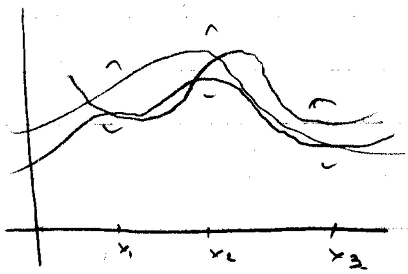
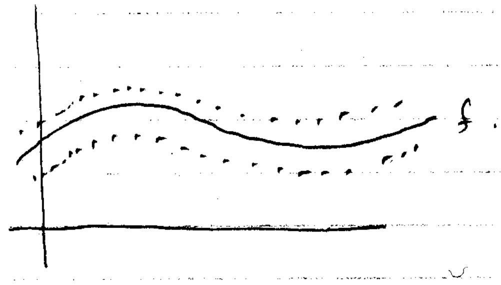
Topologies on Function Spaces

Recall we defined:

Given a point in a set X and an open set U of Y let
$$S(x,U) = \{ f \in Y^X : f(x) \in U \},$$
The sets $S(x,U)$ form a subbasis for the topology of pointwise convergence



set of fns



Say: in uniform convergence top. a limit preserves continuity
not in pointwise cvgence.

We'll find something in between.

Topology of Cpt Convergence

Let (Y,d) be a metric space, X a top. space. Given $f \in Y^X$ and C cpt in X, define
$$B_C(f,\epsilon) = \{ g \in Y^X : \text{lub } \{d(f(x), g(x))\} < \epsilon \ \forall x \in C \}$$

The sets B_C form a basis for a topology on Y^X called the topology of compact convergence

Fact:

A sequence $f_n : X \to Y$ of fcns converges to a fcn f in this topology iff $f_n|C$ converges to $f|C$ uniformly (in the usual sense)

②

Theorem

Let X be compactly generated and let (Y,d) be a metric space. Then $\mathcal{C}(X,Y)$ is closed in Y^X in the top. of cpt cvge.

A space X is said to be compactly generated if

A is open in X iff $A \cap C$ is open for each cpt C in X.

Examples.

pf: of Theorem

Let $f \in Y^X$ be a limit point of $\mathcal{C}(X,Y)$. (We will to show $f \in \mathcal{C}(X,Y)$)

First we show that $f|C$ is continuous for each compact C in X. For each n, $B_C(f, \frac{1}{n})$ contains some $f_n \in \mathcal{C}(X,Y)$. The sequence $f_n|C : C \to Y$ converges uniformly to the fcn $f|C$. Thus by the uniform limit theorem $f|C$ is continuous.

Now we show f is continuous. Let V be an open subset in Y. Then $f^{-1}(V) \cap C = (f|C)^{-1}(V)$

If C is compact, then $f|C$ is continuous so $(f|C)^{-1}(V)$ is open. As X is cpt'ly generated we are done.

Compact-Open Topology

Let X, Y be top. spaces. If C is cpt in X, U open in Y define

$$S(C,U) = \{ f \mid f \in \mathcal{C}(X,Y),\ f(C) \subset U \}$$

The sets $S(C,U)$ form a subbasis for a top. on $C(X,Y)$ called the compact open topology.

Theorem

Let X be a space, (Y,d) be a metric space. For $C(X,Y)$ the cpt open-topology and the topology of cpt convergence coincide.

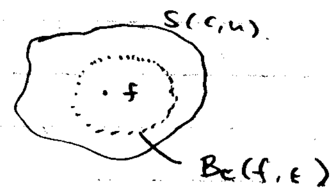
Pf:

Step 1: If A is cpt and V open w/ $A \subset V$, there is an ϵ-neighbourhood of A contained in V.

ie $U(A,\epsilon) = \bigcup_{a \in A} B_d(a,\epsilon) \subset V$ for some $\epsilon > 0$

Step 2: cpt $\overset{\text{dist}}{\text{convergence}} \supset$ cpt open

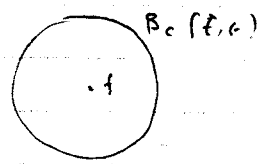
Let $S(C,U)$ be a subbasis element for compact-open top. on $C(X,Y)$ and let $f \in S(C,U)$.



- f is continuous so $f(C) \subset U$ is cpt.
- choose an ϵ-neighbourhood of $f(C)$ in U.

$B_C(f,\epsilon)$ $B(C,U)$

Step 3: cpt-open $\supset$ $\overset{\text{dist.}}{\text{cpt-cvg.}}$



- Consider a basis element $B_C(f,\epsilon)$
- Each $x \in X$ has a neighbourhood V_x st

~~$f(V_x)$ $\subset U_x$~~

$f(\overline{V}_x)$ lies in an open U_x of diameter $< \epsilon$.

Cover C by finitely many V_x, say ~~$\#$~~ for $x = x_1, x_2, \ldots, x_n$

Let $C_x = \overline{V}_x \cap C$. Then $\overline{C}_x$ is compact.

The basis ~~element~~

$S(C_{x_1}, U_{x_1}) \cap S(C_{x_2}, U_{x_2}) \ldots \cap S(C_{x_n}, U_{x_n})$

contains f and lies in $B_C(f,\epsilon)$