PROBLEM 21: If $A \subset X$ is a connected subset of a topological space, show that its closure $\bar{A}$ is also connected. Is this true for "path-conn." replacing "connected?"

A space is locally path-connected if it has a basis consisting of path-connected open sets.

PROBLEM 22: If a space is connected and locally path-connected, then it is path-connected.

Many topological questions can be expressed in terms of map "extensions" or "liftings"

$$\begin{tikzcd} & B \arrow[dr, dashed, "?"] & \\ A \arrow[ur] \arrow[rr] & & C \end{tikzcd} \qquad\qquad \begin{tikzcd} & B \arrow[dl] & \\ A & & C \arrow[ll] \arrow[ul, dashed, "?"'] \end{tikzcd}$$

Extension problem — Lifting problem (dual)

Examples ① B = Hausdorff space, $A = \{p, q\} \subset B$, $C = \{0, 1\}$ discrete, $A \twoheadrightarrow C$ ONTO. Then extension exists $\Leftrightarrow$ p, q separated in B.

② $A = \{0, 1\} \hookrightarrow B = [0, 1]$. Then extension always exists $\Leftrightarrow$ C is path-connected.

Compactness. A family $\mathcal{F}$ of subsets of X is said to cover X if $\bigcup \mathcal{F} = X$, i.e. every $x \in X$ lies in some $F \in \mathcal{F}$. A subset $\mathcal{F}' \subset \mathcal{F}$ is a subcover if $\bigcup \mathcal{F}' = X$. If every set in $\mathcal{F}$ is open
cover has a finite subcover.

Ex: $\mathbb{R}$ (usual topology) is not compact, because the open cover $\mathcal{F} = \{ (n - \frac{3}{4}, n + \frac{3}{4}) \mid n \in \mathbb{Z} \}$ is infinite and has no proper subcover.

THM (Heine-Borel): A closed interval $[a,b] \subset \mathbb{R}$ is compact. (proof in class)

PROP: If X is compact and $A \subset X$ is closed, then A is compact. (converse may not hold, unless..

PROP: If X is a compact Hausdorff space, then any compact subset of X is closed.

PROP: If X is compact and $f: X \to Y$ is a map, then $f(X)$ is compact.

COR: Any map $f: X \to \mathbb{R}$, with X compact, is bounded and closed. In particular the least upper bound u of $f(X)$ is realized ($\exists x, f(x) = u$)

THM: If X is compact and Y is a Hausdorff space, then a map $X \xrightarrow{f} Y$ which is 1-1 and onto must be a homeomorphism.

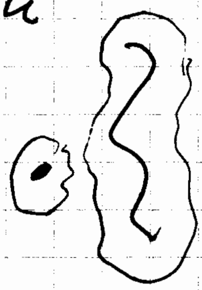
NOTE: This is false ~~for~~ noncompact X -- for example The map $[0,1) \xrightarrow{f} S^1$, $f(t) = e^{2\pi i t}$ is 1-1, onto but not a homeo.

THM: (Bolzano-Weierstrass) Every infinite subset of a compact space has a limit point.

PROBLEM 23: Find an infinite subset $A \subset [0,1]$ with all these properties: A is discrete (with subspace topology); for every $a \in A$ there is an element of A greater than a, and another less than a; between any two elements of A there is another elt. of A.

Separation properties:

There exist open sets $\mathcal{U}, V \subset X$ with $p \in \mathcal{U}$, $A \subset V$ and $\mathcal{U} \cap V = \phi$.



X is <u>normal</u> if it is Hausdorff and for any two disjoint closed sets $A, B \subset X$, there exist disjoint open sets $\mathcal{U}, V$ with $A \subset \mathcal{U}$ and $B \subset V$

Our text proves (Thm 3.6) that a compact Hausdorff space is regular.

<u>PROBLEM 24</u>: A compact Hausdorff space is normal

<u>PROBLEM 25</u>: Show problem 24 does not hold if the "Hausdorff" condition is dropped.

<u>PROBLEM 26</u>: Every metric space is normal.

Def: A point p in a topological space X is isolated if there is a neighbourhood of p which does not contain any other points of X. In other words, $p \in X$ but p is not a limit point of X.

Note that the Cantor set has no isolated points, although it is totally disconnected.

PROBLEM 27: Show that a compact metric space may have infinitely many isolated points, but not uncountably many.