

An application of Brouwer's FPT to Game Theory

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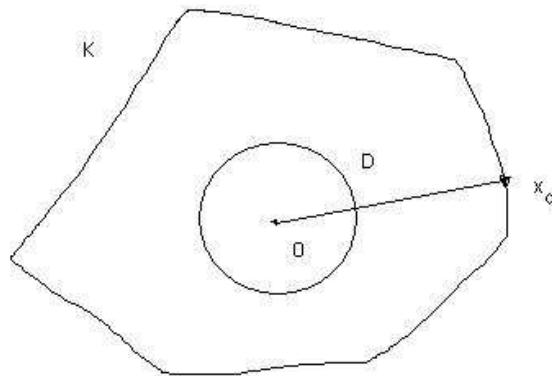
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Brouwer's FPT

Recall Brouwer's fixed point theorem as stated in class: If D^n is the unit disk ($\subset \mathbb{R}^n$), then any map $h : D^n \rightarrow D^n$ has a fixed point.

Another common way to state this theorem, which is more useful in this particular application is as follows: If $K \subset \mathbb{R}^n$ is convex and compact with non-empty interior, a map $f : K \rightarrow K$, has a fixed point.

To see the first implies the second, first claim that such a K is homeomorphic to the unit ball D^m for some $m \leq n$. WLOG remap K so that it contains the origin – pick an interior point q and map $x \rightarrow x - q$. Then there exists $\epsilon > 0$ s.t. $B_\epsilon(0) \subset K$. Then rescale K using $x \rightarrow x/\epsilon$. Then we have $D^m \subset K$. Each ray from the origin intersects ∂K at only one point, say x_0 (by convexity). So, for x on the ray from 0 to x_0 , map $g : x \rightarrow \frac{x}{|x_0|}$. This takes $K \rightarrow D^m$ with $\partial K \rightarrow S^{m-1}$, this map is bijective, continuous and so is its inverse.



Then $g \circ f \circ g^{-1}$ is a map from the unit ball to itself. So there exists v with $g(f(g^{-1}(v))) = v$. Let $x = g^{-1}(v)$, then $g(f(x)) = g(x)$, and so $f(x) = x$ provides the fixed point as g is injective.

Def: The *standard closed n -simplex* is the set

$$\left\{ y \in \mathbb{R}^n : y_i \geq 0, i = 1, \dots, n; \sum_{i=1}^n y_i = 1 \right\}$$

which is closed and bounded (hence compact) and convex (special case of convex combination of n components).

Quick intro to game theory

A standard introductory example is the so-called “battle of the sexes”. A husband and wife need to decide how to spend the evening, but cannot discuss with the other person before making their decision. The husband wants to watch football, and the wife wants to watch ballet. But, neither will be happy if they watch something alone. If we can assign a number to represent their respective happiness (payoff) given the choices, we might write it in a matrix as

		Husband	
		F	B
Wife	F	$H: 2 \quad H: -1$	$W: 1 \quad W: -1$
	B	$H: -1 \quad H: 1$	$W: -1 \quad W: 2$

This is a *two person non-cooperative game*. More generally, we have a n -person game $I = \{1, 2, \dots, n\}$. The i -th person has m_i “choices”, or *pure strategies*. The player can also assign a probability distribution to the pure strategies available to them; this is called a *mixed strategy*. The player then randomly selects one of his m_i pure strategies based on that probability distribution. Note that a pure strategy then is special case of a mixed strategy, with probability 1 assigned to one pure strategy and 0 to all others. Then, with each player having chosen their pure strategy, each receives a *payoff* (a real number), based on strategy choices of all players.

Notation: Let $S_i = \{\sigma_i^1, \sigma_i^2, \dots, \sigma_i^{m_i}\}$ be the set of pure strategies for the i -th player. Let x_i be an arbitrary mixed strategy for the i -th player; then, the probability assigned by x_i to the j -th pure strategy for that player is $x_i(\sigma_i^j)$, so this satisfies

$$x_i(\sigma_i^j) \geq 0, j = 1, \dots, m_i; \sum_{j=1}^{m_i} x_i(\sigma_i^j) = 1$$

Let X_i denote the set of all such mixed strategies x_i , then it is a standard closed m_i -simplex. The payoff P_i for the i -th player is a function of the mixed strategies of all the players,

$$P_i : X_1 \times X_2 \times \cdots \times X_n \rightarrow \mathbb{R}$$

Let $x = (x_1, \dots, x_n)$ denote the mixed strategy n -tuple for all the players together. Note that $X_1 \times X_2 \times \cdots \times X_n$ is then a standard closed $(\sum_{i=1}^n m_i)$ -simplex. Then for a given x , write $x|x'_i = (x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)$, i.e. i -th player changes strategy to x'_i .

$P_i(x)$ is computed using the usual expectation in terms of payoffs for pure strategies:

$$P_i(x_1, \dots, x_n) = \sum_{j_1=1}^{m_1} \cdots \sum_{j_n=1}^{m_n} \left[P_i(\sigma_1^{j_1}, \dots, \sigma_n^{j_n}) \prod_{k=1}^n x_k(\sigma_k^{j_k}) \right]$$

Observe that the probability for x can be decomposed into the n player components since they are acting independently.

Def: A mixed strategy n -tuple x is *admissible* for player i if $P_i(x|x'_i) \leq P_i(x)$ for all $x'_i \in X_i$. x is an *equilibrium point* of the game if it is admissible for all players i .

Nash Theorem

Lemma 1: A necessary and sufficient condition for a mixed strategy n -tuple x to be an equilibrium point is $P_i(x|\sigma_i) \leq P_i(x)$ for all pure strategies $\sigma_i \in S_i$, for all players i .

Pf: Necessary is clear, since pure strategy is a special case of a mixed strategy. Sufficient: choose arbitrary mixed strategy $x'_i \in X_i$, then the claimed inequality is equivalent to

$$\sum_{\sigma_i \in S_i} P_i(x|\sigma_i)x'_i(\sigma_i) \leq \sum_{\sigma_i \in S_i} P_i(x)x'_i(\sigma_i)$$

But this is true, since the left side equals $P_i(x|x'_i)$ and the right side is $P_i(x) \sum_{\sigma_i \in S_i} x'_i(\sigma_i) = P_i(x)$.

Lemma 2: For any mixed strategy n -tuple x , each player i has a pure strategy σ_i^j s.t. $x_i(\sigma_i^j) > 0$ and $P_i(x|\sigma_i^j) \leq P_i(x)$.

Pf: Suppose not. Then all $\sigma_i \in S_i$ with $x_i(\sigma_i) > 0$ have $P_i(x|\sigma_i) > P_i(x)$. Clearly there must be at least one such σ_i . Then the inequality implies

$$\sum_{\sigma_i \in S_i} P_i(x|\sigma_i)x_i(\sigma_i) = P_i(x|x_i) = P_i(x) > \sum_{\sigma_i \in S_i} P_i(x)x_i(\sigma_i)$$

since the terms in the sum with $x_i(\sigma_i) = 0$ do not change the sum on both sides. But the right hand side also simplifies to $P_i(x)$, a contradiction.

Theorem (Nash): Any finite n -person non-cooperative game has at least one mixed strategy equilibrium point.

Pf: Recall that the mixed strategy n -tuples form a standard closed $(\sum_{i=1}^n m_i)$ -simplex, which is a convex and compact subset of $\mathbb{R}^{\sum_{i=1}^n m_i}$. Given a mixed strategy n -tuple x , define for player i and any pure strategy $\sigma_i^j \in S_i$ the non-negative function

$$\psi_{ij}(x) = \max\{P_i(x|\sigma_i^j) - P_i(x), 0\}$$

This represents the increase in payoff (if any) for player i when they replace strategy x_i by the pure strategy σ_i^j .

We want to define a function $f : (x_1, \dots, x_n) \rightarrow (y_1, \dots, y_n)$ where (y_1, \dots, y_n) is again a mixed strategy n -tuple. For each $i = 1, \dots, n$ and $j = 1, \dots, m_i$ we choose the map

$$y_i(\sigma_i^j) = \frac{x_i(\sigma_i^j) + \psi_{ij}(x)}{1 + \sum_{j=1}^{m_i} \psi_{ij}(x)}$$

Notice that $\sum_{j=1}^{m_i} y_i(\sigma_i^j) = 1$, which again defines a probability distribution (i.e. mixed strategy) on S_i for player i . Doing this for $i = 1, \dots, n$, we thus see that this function maps the space of strategy n -tuples to itself.

Next, we argue that this f is continuous: the payoff $P_i(x)$ is a polynomial in the $x_i(\sigma_i^j)$'s with coefficients determined by $P_i(\sigma_1, \dots, \sigma_n)$ (where $\sigma_i \in S_i$ for each i). Thus $P_i(x)$ is continuous, and so are $\psi_{ij}(x)$ and $x_i(\sigma_i^j) + \psi_{ij}(x)$. Now the denominators $1 + \sum_{j=1}^{m_i} \psi_{ij}(x)$ are strictly positive. So f is continuous.

The conditions for Brouwer's FPT are thus satisfied, and there exists a mixed strategy n -tuple $x^0 = (x_1^0, \dots, x_n^0)$ with $f(x^0) = x^0$. For this n -tuple, we have for all i, j that

$$x_i^0(\sigma_i^j) = \frac{x_i^0(\sigma_i^j) + \psi_{ij}(x^0)}{1 + \sum_{j=1}^{m_i} \psi_{ij}(x^0)}$$

By Lemma 2 and the definition of $\psi_{ij}(x)$, there exists for each player i a pure strategy σ_i^k s.t. $x_i^0(\sigma_i^k) > 0$ and $P_i(x^0|\sigma_i^k) \leq P_i(x^0)$, and so $\psi_{ik}(x^0) = 0$. We thus have

$$x_i^0(\sigma_i^k) = \frac{x_i^0(\sigma_i^k)}{1 + \sum_{j=1}^{m_i} \psi_{ij}(x^0)}$$

Rearranging, this implies that $x_i^0(\sigma_i^k) \sum_{j=1}^{m_i} \psi_{ij}(x^0) = 0$. But $x_i^0(\sigma_i^k) > 0$, so $\sum_{j=1}^{m_i} \psi_{ij}(x^0) = 0$. But each term in the sum is non-negative, so we must have for all i, j that $\psi_{ij}(x^0) = 0$, i.e. $P_i(x^0|\sigma_i^j) \leq P_i(x^0)$. By Lemma 1, x^0 is an equilibrium point.