# Ma 426 - Topology

A modern form of geometry

Basic objects: topological spaces

" concept: continuity

Applications:

- other branches of math
    - analysis
    - algebra (e.g. fund thm of alg.)
    - logic
    - diff eqns.
- physics
    - quantum theory
    - cosmology (CMB experiments, shape of univ.)
- biology / dna structure & combinatorics
- economics, game theory
    - existence of equilibria
- computer science
- medicine
- chemistry (chirality e.g. R-thalidomide, S-thalidomide ← caused birth defects
- electronics (circuit analysis)

Some (amazing) theorems of topology:

Jordan curve theorem: If $C$ is a simple closed curve in the plane $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus C$ consists of two connected ~~pieces~~ components & $C$ is bdry of each

Brouwer fixed point thm: If $f: D \to D$ is a cont. map of the disk $D$, then there is $x \in D$ with $f(x) = x$.

Borsuk-Ulam thm (one version) Suppose $S^2$ is the sphere, and $f$ & $g$ are cont. fns: $S^2 \to \mathbb{R}$. Then there are two antipodal points $x, -x$ such that $f(x) = f(-x)$ and $g(x) = g(-x)$.
e.g. Temp. & Pressure.

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Algebraic topology:

convert a top. problem to an algebraic one, which can be rigorously solved.

Example: Euler characteristic

~~Gr~~ Graph: def.

# 1. Euler characteristic — an algebraic topology example

Consider a "polyhedral" surface: a union of (planar) polygons (faces), with these properties:

- each edge belongs to exactly two faces
- each vertex belongs to faces $F_1, \dots, F_k$ so that $F_i$ & $F_{i+1}$ share an edge & so do $F_k$ & $F_1$

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__Euler's thm__; If $P$ is a polyhedral surface st

(a) any two vertices can be connected by a chain of edges (connected)

(b) any loop on $P$ made of straight line segments (not necessarily edges) separates $P$ into two pieces.

Then $v - e + f = 2$ where $v = \#$ vertices $= v(P)$
$e = \#$ edges $= e(P)$
$f = \#$ faces $= f(F)$

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__Proof uses graphs:__

$G$ graph $= (V, E)$ where $E =$ (finite) set of points
$V =$ ( " ) set of "edges" with ends in $E$

A *chain* of edges $e_1, \ldots, e_k$ is a sequence in $E$ so that $e_i$ and $e_{i+1}$ share a vertex.

$G$ is *connected* if each pair of vertices $v, v'$ can be joined by a sequence of edges $e_1, \ldots, e_k$ (so $v$ and $v'$ are vertices of $e_1$ and $e_k$ resp.)

A *loop* in $G$ is a chain of pairwise distinct edges $e_1, \ldots, e_k$ so that $e_1$ and $e_k$ also share a vertex.

A *tree* is a connected graph with no loops.

PROBLEM 1: Show that for any connected graph $G = (V, E)$, $v - e \leq 1$ with equality if and only if $G$ is a tree.

Problem 2: For any connected graph $G$, there is a sub-graph $T$ which contains all the vertices of $G$ and is a *tree*. (called a maximal tree.

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Euler's Theorem, proved in class, uses the above calculations for graphs.

Now... we turn to ~~pointset~~ topology...

**Def**: Let $X$ be a set, $\mathcal{B}$ = a collection of subsets of $X$.
Then $\mathcal{B}$ is a *basis* (for a top. on $X$) $\Leftrightarrow$

(i) for each $x \in X$, $\exists B \in \mathcal{B}$ st $x \in B$

(ii) if $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, $\exists B_3 \in \mathcal{B}$ st.
$x \in B_3 \subset B_1 \cap B_2$

**Example**: $X = \mathbb{R}$, $\mathcal{B} = \{(a,b) \subset \mathbb{R} \mid a<b\}$
$X = \mathbb{R}$, $\mathcal{B} = \{[a,b) \quad \cdots \quad\}$
$X$ = any set, $\mathcal{B} = \{\{x\}\}$
$X$ = any set, $\mathcal{B} = \{X\}$.

**Def**: If $\mathcal{B}$ is a basis, then let $\mathcal{T}$ = the set of unions of elements of $\mathcal{B}$. Then $\mathcal{T}_{\mathcal{B}}$ is the top. gen by $\mathcal{B}$.

**Problem 4**: Show that $\mathcal{T}_{\mathcal{B}}$ is a topology on $X$.

**DEF**: If $(X, \mathcal{T})$ is a top. space, any open $\mathcal{U}$ with $x \in \mathcal{U}$ is called a *neighborhood* of $x$.

**PROP**: $A \subset X$ is open $\Leftrightarrow$ $\forall x \in A$, $\exists$ nbd of $x$ contained in $A$.

<u>Definition</u>: A <u>topological space</u> is a pair $(X, \mathcal{T})$:

$X$ = a set (called the set of <u>points</u>)

$\mathcal{T}$ = a collection of subsets of $X$ (called the <u>open</u> sets)

satisfying:

(1) the empty set $\phi$ and $X$ itself belong to $\mathcal{T}$.

(2) The union of every collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

(3) The intersection of each <u>finite</u> collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

EXAMPLES:

(1) $X$ any set, $\mathcal{T} = \{\phi, X\}$ (the indiscrete or 'trivial' topology)

(2) $X$ any set, $\mathcal{T}$ = all subsets (discrete topology)

(3) $X$ any set, $\mathcal{T} = \{Y \subset X \mid X \setminus Y \text{ is finite}\} \cup \{\phi\}$ (cofinite topology)

<u>PROBLEM 3</u>: Verify that this (3) is a topological space

(4) fix $p \in X$ = any set. $\mathcal{T} = \{Y \subset X \mid p \in Y \text{ or } Y = \phi\}$

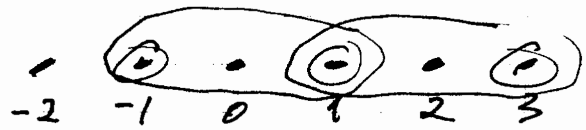
(5) $X = \mathbb{R}$, $\mathcal{T} = \{\phi, \mathbb{R}, (-\infty, p) \text{ for any } p \in \mathbb{R}\}$

## Digital line topology on $\mathbb{Z}$.

Space $X = \mathbb{Z}$. $\mathcal{B} = \{ B(n) \mid n \in \mathbb{Z} \}$

where

$$B(n) = \begin{cases} \{n\} & , n \text{ odd} \\ \{n-1, n, n+1\}, & n \text{ even} \end{cases}$$



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Metric spaces: If $X$ is a set, then a function $d : X \times X \to \mathbb{R}$ is called a metric (or distance fu

if
ⓐ $d(x,y) = d(y,x)$
ⓑ $d(x,y) \geq 0$, with equality $\Leftrightarrow x = y$
ⓒ $d(x,y) + d(y,z) \geq d(x,z)$.

Def: Given a metric space $(X, d)$ there is a corresp. top. space $(X, \mathcal{T}_d)$ whose basis is the set

$$\mathcal{B} = \{ N_\varepsilon(x) \mid x \in X, \ \varepsilon > 0 \}$$

where $N_\varepsilon(x) = \{ y \in X \mid d(x,y) < \varepsilon \}$ = the $\varepsilon$-neighbourhood.

Examples:

This is the usual (Euclidean) distance. For $n=1$ it reduces to $|x-y|$.

Then $\mathcal{T}_d$ is the "usual" topology on $\mathbb{R}^n$.

(2) $X=\mathbb{R}^n$, $\quad d'(x,y)=\sum_{i=1}^{n}|x_i-y_i|$

This is sometimes called the 'taxicab' metric, as (in 2 dimensions) it measures distance if one is constrained (as on streets) to going only N–S or E–W.

PROBLEM 5: Verify that both $d$ and $d'$ are metrics on $\mathbb{R}^n$. Show moreover that $\mathcal{T}_d$ and $\mathcal{T}_{d'}$ coincide.

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If two metrics on a space $X$ give rise to the same topology, they are said to be (topologically) equivalent.

(3) $X$ = any set. $\quad d(x,y)=\begin{cases}0 & \text{if } x=y\\ 1 & \text{if } x\neq y\end{cases}$

Here the metric topology is the discrete topology.

A topological space $(X, \mathcal{T})$ is <u>metrizable</u> if there exists a metric $d: X \times X \to \mathbb{R}$ with $\mathcal{T}_d = \mathcal{T}$.

<u>PROBLEM 6</u>: Show that the digital line topology on $\mathbb{Z}$ is <u>not</u> metrizable.

<u>Convention</u>: If the topology $\mathcal{T}$ is understood, a top. space $(X, \mathcal{T})$ may be denoted just $X$.

<u>Order topology</u>: A <u>strict</u> <u>total</u> <u>ordering</u> $<$, on a set $X$ is a binary relation such that

- for every $x, y \in X$, exactly one of $x<y$, $y<x$, $y=x$ is true.
- If $x<y$ and $y<z$, then $x<z$.

Given an ordered set $(X, <)$, one defines the <u>order topology</u> by taking as <u>basis</u> the set of open intervals $(a,b) = \{x \in X \mid a<x<b\}$, where $a, b$ range over all pairs of points w/ $a<b$.

<u>Example</u>: With the usual order of $\mathbb{R}$, the order topology coincides with the usual topology.

Call the order topology $\mathcal{T}_<$

*Example*: Give $\mathbb{R}^2 = \{x = (x_1, x_2)\}$ the *lexicographic ordering*, defined by

$$x < y \iff x_1 < y_1 \quad \text{or} \quad x_1 = y_1 \ \& \ x_2 < y_2$$

*Problem 7*: Verify that this is a strict total ordering on $\mathbb{R}^2$, and that the corresponding topology is not comparable with the usual topology, in that in each topology there is an open set which is not open in the other.

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*Subspace topology*:

If $(X, \mathcal{T})$ is a topological space and $Y$ any subset of $X$, define the topology $\mathcal{T}_Y$ by:

$$\mathcal{T}_Y = \{\mathcal{U} \cap Y \mid \mathcal{U} \in \mathcal{T}\}$$

This is called the *subspace*, or *relative* topology.

*Problem 8*: If $(X, d)$ is a metric space and $Y \subset X$, the restriction of $d$ to $Y \times Y$ is a metric on $Y$. Does this define the same topology on $Y$ as the subspace topology on $Y$, using $\mathcal{T}_d$ on $X$?

Problem 9: Similar to problem 8, but start with a strict total order, $<$, on $X$. Consider $Y \subset X$, which is also ordered by (the restriction of) $<$. Is the subspace topology on $Y$, as subset of $(X, \mathcal{T}_<)$, the same as the order topology on $Y$?

In a topological space $X$, a subset $A \subset X$ is closed if $X \setminus A$ is open. Note that a set can be both open and closed, or neither.

$X$ and $\phi$ are both open & closed. The family of closed sets is "closed" under finite unions and arbitrary intersections.

Def: If $A \subset X$, $X$ a top. space. A point $p \in X$ is a limit point of $A$ if every neighbourhood of $p$ contains at least one point of $A - \{p\}$.

PROP: A set is closed if & only if it contains all of its limit points.

Def: If $A \subset X$, the closure $\bar{A}$ is the union of $A$ and its limit points

PROP: $\bar{A}$ is the intersection of all closed sets containing $A$.