Tychonoff's theorem

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Theorem. Product of compact topological spaces is again compact.

[sketch of the proof] Let X_α ($\alpha \in I$) be compact topological spaces and let $\{\mathbb{O}_\beta\}_{\beta\in\mathbb{A}}$ be an open cover for $\prod_{\alpha\in I} X_\alpha$. Without loss of generality we can asume that all $\mathbb{O}_\beta$'s are basic open sets. So

$$\mathbb{O}_\beta = \mathbb{U}^\beta_{\alpha_1} \times \ldots \times \mathbb{U}^\beta_{\alpha_{n(\beta)}} \times \prod_{\alpha_i \neq \alpha \in I} X_\alpha$$

where $\mathbb{U}^\beta_{\alpha_i}$ is an open subset of X_{α_i}. If there was a finite open subcover then we were done, so assume that such an open subcover doesn't exist. So if we look at the complements of $\mathbb{O}_\beta$ which are closed sets, the assumption means that any finite number of them has a non-empty intersection. Having this I'll show that intersection of all those closed sets must be non-empty, which when you translate it back to $\mathbb{O}_\beta$'s, you get that they don't cover the product space. There is a very neat formula for complement of $\mathbb{O}_\beta$s

$$\mathbb{O}^c_\beta = \bigcup_{i=1}^{n(\beta)} ((\mathbb{U}^\beta_{\alpha_i})^c \times \prod_{\alpha_i \neq \alpha \in I} X_\alpha)$$

The trick here is to somehow choose an appropriate $(\mathbb{U}^\beta_{\alpha_i})^c \times \prod_{\alpha_i \neq \alpha \in I} X_\alpha$ for all $\mathbb{O}^c_\beta$ so that their intersections is non-empty. For getting these collection of sets we only need to look at these specific sets component-wise sinse we have this equality

$$\bigcap_{\gamma\in J} (\prod_{a\in S} A^\gamma_a) = \prod_{a\in S} (\bigcap_{\gamma\in J} A^\gamma_a)$$

So if we choose $(\mathbb{U}^\beta_{\alpha_i})^c \times \prod_{\alpha_i \neq \alpha \in I} X_\alpha$ such for every component they have finite intersection property, since all X_αs are compact their intersections is non-empty and hence product of all those points in the product space is not covered by any of the $\mathbb{O}_\beta$s.

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Other Problems

which can be solved using axiom of choice

Problem 1. Every vector space has a basis.
Problem 2. Every connected manifold is metrizable (also it can be chosen so that the metric becomes path-metric).
Problem 3. Let R be a commutative ring and $I \neq R$ an ideal. Then there exists a maximal ideal J of R containing I. [Re]
Problem 3'. Let R be a commutative ring, S a multiplicative set, and I an ideal of R disjoint from S. Then there exists a prime ideal P of R containing I and disjoint from S. [Re]
Problem 4. Let A and B be any two bounded sets in three-dimentional space with non-empty interior. Then there is a partition of A into finitely many sets which can be reassembled to yield B. [Ru]
Problem 5. Every hausdorff topological tree is metrizable (strongly, geodesicly metrizable).
Problem 6. Every set is well-orderable.
Problem 7. Any product of complete uniform spaces is complete.
Problem 8. Every field has an algebriac closure.
Problem 9. Every subgroup of a free group is free.
Problem 10. Every Hilbert space has orthonormal basis.
Problem 11. A partial right order $\mathbb{P}$ of a group G can be extended to a total right order if and only if it satisfies the following property:
for every finite set of non-identity elements $x_1, \ldots, x_n \in G$, semigroup generated by $\mathbb{P} \setminus \{e\}, x_1^{t_1}, \ldots, x_n^{t_n}$ does not contain e for some choice of signs $t_i \in \{-1, +1\}$. [Kop-Med]
Problem 12. Every hausdorff topologcial tree is strongly contractible and hence all its homotopy groups are trivial.
Problem 13. Every graph has a spanning tree.
Problem 14. Space of all right-ordering (ordering) of a group with its natural topology is compact. [Sik]

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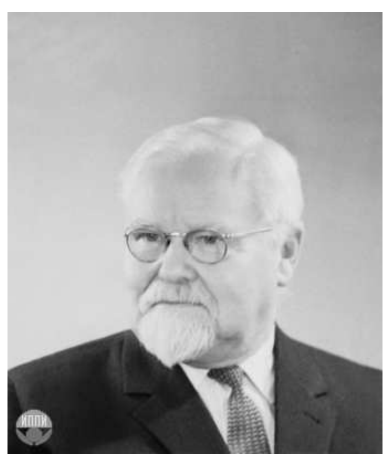

Figure 1: Andrey Nikolayevich Tychonoff