Weeks manifold:

Question: Suppose $G = \pi_1(M)$ is the fundamental group of a compact hyperbolic 3-manifold M (a.k.a. Kleinian group). Does G have a finite index subgroup which is left-orderable?

If one could find an M as above for which the answer is no, then one would have a counterexample to both of the following:

Conjectures of Thurston: (1) Every compact hyperbolic 3-manifold is finitely covered by a manifold which has positive first Betti number.

(2) Every compact hyperbolic 3-manifold is finitely covered by a manifold which fibres over S^1.

An application of orderable groups to foliations of 3-manifolds:

A foliation $\mathfrak{F}$ (of dimension k) of a manifold M^n is a partition of M^n into sets (called "leaves"), so that each point of M has a neighborhood homeomorphic with $\mathbb{R}^n$, so that the leaves meet this neighborhood in sets which correspond to parallel k-hyperplanes in $\mathbb{R}^n$.

A similar definition applies to manifolds with boundary.

Example: A foliation of the Klein bottle.

Note that the family of horizontal lines in $\mathbb{R}^2$ is preserved by the action of the group $G \subset Isom(\mathbb{R}^2)$ described earlier. So under the mapping $\mathbb{R}^2 \to K^2$ it descends to a foliation of K^2 by circles which look locally like parallel lines.

However the image of the x-axis is a circle whose neighboring circles wrap "twice." Similarly for the image of the line $y = 1/2$.

A codimension-one foliation $\mathfrak{F}$ of a manifold M is transversely oriented if there is a continuous choice of normal vector at each point of each leaf.

A codimension-one foliation $\mathfrak{F}$ is said to be $\mathbb{R}$-covered if the pullback foliation $\tilde{\mathfrak{F}}$ of the universal cover $\tilde{M}$ has space of leaves homeomorphic with $\mathbb{R}$.

Example: The Klein bottle foliation described above is $\mathbb{R}$-covered but not transversely-oriented. On the other hand, the foliation of K defined by the vertical lines $x =$ constant is both $\mathbb{R}$-covered and transversely orientable.

We now turn to the special case of compact orientable 3-manifolds and 2-dimensional foliations.

Theorem: (Lickorish, Zieschang) Every compact orientable 3-manifold has a 2-dimensional foliation.

This is contrast to the situation for 2-manifolds (surfaces) – the only compact surfaces which have codimension-one foliations are the torus and Klein bottle.

Proposition: If an orientable M^3 has a 2-dimensional foliation which is $\mathbb{R}$-covered and transversely oriented, then $\pi_1(M)$ is left-orderable.

Reason: Let $\mathfrak{F}$ be such a foliation of M. Consider the universal cover $\tilde{M}$, which has a "lifted" foliation $\tilde{\mathfrak{F}}$.

$\pi_1(M)$ acts on $\tilde{M}$ as the covering translations, and also acts on the set $\tilde{\mathfrak{F}}$. And therefore $\pi_1(M)$ acts on the space of leaves of $\tilde{\mathfrak{F}}$ which is homeomorphic to $\mathbb{R}$. Also, the action respects the transverse orientation, which also lifts to $\tilde{\mathfrak{F}}$.

Thus we have a homomoprhism $\pi_1(M) \to Homeo_+(\mathbb{R})$.

The kernel may be nontrivial, but it acts freely on each leaf, which is an orientable surface. Hence the kernel is left-orderable, and by the extension property, $\pi_1(M)$ is left-orderable.

A construction:

Let $\tilde{Q} = \{(x, y, z) \in \mathbb{R}^3 \mid -1 \leq z \leq 1\}$. Define $X, Y : \tilde{Q} \to \tilde{Q}$ by:

$$X : (x, y, z) \to (1 + x, -y, -z) \quad \text{and} \quad Y : (x, y, z) \to (x, 1 + y, z).$$

Let $G =$ the group of isometries of $\tilde{Q}$ generated by X and Y.

Define $Q = \tilde{Q}/G$.

Then $\tilde{Q} \to Q$ is a covering space and $\pi_1(Q) \cong G$.

Note that Q is orientable and has boundary $\partial Q \cong S^1 \times S^1$.

Moreover, Q contains a Klein bottle, the image of the plane $z = 0$.

In fact, $\pi_1(Q) \cong \langle a, b \mid a^2 = b^2 \rangle$ is just the Klein bottle group.

Its boundary has $\pi_1(\partial Q) \cong \mathbb{Z} \times \mathbb{Z} =$ the subgroup consisting of words in a, b with total exponent even.

We take as basis for $\pi_1(\partial Q)$ the words

$$m = a^2 \quad \text{and} \quad l = ab.$$

We recall that the orientation-preserving homeomorphisms of the torus $S^1 \times S^1$ are parametrized by $SL(2, \mathbb{Z})$.

Now take two copies Q_1, Q_2 of Q and glue their boundaries together by a homeomorphism whose matrix in the m, l bases is

$$\varphi = \begin{pmatrix} p & r \\ q & s \end{pmatrix}.$$

This produces the closed orientable manifold

$$M_\varphi = Q_1 \cup_\varphi Q_2$$

Proposition: Suppose p, q are non-negative and r, s are non-positive integers (or vice-versa). Then $\pi_1(M_\varphi)$ is not left-orderable.

Proof: The group $\pi_1(M_\varphi)$ has presentation with generators a, b, x, y and relations:

$$a^2 = b^2,\ x^2 = y^2,\ a^{2p}(ab)^q = x^2,\ a^{2r}(ab)^s = xy$$

Assume $\pi_1(M_\varphi)$ is left-ordered. Then the first relation implies that a and b have the same "sign" — that is, either both are greater than the identity or less than the identity. The same is true of x and y. If a, b have the same sign as x, y we contradict the fourth equation. If a, b have the opposite sign as x (and y) we contradict the third. QED

Remark: These examples all have nontrivial, finite first homology (recalling H_1 is the abelianization of π_1):

$$|H_1(M)| = 16|p + q - r - s|.$$

Therefore this construction gives infinitely many distinct examples of 3-manifolds with non-LO fundamental groups.

Their fundamental groups are, however, torsion-free, as they are amalgamated free products of torsion-free groups.

Note that, by construction, they are foliated by (two) Klein bottles and (infinitely many) tori. The universal cover of M_φ is Euclidean space, foliated by planes, so it is an $\mathbb{R}$-covered foliation. However, the foliation is not transversely oriented.

Proposition: The manifolds M_φ constructed above cannot be given transversely oriented $\mathbb{R}$-covered foliations.

The same is true of the Weeks manifold, as well as examples constructed by Roberts, Shareshian and Stein.

Summary: We have discussed three applications of group orderability to 3-dimensional topology:

- an obstruction to the existence of maps $M^3 \to N^3$ of finite nonzero degree.

- an approach to the Thurston conjectures.

- an obstruction to the existence of very nice foliations for M^3.

MUCHAS GRACIAS!