The Mandelbrot Set

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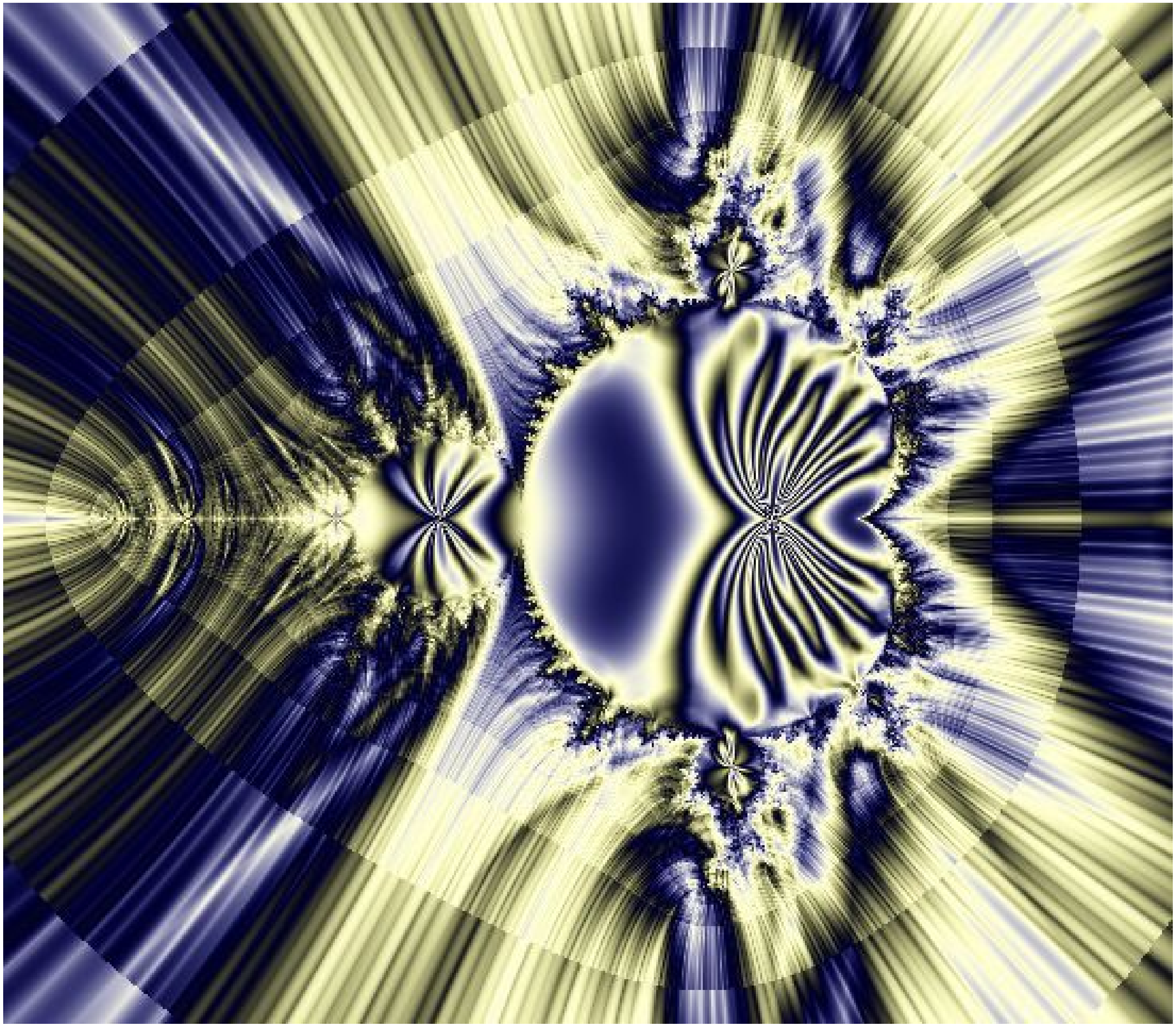
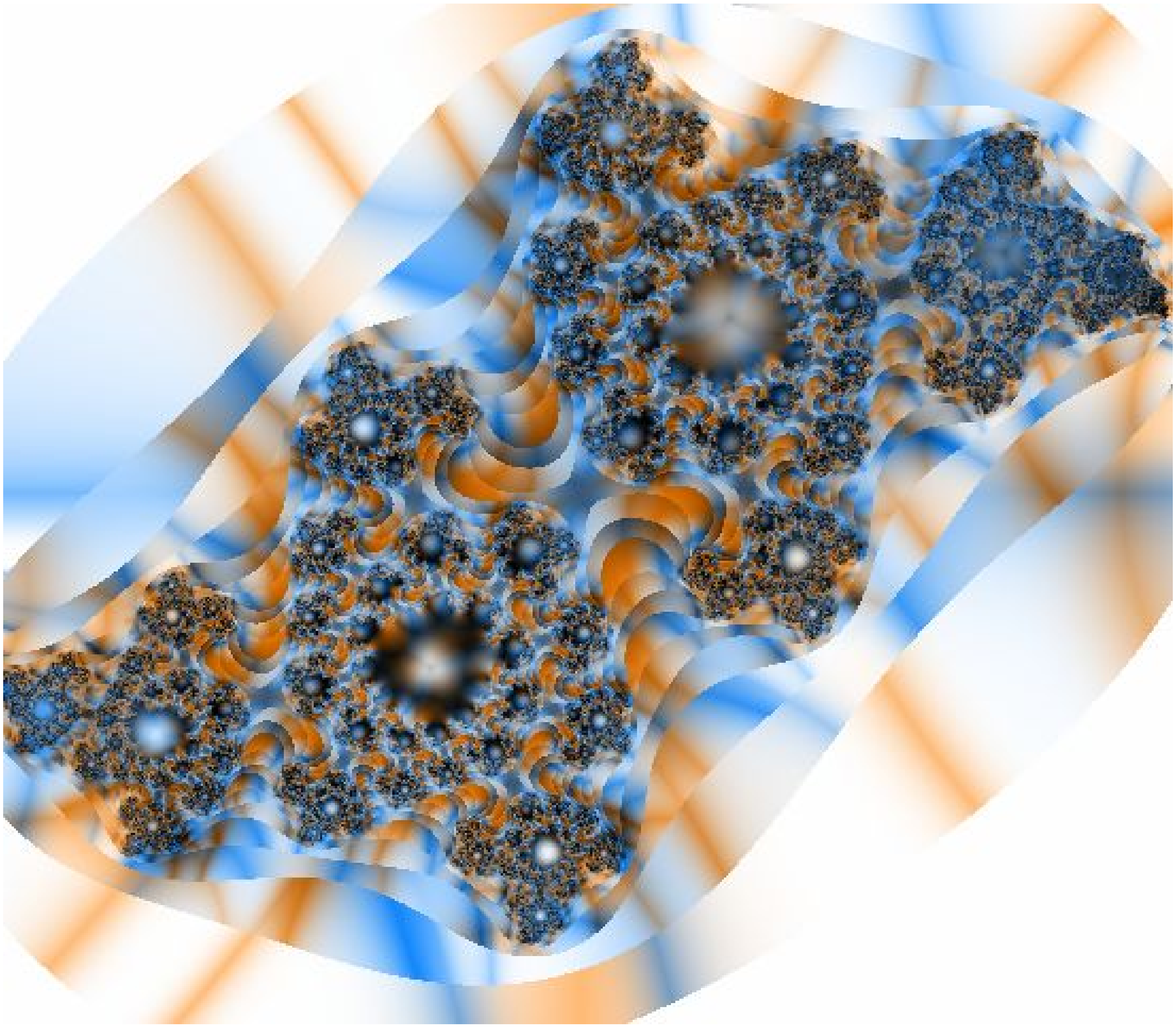
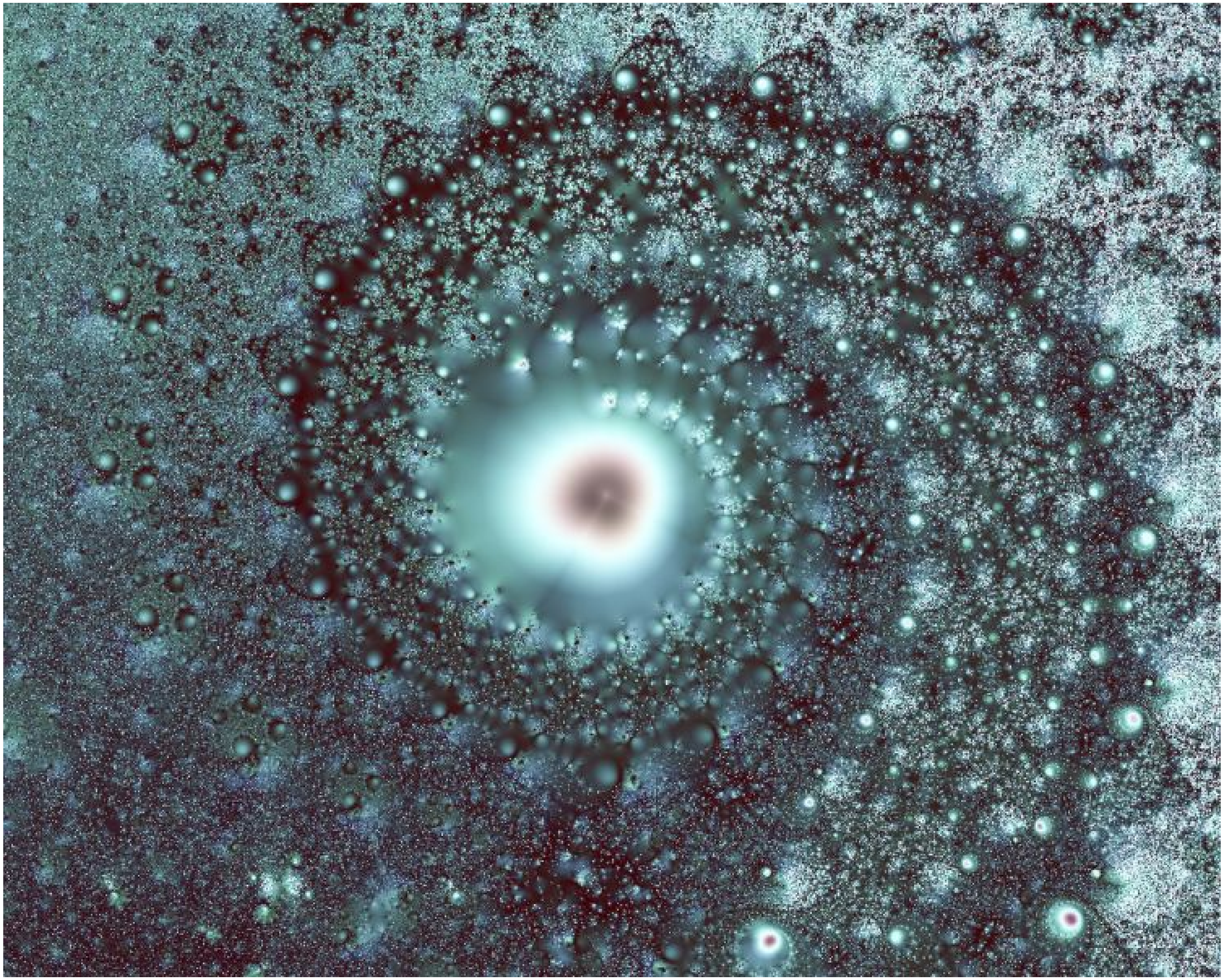
The Mandelbrot Set and other Fractals are Cool







But What are They?

To understand Fractals, we must first understand some things about iterated polynomials on $\mathbb{C}$.

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. Then nth iterate of f is

$$f^n(z) = \underbrace{f(f(f(\cdots f(z)\cdots)))}_{n \text{ times}}$$

Note: The n here is not an exponent.

The orbit launched from $z_0 \in \mathbb{C}$ is the sequence

$$\mathcal{O}_f(z_0) = \{f^n(z_0)\}.$$

The Julia Set of a Polynomial $f(z)$

The Basin of Attraction for ∞ is the set

$$A_f(\infty) = \{z \in \mathbb{C} \mid \mathcal{O}_f(z) \to \infty\}.$$

The Julia Set of $f(z)$ is the the boundary of $A_f(\infty)$,

$$\mathcal{J}(f) = \delta A_f(\infty).$$

Example: Take $f(z) = z^m$. Then $\mathcal{J}(f)$ is the unit circle.

The Mandelbrot Set

For the Mandelbrot set, we consider polynomials of the form

$$p_c(z) = z^2 + c,$$

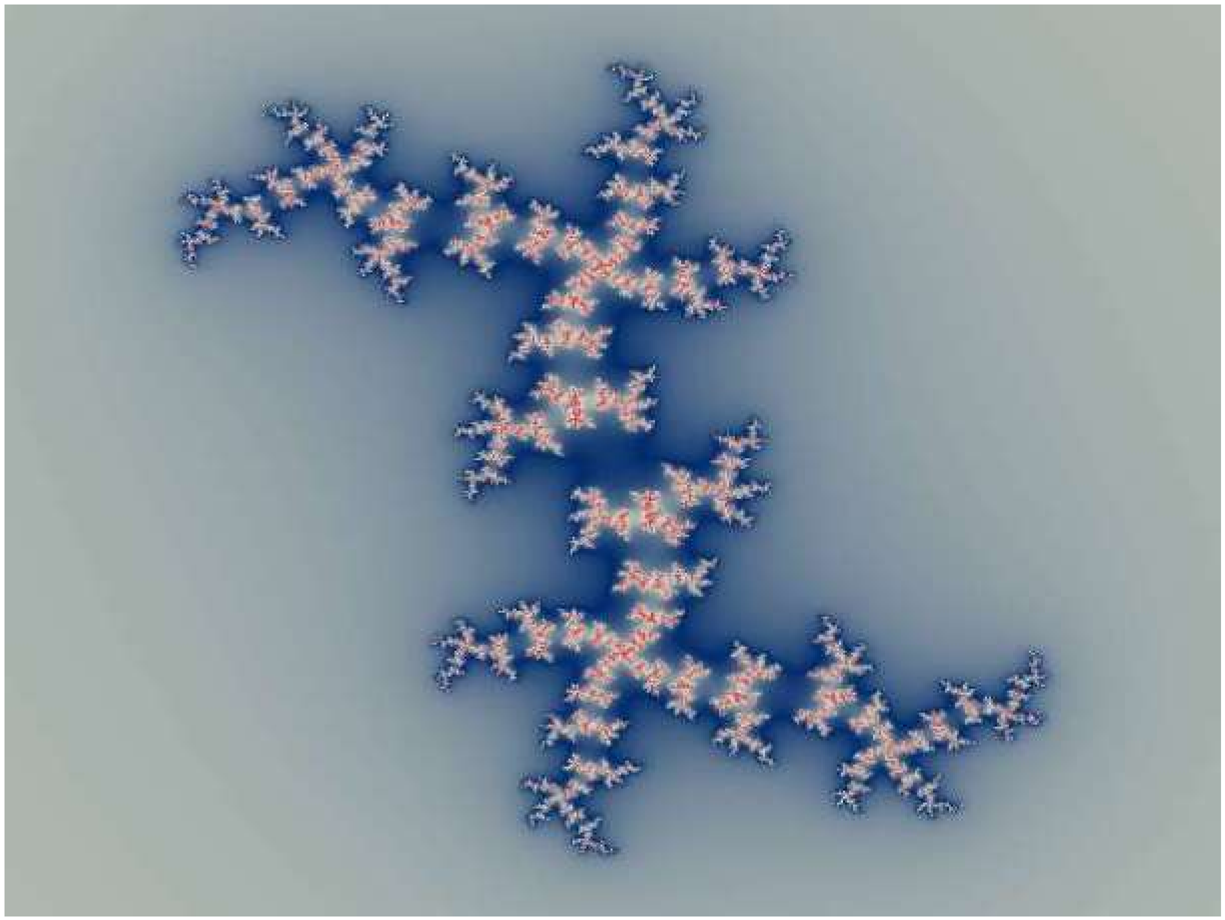
with parameter $c \in \mathbb{C}$. We'll write $\mathcal{O}_{p_c}(z) = \mathcal{O}_c(z)$ for the orbits of p_c.

The Mandelbrot Set is

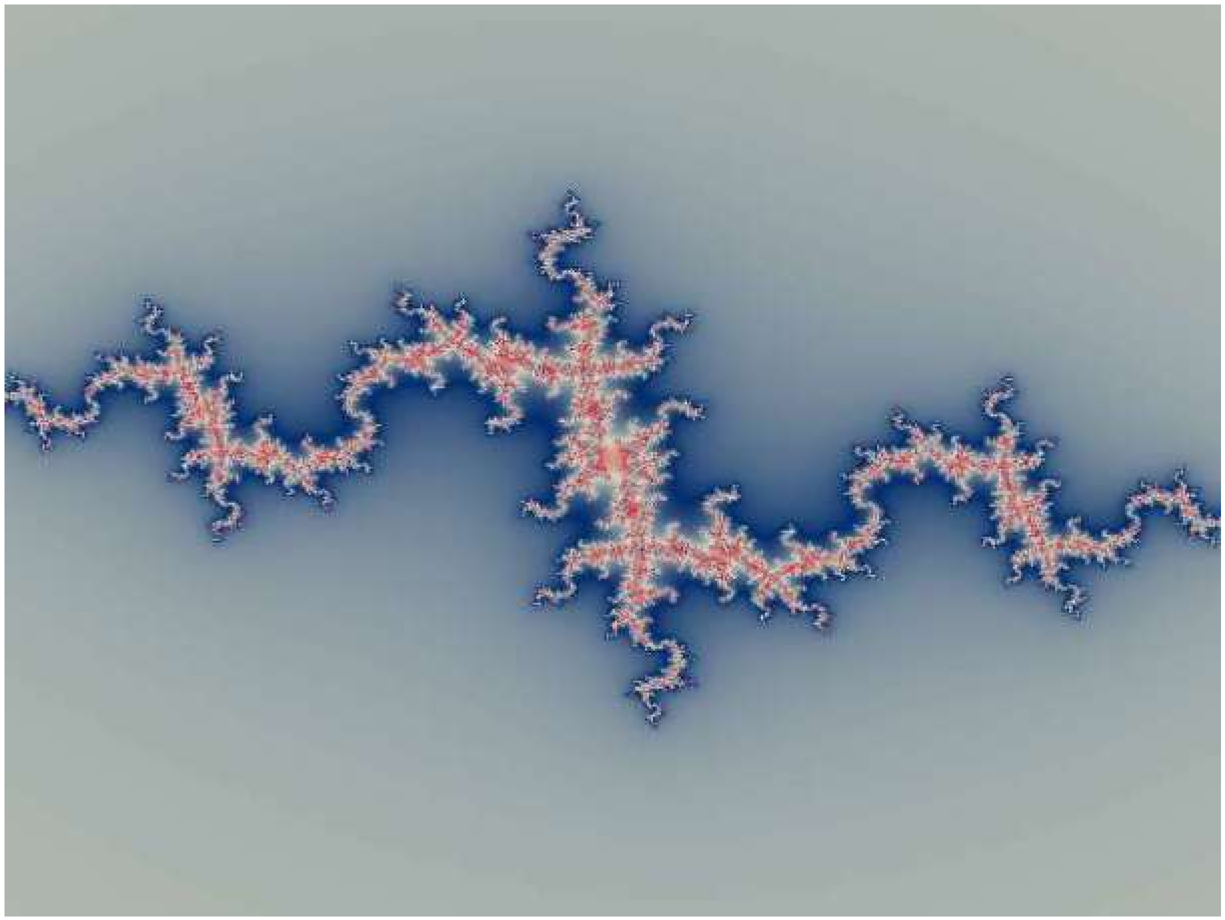
$$\mathcal{M} = \{c \in \mathbb{C} \mid \mathcal{J}(p_c) \text{ is connected}\}.$$

This definition appears simple enough, but Julia Sets are not simple objects.

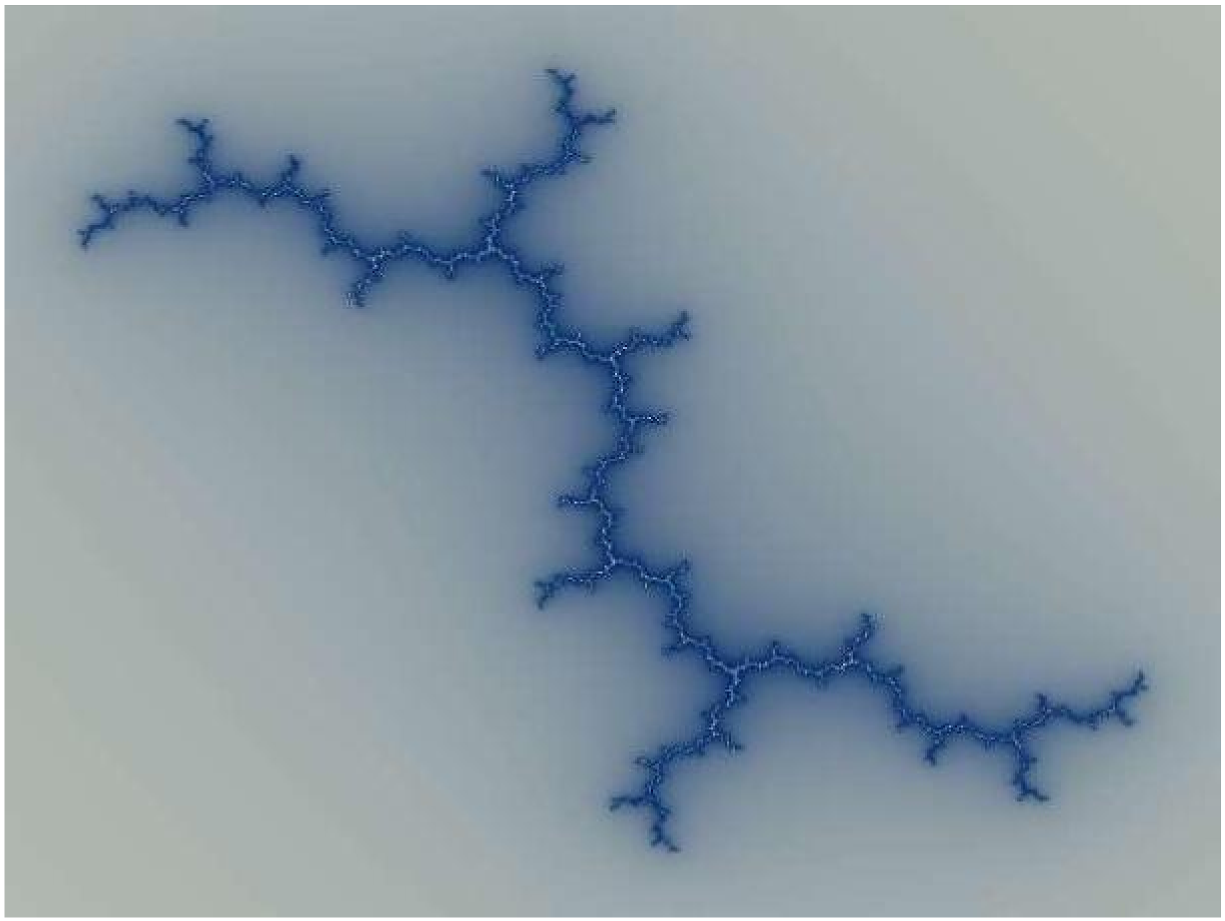
$\mathcal{J}(p_c)$ for $c = \frac{3+6i}{10}$



$\mathcal{J}(p_c)$ for $c = -1 + \frac{3i}{10}$



$\mathcal{J}(p_c)$ for $c = i$



$\mathcal{J}(p_c)$ Connectedness Characterization

Luckily, there is a nice characterization of $\mathcal{J}(p_c)$ being connected.

Theorem. The Julia Set, $\mathcal{J}(p_c)$, is connected if and only if there is an $R \in \mathbb{R}$ such that

$$|p_c^n(0)| \leq R \text{ for all } n.$$

If the Julia set is disconnected, it is totally disconnected. ■

This characterization shows that $c \notin \mathcal{M} \Leftrightarrow \mathcal{O}_c(0) \rightarrow \infty$.

A useful Fact

Lemma. Let $|p_c^n(0)| > 2$, and $|p_c^n(0)| \geq |c|$ for some $n \geq 1$. Then $\mathcal{O}_c(0) \to \infty$.

Proof Take n be the smallest such integer. We have that

$$|p_c^{n+1}(0)| = |p_c^n(0)^2 + c| \geq |p_c^n(0)|^2 - |c| \geq (|p_c^n(0)| - 1)|p_c^n(0)|.$$

Now, if $|p_c^{n+k}(0)| \geq (|p_c^n(0)| - 1)^k |p_c^n(0)|$, then

$$|p_c^{n+k+1}(0)| \geq (|p_c^{n+k}(0)| - 1)|p_c^{n+k}(0)| \geq (|p_c^n(0)| - 1)^{k+1}|p_c^n(0)|.$$

Hence, by induction, we have that $|p_c^n(0)| \to \infty$. ■

Characterization of $\mathcal{M}$

Theorem. $c \in \mathcal{M} \Leftrightarrow |p_c^n(0)| \leq 2$ for all $n \geq 1$.

Proof $\Leftarrow$: By the $\mathcal{J}(p_c)$ Connectedness Characterization, $|p_c^n(0)| \leq 2$ for all $n \geq 1 \Rightarrow c \in \mathcal{M}$.

$\Rightarrow$: Say $|p_c^k(0)| > 2$ for some k. If $|c| > 2$, then $|p_c^n(0)| \geq |c| > 2$. If $|c| \leq 2$, $|p_c^n(0)| \geq |c|$.

In either case, we can apply the above Lemma for some n to get that $|p_c^k(0)| \to \infty$. Hence, $c \notin \mathcal{M}$. ■

Corollary. $\mathcal{M} \subset \mathcal{D}_2$, the disc of radius 2.

Proof $|p_c(0)| = |c|$. ■

This bound is sharp. $\mathcal{J}(p_{-2}) = [-2, 2]$ is connected, so $-2 \in \mathcal{M}$.

Compactness

Theorem. $\mathcal{M}$ is compact.

Proof Let M_n be the set of parameter values with $|p_c^n(0)| \leq 2$.

Let $\{c_k\} \subset M_n$ be a sequence in M_n, and $c_k \to c$.
Then $|p_{c_k}^n(0)| \to |p_c^n(0)|$. Since $|p_{c_k}^n(0)| \leq 2$, $|p_c^n(0)| \leq 2$, so for each n, M_n is a closed set.

The Characterization of $\mathcal{M}$ above gives that

$$\mathcal{M} = \bigcap_{n=1}^{\infty} M_n,$$

so $\mathcal{M}$ is also closed.

$\mathcal{M}$ is closed and bounded, so $\mathcal{M}$ is compact. $\blacksquare$

The Mandelbrot Set is Connected?

We outline the Proof given by Douady and Hubbard that $\mathcal{M}$ is a connected subset of $\mathbb{C}$.

Our first stop is a result of Böttcher's that underlies the proof.

Theorem. Let $f(z)$ be a polynomial of degree $n \geq 2$. Then there is an conformal change of coordinates $w = \psi(z)$ such that

$$\psi \circ f \circ \psi^{-1} : w \mapsto w^n$$

on some neighbourhood of ∞. ψ is unique up to multiplication by an $n-1$ root of unity.

We say that f is conformally conjugate to z^n, with conjugacy ψ.

The Böttcher Coordinate for p_c

The above Theorem gives that p_c is conformally conjugate to the map $z \mapsto z^2$ in some neighbourhood, $\mathcal{U}_c$, of ∞ and the conjugacy is unique.

Douady and Hubbard go further and calculate the explicit form of the conjugacy.

Theorem. Let $B_c : \mathcal{U}_c \to \mathbb{C} \setminus \overline{\mathcal{D}}_R$ be the conformal conjugacy associated with the polynomial p_c. Then

$$B_c(z) = \lim \left[p_c^n(z)\right]^{1/2^n},$$

where the root on the RHS is choosen so that $\left[p_c^n(z)\right]^{1/2^n} \sim z$. Moreover, $B_c(z) \sim z$ near ∞.

Analytic Continuation of the Böttcher Coordinate for p_c

It is straightforward to show that the Böttcher coordinate obeys the following formula.

$$B_c\Big[p_c(z)\Big] = \Big[B_c(z)\Big]^2.$$

We can see that B_c obeys the conjugacy relationship with p_c.

Now, if the critical point c is not in $\mathcal{U}_c$, then p_c has an analytic inverse that sends $\mathcal{U}_c$ to the pre-image $p_c^{-1}(\mathcal{U}_c)$, by the Inverse Function Theorem.

The above equation then gives us a way to conformally extend B_c to the bigger domain $p_c^{-1}(\mathcal{U}_c)$.

If c is not in $p_c^{-1}(\mathcal{U}_c)$ either, we can extend to $p_c^{-2}(\mathcal{U}_c)$, and so on, until $c \in p_c^{-k}(\mathcal{U}_c)$.

Analytic Continuation of the Böttcher Coordinate for p_c

These extensions give us the the conformal map

$$B_c : \Omega_c \to \mathbb{C} \setminus \overline{\mathcal{D}}_{R_c},$$

where the new domain Ω_c depends on $J(p_c)$.

If $\mathcal{J}(p_c)$ is connected, $c \notin A(\infty)$, so the extending process outlined for B_c can be repeated infinitely, so $\Omega_c = A(\infty)$.

If $\mathcal{J}(p_c)$ is disconnected, $c \in A(\infty)$, so the extended domain will eventually contain c. The Inverse Function Theorem then fails to provide an inverse.

Though it is not obvious from this discussion, $R_c = 1$ when $\mathcal{J}(p_c)$ is connected, and $R_c > 1$ otherwise.

The Mandelbrot IS Connected.

We now consider the function

$$\Phi : \mathbb{C}^* \setminus \mathcal{M} \to \mathbb{C}^* \setminus \overline{\mathcal{D}}_1, \ \text{ where } c \mapsto B_c(c).$$

Φ is well defined as $c \in \Omega_c \Leftrightarrow \mathcal{J}(p_c)$ disconnected $\Leftrightarrow c \notin \mathcal{M}$, and $B_\infty(\infty) = \infty$.

Φ is also conformal. (not proved here)

Hence, $\mathbb{C}^* \setminus \mathcal{M}$ and $\mathbb{C}^* \setminus \overline{\mathcal{D}}_1$ are conformally equivalent via Φ, so $\mathbb{C}^* \setminus \mathcal{M}$ is simply connected.

Thus, $\mathcal{M}$ is connected.