A subset $A \subset X$ is <u>dense</u> if $\bar{A} = X$.

<u>PROBLEM 10</u>: Show that if $(X, \mathcal{T})$ has a countable basis, then it contains a countable dense subset. What about the converse?

A space with countable dense subset is called <u>separable</u>; if it has a countable basis it is called <u>second countable</u>. "First countable" means that each point has a countable collection of neighbourhoods $\mathcal{N}$ such that every neighbourhood of the point contains a sub-neighbourhood belonging to $\mathcal{N}$.

<u>DEF</u>: If $\mathcal{C}$ is any collection of subsets of X, one can construct a basis $\mathcal{B}$ by taking

$$\mathcal{B} = \{\text{finite intersections of elements of } \mathcal{C}\}$$

If $\mathcal{C}$ <u>covers</u> X (meaning $\cup \mathcal{C} = X$), then $\mathcal{B}$ as above is a basis, and determines a topology $\mathcal{T}_{\mathcal{B}}$. $\mathcal{C}$ is said to be a <u>sub-basis</u> for this topology, which is the "smallest" topology on X such that members of $\mathcal{C}$ are open sets.

CONTINUITY: If X and Y are top. spaces, a function $f: X \to Y$ is continuous if for every open $\mathcal{U} \subset Y$, the set $f^{-1}(\mathcal{U})$ is open in X. Here $f^{-1}(\mathcal{U}) = \{x \in X \mid f(x) \in \mathcal{U}\}$

PROP: If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are continuous, so is $X \xrightarrow{g \circ f} Z$.

PROP: If $X \xrightarrow{f} Y$ is continuous and $A \subseteq X$ has the subspace topology, then $f|_A : A \to Y$ is continuous.

PROBLEM 11. If X, Y are metric spaces, verify that $f: X \to Y$ is continuous if & only if for all $x_0 \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ s.t.

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$$

PROBLEM 12. If $f: X \to Y$ is continuous, and x is a limit point of A, where $A \subset X$, does it follow that $f(x)$ is a limit point of $f(A)$ in Y?

PROP: If $f: X \to Y$ is a function and X, Y topological spaces, the following are equivalent:

a) f is continuous

b) If $\mathcal{B}$ is a basis for the topology of Y, then $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.

c) $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$

d) inverse images of closed sets are closed

proof: see the text, p. 39.

JARGON: A continuous function is also called a map or mapping. If $f: X \to Y$ is 1-1 and onto, and if $f^{-1}: Y \to X$ is also continuous, then f (and f^{-1}) is a homeomorphism or topological equivalence. Under f, the open sets of X and Y are also in a 1-1 correspondence

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A sequence $\{x_n\}$, $n = 1, 2, \ldots$ of points $x_n \in X$ is said to converge to x_0 (written $x_n \to x_0$) if every neighbourhood of x_0 contains all but finitely many x_n.

PROBLEM 13: If $f: X \to Y$ is continuous and $x_n \to x_0$ in X, does $f(x_n) \to f(x_0)$ in Y? Compare with problem 12.

PROBLEM 14: If X is a metric space with metric d, and $x_0 \in X$ is the function $f: X \to \mathbb{R}$ (usual topology), defined by $f(x) = d(x, x_0)$, a continuous function? Why?

DEF: $\mathbb{E}^n$ denotes $\mathbb{R}^n$ with the usual topology.

If $\phi: X \to X$ is continuous, is

$$x \longmapsto d(x, \phi(x)) \text{ continuous?}$$

If $f: X \to X$ is a function, then $\mathrm{Fix}\, f = \{x \in X \mid f(x) = x\}$

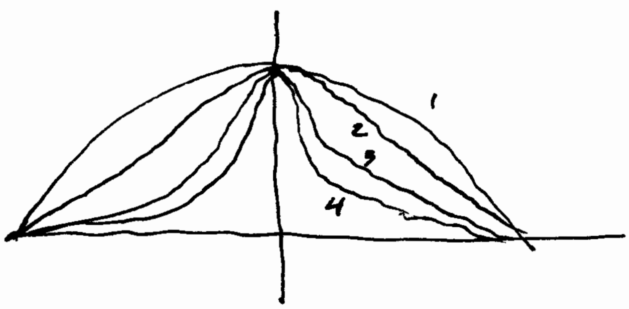
By the above, if f is continuous and X is a metric space, $\mathrm{Fix}\, f = F^{-1}(0)$ where $F(x) = d(x, f(x))$ and so $\mathrm{Fix}\, f$ is closed.

PROBLEM 15: If X is a Hausdorff space and $f: X \to X$ continuous, then $\mathrm{Fix}(f)$ is closed. Show by example that this does not hold in arbitrary topological spaces.

<u>Def</u>: Suppose (X, d) and (X', d') are metric spaces, and $f_n : X \to X'$, $n = 1, 2, \ldots$ is a sequence of functions, with (pointwise) $\lim_{n\to\infty} f_n(x) = f(x)$. We say f_n converges to f <u>uniformly</u> if for all $\varepsilon > 0$, there exists N such that $d'(f_n(x), f(x)) < \varepsilon$ for all $n > N$ and all $x \in X$. [Note: for this def. we do not need a metric or even a topology for X].

<u>PROB 16</u>: If $f_n : X \to X'$ is a sequence of <u>continuous</u> functions between metric spaces and $f_n \to f$ uniformly, then f is also continuous.

<u>Example</u>: Here $X = [-\pi/2, \pi/2]$, $X' = [0, 1]$ with the usual metrics and $f_n(x) = \cos^n(x)$, which are continuous, even infinitely differentiable.



But $f_n \to f(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases}$ <u>Discontinuous!</u>

The interval $[0,1]$, usual topology, then a cont. ftn $\alpha : I \to X$ is called a path or curve in X from $\alpha(0)$ to $\alpha(1)$. If $\alpha(0) = \alpha(1)$ then α is called a loop or closed curve.

A space X is said to be path-connected if for every $x, x' \in X$ there exists a path in X from x to x'.

Examples: $\mathbb{E}^n$, convex sets in $\mathbb{E}^n$, star-shaped sets in $\mathbb{E}^n$. The space $\mathbb{Q}$ is not path-conn. (why?)

It is easy to check that if X, Y are spaces and $f : X \to Y$ continuous, then X path-conn $\Rightarrow f(X)$ path-conn.

Example (Peano 1890): If I^2 denotes the square $\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1, 0 \le y \le 1\}$, there is a path $\alpha : I \to I^2$ which is surjective. [a.k.a. space-filling curve]

Connectedness:

Def: A space X is connected if it is not the union of two disjoint nonempty open sets.

PROP: X is connected iff there is no continuous function $X \to \{0,1\}$ (discrete topology)

In the other direction, if $X = \mathcal{U} \cup V$, open nonempty sets with $\mathcal{U} \cap V = \phi$, then $p \in \mathcal{U}$ and $q \in V$ are said to be separated in X. Note that the map $f: X \to \{p, q\}$ (discrete top) $f(x) = \begin{cases} p & \text{if } x \in \mathcal{U} \\ q & \text{if } x \in V \end{cases}$ is continuous.

Def: A space X is totally disconnected if every pair of points are separated in X.

PROB 17 A subspace $X \subset \mathbb{E}^1$ is connected if & only if it is an interval (a,b), $(a,b]$ (allowing $a = -\infty$, $b = \infty$) $[a,b]$ or $[a,b)$

PROP: The continuous image of a connected space is connected: if $f: X \to Y$ is cont., then $f(X)$ is connected (with subspace topology from Y). The same is true for "path connected."

COR: The "intermediate value theorem" of calculus.

PROP: If X is path-connected, then X is connected.

Example: To see the converse of this is false, consider $X =$ the closure of $\{(x,y) \in \mathbb{R}^2 \mid 0 < x \leq 2\pi,\ y = \sin\frac{1}{x}\}$

PROBLEM 18: This space X is connected but not path-connected.

Def: If $p \in X$, the component of p (in X) is the set of all $x \in X$ which is not separated from p in X. The path-component is the set of x such that there exists a path in X from p to x.

A subset $A \subset X$ is "connected" if with the subspace topology it is a connected space.

PROBLEM 19: Being in the same component is an equivalence relation: if x and y are in the component of p, then y is in the component of x and vice-versa. (Similarly for path-component.) A component is connected

Def: A space is locally connected if it has a basis consisting of connected sets. (using the subspace topology)

PROBLEM 20: If X is locally connected, then components of X are open. This is not true in general

Def: A space is totally disconnected if every pair of points in the space is separated in the space, i.e. components are just points.

EXAMPLES: The set $\mathbb{Q}$ of rational numbers is totally disconnected, with the usual topology. Similarly for the irrationals. A closed, bounded example in $\mathbb{R}$ is the Cantor set: obtained by successively deleting middle $\frac{1}{3}$'s from $I = [0,1]$