PROBLEM 28: Is the "bubble space" normal?

This is the space whose points are the upper half-plane $\{(x,y) \mid y \geq 0\}$ and basic open sets:

PRODUCT SPACES

If X_1 and X_2 are sets, then $X_1 \times X_2 = \{(x_1, x_2) \mid x_i \in X_i\}$ is the usual Cartesian product. If they have topologies, then the product topology has basis $\{\mathcal{U}_1 \times \mathcal{U}_2 \mid \mathcal{U}_i \text{ open in } X_i\}$.

There are natural projections $p_1 : X_1 \times X_2 \to X_1$ and $p_2 : X_1 \times X_2 \to X_2$ defined by $p_1(x_1, x_2) = x_1$, $p_2(x_1, x_2) = x_2$

PROP: With the product topology, each p_i is continuous. Moreover, that's the smallest topology on the set $X_1 \times X_2$ for which that holds.

PROP: If Z is a topological space, then a function $f: Z \to X_1 \times X_2$ is continuous if and only if $p_i \circ f$ is continuous, $i=1,2$.

$$\begin{tikzcd} Z \arrow[r, "f"] & X_1 \times X_2 \arrow[dl, "p_1"'] \arrow[d, "p_2"] \\ X_1 & X_2 \end{tikzcd}$$

PROP: $X_1 \times X_2$ is Hausdorff $\Leftrightarrow$ each X_i is.

THM: $X_1 \times X_2$ is compact $\Leftrightarrow$ each X_i is compact.

PROBLEM 29: If $\mathcal{B}_i$ is a basis for the topology on X_i, then $\{B_1 \times B_2 \mid B_i \in \mathcal{B}_i\}$ is a basis for the product topology on $X_1 \times X_2$.

PROBLEM 30: If $A_i \subset X_i$, then $\bar{A}_1 \times \bar{A}_2 = \overline{A_1 \times A_2}$

PROBLEM 31: Suppose X is a Hausdorff space. Then

(a) X is regular $\Leftrightarrow$ for each $x \in \mathcal{U}^{\text{open}}$, there exists V open with $x \in V \subset \bar{V} \subset \mathcal{U}$.

(b) X is normal $\Leftrightarrow$ for each closed $A \subset X$ and open $\mathcal{U}$, with $A \subset \mathcal{U}$, there exists V open with $A \subset V \subset \bar{V} \subset \mathcal{U}$.

PROBLEM 32: The product of two regular spaces is regular.

PROBLEM 33: The product of two normal spaces may not be normal.

Arbitrary products: Let J be an (index) set, and suppose for each $j \in J$, X_j is a set. Then $\prod_{j\in J} X_j$ is the set of all functions

$$x : J \to \bigcup_{j\in J} X_j$$

such that $x(j) \in X_j$. We usually write $x_j = x(j)$. Define $p_i : \prod X_j \to X_i$ by $p_i(x) = x_i$

In the special case that all X_j are equal (to X), the product space is written X^J $= \{\text{functions } J \to X\}$.

If each X_j has a topology, we give $\prod X_j$ the smallest topology s.t. the p_i are continuous; this is the product topology on $\prod_{j\in J} X_j$.

A sub-basis is, for each fixed $i \in J$, the sets $p_i^{-1}(\mathcal{U}_i) \subset$ where $\mathcal{U}_i$ is open in X_i. Thus a basis for the product topology = finite intersections.

Thus a typical basis element for the product topology is obtained by choosing $\{j_1, \dots, j_k\}$ finite $\subset J$ and $\mathcal{U}_{j_q}$ open in X_j. The corresponding basic open set is then:

$$\{x \in \prod X_j \mid x_{i_q} \in \mathcal{U}_{i_q},\ q \in \{j_1, \dots, j_k\},\ x_j \text{ arbitrary if } j \notin \{j_1, \dots, j_k\}\}$$

<u>PROP</u>: If Z is a space, $f: Z \to \prod_{j \in J} X_j$ is continuous $\Leftrightarrow$ each $p_i \circ f$ is continuous.

If we replace the finite set $\{j_1, \dots, j_k\}$ by all of J, we get the so-called <u>box</u> topology, which is a larger topology on $\prod X_j$.

<u>PROBLEM 34</u>: If $\{0, 1\}$ has the discrete topology and $\omega = \{0, 1, 2, \dots\}$, then $\{0, 1\}^\omega$ is homeomorphic with the Cantor set!

Note: The projection maps p_i are often denoted π_i

Identification spaces:

Let $\mathcal{P}$ be a partition of X (← top. space), i.e. a collection of disjoint subsets with $\cup\mathcal{P} = X$.

Form a new space Y, whose points are the elements of $\mathcal{P}$, and open sets are ~~unions~~ sets of elements of $\mathcal{P}$ whose union is open in X.

There's a projection $\pi : X \to Y$ taking $x \in X$ to the unique $P \in \mathcal{P} = Y$ with $x \in P$. Then $\mathcal{U}$ is open in $Y \Leftrightarrow \pi^{-1}(\mathcal{U})$ is open in X.

Notation: $\mathcal{P}$ is often specified by an equiv. relation: $x_1 \sim x_2 \Leftrightarrow x_1 \,\&\, x_2$ belong to the same element of $\mathcal{P}$. Then we write $Y = X/\sim$. (This is the largest top. on Y making π continuous)

PROP: X, Y as above, and Z any space. Then a function $f : Y \to Z$ is continuous $\Leftrightarrow$ πf is.

If $f : X \to Y$ is a surjective map of spaces, it's called an identification map if $\mathcal{U} \subset Y$ is open $\Leftrightarrow f^{-1}(\mathcal{U})$ open.

In this case, the partition of X is the set

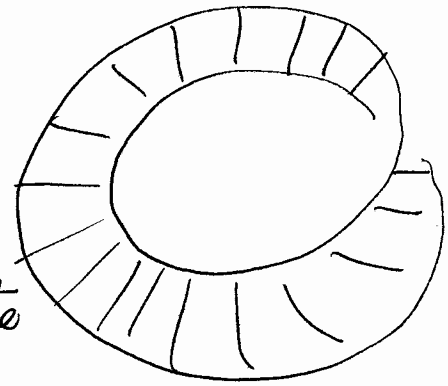
$$\mathcal{P} = \{ f^{-1}(y) \mid y \in Y \}$$

The equivalence is: $x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2)$.
And $Y \cong X/\sim$.

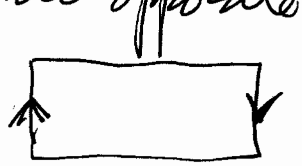
PROP: Suppose $f: X \to Y$ is a surjective map of spaces. If f maps open sets to open sets (or closed to closed), then f is an identification map.

COR: If $f: X \to Y$ is surjective, X compact & Y Hausdorff, then f is an identification.

EXAMPLES: The Mobius strip



can be regarded as a rectangle with two opposite edges identified as shown:



Or, if $X = \{(x,y) \mid -1 \leq y \leq 1\}$ and $f: X \to X$ is $f(x,y) = (x+1, -y)$. Define $p \sim q \Leftrightarrow p = f^n(q)$, some n.
Then $X/\sim$ = Möbius strip.

<u>Projective space</u>: let S^n = unit sphere $\subset \mathbb{R}^{n+1}$.
(usual topology)

define $x \sim y \iff x = \pm y$
Then $S^n/\sim = \mathbb{R}P^n$ (real projective space).

<u>Cone</u>: If X is a space, $CX = X \times [0,1]/\sim$
where $(x,t) \sim (x',t') \iff \begin{cases} t = t' = 0 \ \& \ x, x' \in X \\ \text{or} \\ t = t' \neq 0 \ \& \ x = x' \end{cases}$

(Thus $X \times \{0\}$ is "smashed to a point")

<u>Suspension</u>: $SX = X \times [0,1]/\sim$ where now
$$(x,t) \sim (x',t') \iff \begin{array}{ll} t = t' = 0, & x, x' \in X \\ \text{or } t = t' = 1, & x, x' \in X \\ \text{or } t = t' \notin \{0,1\}, & x = x' \end{array}$$

both ends of $X \times I$ smashed to points

<u>Example</u>: $id : \underline{I}^{\text{usual}} \longrightarrow I^{\text{indiscrete}}$
is a surjective map, but not an identification.

<u>Ex</u>: $X = \mathbb{R}^2$, $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ def by $F(x,y) = (x+1, y)$
$G(x,y) = (x, y+1)$

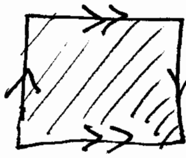
$p \sim q \iff p = G^m F^n(q)$ some $m, n \iff$
coords. differ by integers. $\mathbb{R}^2/\sim = T^2 = S^1 \times S^1$ "torus"

Now let $H:\mathbb{R}^2\to\mathbb{R}^2$ be $H(x,y)=(x+1,-y)$

Then G, H generate a discrete subgp. $\mathcal{G}$ of $\mathrm{Isom}\,\mathbb{R}^2$. Let $\mathcal{P}=\{$orbits of this group$\}$, i.e. $p\sim q \Leftrightarrow \exists K\in\mathcal{G}$ with $p=K(q)$

The space $\mathbb{R}^2/\sim\ =\mathbb{R}^2/$ is a <u>Klein bottle</u>.

Alternate descriptions:

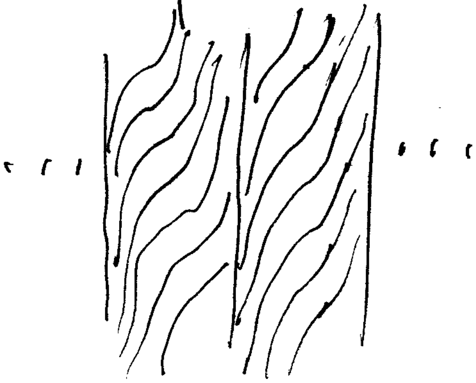


Klein bottle = union of two Möbius bands, sewn along their S^1 boundaries.

<u>Def</u>: A <u>manifold of dimension</u> d is a top. space such that each point has a nbd. homeo to $\mathbb{R}^d$. Manifold w/ bdry: also allow nbds $\cong$ $\mathbb{R}^d_+=\{(x_1,\dots,x_d)\mid x_d\geq 0\}$

A k-dim. <u>foliation</u> $\mathcal{F}$ of manifold M^d is a partition of M s.t. each $p\in M$ has a nbd $\cong\mathbb{R}^d$ by a homeo under which elts of $\mathcal{F}$ (leaves) intersect the nbd. in sets corresp to parallel hyperplanes $\mathbb{R}^k\subset\mathbb{R}^d$, $\mathbb{R}^k=\{(\underbrace{x_1,\dots,x_k}_{\text{(vary)}},\overbrace{c_{k+1},\dots,c_d}^{\text{const}})$

Examples: $\mathbb{R}^2$ has 1-dim foliations:



$\{y = \tan x + c\} \cup$
$\{x = (\text{odd}) \cdot \frac{\pi}{2}\}$

Also $\{y = \sec x + c\} \cup \{x = (\text{odd}) \cdot \frac{\pi}{2}\} = \mathcal{F}$

<u>PROBLEM</u> 35: In this example, the space of leaves $\mathbb{R}^2/\mathcal{F}$ is a 1-manifold, but not Hausdorff.

<u>Covering spaces</u> (not to be confused with "open coverings"):

A covering space is a triple $(\tilde{X}, p, X)$ where $\tilde{X}, X$ are spaces, $p : \tilde{X} \to X$ a surjective map and each $x \in X$ has a (connected) nbd $\mathcal{U} \ni x$ such that each component of $p^{-1}(\mathcal{U})$ is mapped homeomorphically to $\mathcal{U}$ by p. Also assume both X and $\tilde{X}$ are connected & locally connected.

<u>Exs:</u> $\tilde{X} = \mathbb{R}$, $X = S^1 \subset \mathbb{C}$ $\quad p(t) = e^{it}$

$\tilde{X} = S^1$, $X = S^1$, $\quad p(z) = z^n$

$\tilde{X} = \mathbb{C} \setminus 0$, $X = \mathbb{C} \setminus 0$, $\quad p(z) = z^n$

$\tilde{X} = \mathbb{C}$, $X = \mathbb{C} \setminus \mathbb{C}$ $\quad p(z) = e^z$

The maps $\mathbb{R}^2 \to T^2$ and $\mathbb{R}^2 \to$ Klein bottle defined previously, are covering spaces.
Also $S^n \to \mathbb{R}P^n$

<u>PROB 36:</u> <u>If</u> $p : \tilde{X} \to X$ is a covering space, $\alpha : I \to X$ a map and $y \in \tilde{X}$ satisfies $p(y) = \alpha(0)$, Then $\exists!$ map $\tilde{\alpha} : I \to \tilde{X}$ with $\tilde{\alpha}(0) = y$ & $p \circ \tilde{\alpha} = \alpha$. (path lifting)

$I = [0,1]$

Lemma: If X is a normal space and $\mathcal{U}, W$ are open sets such that $\overline{\mathcal{U}} \subset W$, then there exists an open $V \subset X$ such that $\overline{\mathcal{U}} \subset V$ and $\overline{V} \subset W$.

proof: The sets $\overline{\mathcal{U}}$ and $X \setminus W$ are disjoint closed sets, so there exist disj. open V, V' with $\overline{\mathcal{U}} \subset V$, $X \setminus W \subset V'$. But then $\overline{V} \subset W$: no point of V is in $X \setminus W$ as V & V' are disjoint. Also no limit point of V is in $X \setminus W$ as each pt of $X \setminus W$ lies in the open V' disjoint from V.

Urysohn's lemma: If A and B are disjoint closed sets in the normal space X, then there is a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.

— proof given in class —

<u>Tietze extension thm</u>: Supose X is normal and A is a closed subset of X. Then

(1) any cont ftn $A \to [a,b]$ can be extended to a cont. ftn $X \to [a,b]$.

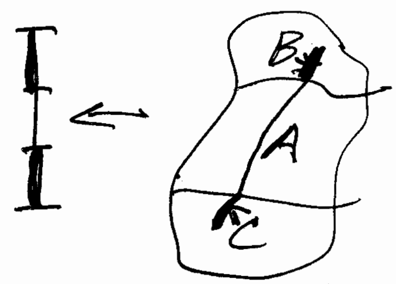
(2) any cont ftn $A \to \mathbb{R}$ - - - - - - - - - - - - - - - - - $X \to \mathbb{R}$

<u>proof</u> (of (1)): Step 1: We may assume w.l.o.g. that $[a,b] = [-r, r]$ and that $-r = \inf\{f(x) \mid x \in A\}$

$+r = \sup\{f(x) \mid x \in A\}$.

Consider $B = f^{-1}[-r, -\frac{r}{3}]$

$C = f^{-1}[\frac{r}{3}, r]$



Then $B \subset A$, $C \subset A$ and are closed in A & also in X (as A is closed).

Urysohn $\Rightarrow \exists g: X \to [-\frac{r}{3}, \frac{r}{3}]$ such that

$$g(b) = -\frac{r}{3}, \; b \in B, \quad g(c) = \frac{r}{3}, \; c \in C$$

Then $|g(x)| \le \frac{1}{3} r$, $|g(x) - f(x)| \le \frac{2}{3} r$

Step 2: Now, w.l.o.g. replace $[a,b]$ by $[-1,1]$.
by the above, there's $g_1 : X \to [-1/3, 1/3]$
with $|f(a) - g_1(a)| \leq 2/3$ for $a \in A$

Consider $f - g_1 : A \to [-2/3, 2/3]$

By step 1, $\exists\, g_2$ real val. cont fcn on X with

$$|g_2(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right) \qquad x \in X$$

$$|f(a) - g_1(a) - g_2(a)| \leq \left(\frac{2}{3}\right)^2 \quad a \in A$$

apply again, etc. inductively apply to

$$f - g_1 - g_2 \cdots - g_n : A \to \left[\left(-\frac{2}{3}\right)^n, \frac{2}{3}^n\right]$$

to get $g_{n+1} : X \to \mathbb{R}$ with

$$|g_{n+1}(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n$$

$$|f(a) - g_1(a) - \cdots - g_{n+1}(a)| \leq \left(\frac{2}{3}\right)^{n+1} \quad a \in A.$$

Define $g(x) = \sum_{n=1}^{\infty} g_n(x)$

This series converges uniformly: let $s_n =$ $n^{\underline{th}}$ partial sum: then if $k > n$

$$|s_k(x) - s_n(x)| = \left|\sum_{i=n+1}^{k} g_i(x)\right|$$

$$\le \frac{1}{3}\sum_{i=n+1}^{k}\left(\frac{2}{3}\right)^{i-1} < \frac{1}{3}\sum_{i=n+1}^{\infty}\left(\frac{2}{3}\right)^{i-1}$$

$$= \left(\frac{2}{3}\right)^{n}$$

Fix n, let $k \to \infty$

$$|g(x) - s_n(x)| \le \left(\frac{2}{3}\right)^n$$

and so $s_n \to g$ uniformly on X.
So g is continuous

Also $\left|f(a) - \sum_{i=1}^{n} g_i(a)\right| \le \left(\frac{2}{3}\right)^n \quad a \in A$

$\Rightarrow g(a) = f(a)$ for $a \in A$.

<u>COR</u>: If $A \subset X$ normal is homeo with I, then there is a retraction $r: X \to A$.

<u>COR</u>; The Tietze thm applies to I^n, $\mathbb{R}^n$ in place of $I, \mathbb{R}$.

PROBLEM : If $X_1, X_2, \ldots$ is a countable family of metric spaces, then the product $\prod_{n=1}^{\infty} X_i$ is metrizable.

In particular, if we take $X_i = \mathbb{R}$ for each $i = 0, 1, 2, \ldots$ the product $\mathbb{R}^\omega$ is metrizable. Likewise $[0,1]^\omega$ is metrizable.

<u>VRYSOHN METRIZATION THEOREM</u>: If X is a normal space with a countable basis, then X is metrizable.

<u>proof</u>: First we find a <u>countable</u> family of continuous functions $f_n : X \to [0,1]$ such that:

* if $x \in \mathcal{U} \subset X$ with $\mathcal{U}$ open, there exists n such that $f_n(x) > 0$ and $f_n(y) = 0$ if $y \in X \setminus \mathcal{U}$.

This can be done by considering the countable basis $\mathcal{B} = \{B_i \,;\, i \in \omega\}$. For each pair i, j such that $\bar{B}_i \subset B_j$, Vrysohns lemma gives a function $f_{ij} : X \to [0,1]$ with $f_{ij}(\bar{B}_i) = 0$, $f_{ij}(X \setminus B_j) = 1$. This is a countable family, so they may be re-indexed by $n \in \omega$.

Now, we define a map $F: X \to \mathbb{R}^\omega$ by

$$F(x) = (f_0(x), f_1(x), \dots).$$

<u>Claim</u>: F is an embedding, that is a homeo. from X to its image $F(X) =: Z$ in $\mathbb{R}^\omega$.

- F is continuous, as each coordinate map $\pi_n \circ F = f_n$ is continuous.

- F is injective: for if $x, y \in X$, $x \neq y$, there is an n with $f_n(x) > 0$, $f_n(y) = 0$ and therefore $F(x) \neq F(y)$.

- F^{-1} is continuous: Need to show that if $\mathcal{U} \subset X$ is open, then $F(\mathcal{U})$ is open in Z. Let $z_0 \in F(\mathcal{U})$. We'll find an open $W \subset Z$ with $z_0 \in W \subset F(\mathcal{U})$. Choose $x_0 \in \mathcal{U}$ with $F(x_0) = z_0$. Then there's n such that $f_n(x_0) > 0$ and $f_n(X \setminus \mathcal{U}) = 0$. Let $V = \pi_n^{-1}((0,\infty))$, which is open in $\mathbb{R}^\omega$ and let $W = V \cap Z$, open in Z (subspace topology). Then $z_0 \in W$ because $\pi_n(z_0) = \pi_n(F(x_0)) = f_n(x_0) > 0$. Also $W \subset F(\mathcal{U})$ as $z \in W \Rightarrow z = F(x)$, some $x \in X$. But $\pi_n(z) = \pi_n(F(x)) = f_n(x) > 0$ and so $x \in \mathcal{U}$. Thus $W \subset F(\mathcal{U})$. $\square$