NOTES ON WHITNEY'S EMBEDDING THEOREM
by Tristan Collins

<u>Definition</u>: A topological space M is called an n-dimensional <u>manifold</u> if it is locally homeomorphic to $\mathbb{R}^n$. That is, there is an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M such that, for each $i \in I$ there is a map $\varphi_i : U_i \to \mathbb{R}^n$ which maps U_i homeomorphically onto an open subset of $\mathbb{R}^n$.

We call the pair (φ_i, U_i) a <u>coordinate patch</u>, and the set $\{(\varphi_i, U_i)\}_{i\in I}$ an <u>atlas</u>

Two patches (φ_i, U_i) and (φ_k, U_k) are said to have <u>C^r overlap</u> if the map:

$$\varphi_k \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_k) \longrightarrow \varphi_k(U_i \cap U_k)$$

is C^r. An atlas Φ on M is called C^r if every pair of charts in Φ has C^r overlap.

<u>Claim 1</u>: Given a C^r atlas Φ, there is a unique maximal atlas Γ which contains Φ.

<u>pf</u>: Idea: " Every patch in ___ has C^r overlap with every patch in ___ " is an equivalence relation on the set of all atlases for M. Take union of all atlases in the equivalence class of Φ.

Def'n: A manifold of class C^r is a topological space M together with a maximal C^r atlas

If M and N are two manifolds, and $f: M \to N$ then f is C^r at $m \in M$ if there exist patches (φ, U) with $m \in U$ at (ψ, V) with $f(m) \in V$ s.t. $\psi \circ f \circ \varphi^{-1}$ is C^r at $\varphi(m)$.

Def'n: Let $\gamma: (-\varepsilon, \varepsilon) \to M$ and $\tilde{\gamma}: (-\tilde{\varepsilon}, \tilde{\varepsilon}) \to M$ and $\gamma(0) = \tilde{\gamma}(0) = p \in M$. Then γ and $\tilde{\gamma}$ are said to be equivalent curves if, for some patch $(\varphi_\alpha, U_\alpha)$ with $p \in U_\alpha$

$$(\star) \qquad \frac{d}{dt} \varphi_\alpha(\gamma(t)) \Big|_{t=0} = \frac{d}{dt} \varphi_\alpha(\tilde{\gamma}(t)) \Big|_{t=0}$$

Then this defines an equivalence relation on the set of all curves through p. [To check this just note that there is a prescribed, linear transformation between local coordinate systems, so that if equality $(\star)$ holds in one coordinate system, it holds in all coordinate systems]

Then a tangent vector to M at p is an equivalence class of curves.

Def'n: The tangent space to M at x, TM_x is the set of all tangent vectors to M at x.

Def'n: The tangent bundle TM is:

$$TM = \{(x, \vec{v}) : x \in M, \vec{v} \in TM_x\}$$

Proposition: If M is a manifold, then TM is another manifold with $\dim(TM) = 2\dim(M)$
[Note: if M is C^r, then TM is C^{r-1}].

Idea:
given coordinates (φ_i, u_i), $u_i \ni x$, then send $v \in TM_x$ to $\frac{d}{dt}\varphi_i(\gamma(t))$ where $\gamma(t)$ is a representative curve for v.
$\in \mathbb{R}^n$

Given a C^1 function $f: M \to N$, M, N manifolds, f defines a linear map $T_x f: TM_x \to TN_{f(x)}$ by sending $[\gamma] \to [f \circ \gamma]$.

Def'n: Let M, N be Manifolds of class C^r, $r \geq 1$, Then f is <u>immersive</u> at $x \in M$ if the linear map $T_x f: TM_x \to TN_{f(x)}$ is injective. If f is immersive at every point of M it is an <u>immersion</u>. We call f an <u>embedding</u> if it is an immersion which maps M homeomorphically onto $f(M)$.

The "easy" Whitney Embedding Theorem

<u>Theorem</u>: Let M be a compact manifold of class C^r, $1 \leq r \leq \infty$. Then there exists a C^r embedding of M into $\mathbb{R}^q$ for some q.

<u>Proof</u>: Let $n = \dim M$. Let $D^n(\rho) = \{x \in \mathbb{R}^n : |x| \leq \rho\}$

<u>Thing 1</u>: Constructing a good cover

Let $x \in M$. Choose $(\varphi_\alpha, U_\alpha)$ from the atlas for M such that $x \in U_\alpha$. By translation and scalar multiplication we may construct a new $\tilde{\varphi}_x$ s.t. $\tilde{\varphi}_x(U_\alpha) \supset D^n(2)$, and $\tilde{\varphi}_x(x) = \vec{0}$. Since $\tilde{\varphi}_x = a\varphi_\alpha + c$ for some $a \in \mathbb{R}$, $c \in \mathbb{R}^n$, $\tilde{\varphi}_x$ is clearly compatible with every patch in the atlas. Thus, since by defn our atlas is maximal, $(\tilde{\varphi}_x, U_\alpha)$ is in our atlas for each $x \in M$.

Consider: $\displaystyle\bigcup_{x \in M} \operatorname{Int} \tilde{\varphi}_x^{-1}(D^n(1))$

This is clearly an open cover for M.

$\Rightarrow \exists \{(\varphi_i, U_i)\}_{i=1}^{k}$, a family of patches such that:

(1) $\varphi_i(U_i) \supset D^n(2)$ for $1 \leq i \leq k$.

(2) $M = \displaystyle\bigcup_{i=1}^{k} \operatorname{Int} \varphi_i^{-1}(D^n(1))$.

<u>Lemma 1</u>: There exists a C^∞ map $\lambda : \mathbb{R}^n \to [0,1]$ s.t.

(1) $\lambda : D^n(1) \to \{1\}$

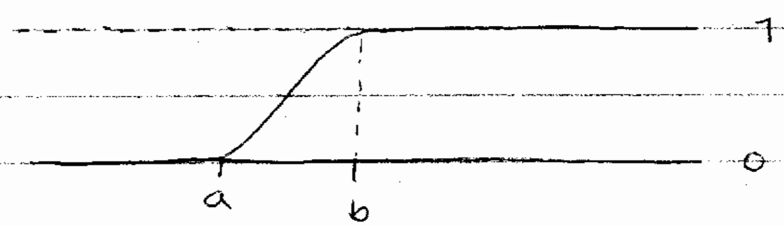
(2) $\lambda : \mathbb{R}^n - D^n(2) \to \{0\}$.

<u>pf</u> <u>Idea</u>: Let $\Psi(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \le 0 \end{cases}$

$\Psi(x)$ is then C^∞. define:

$$g(t) = \frac{\int_{-\infty}^{t} \Psi(x-a)\Psi(b-x)\,dx}{\int_{-\infty}^{\infty} \Psi(x-a)\Psi(b-x)\,dx}$$

$g(t) =$



Take differences of such $g(t)$'s. If we take x to be the radial coordinate, and arrange the boundaries, a, and b accordingly, then we can easily construct such a λ.

Define C^r maps: $\lambda_i = \begin{cases} \lambda \circ \varphi_i & \text{on } U_i \\ 0 & \text{on } M - U_i \end{cases}$

$\lambda_i : M \to [0,1] \subseteq \mathbb{R}$. The λ_i's are the "pull back" of the function λ to the manifold M. To see that the λ_i's are C^r, just check in local coordinates and split up the domains by intersecting with $\psi(\varphi^{-1}(D^n(2) \cap V_\alpha))$ and $\psi(\varphi^{-1}(D^n(2) \cap V_\alpha))^c$ where (ψ, V_α) is the coordinate system you are interested in.

Notice that $\lambda_i(x) = \lambda \circ \varphi_i(x) = 1$ when $\varphi_i(x) \in D^n(1)$ so: let $B_i = \lambda_i^{-1}(\{1\})$. Then the B_i's cover M.

Define maps $f_i(x) : M \to \mathbb{R}^n$ by:

$$f_i(x) = \begin{cases} \lambda_i(x)\varphi_i(x) & \text{if } x \in U_i \\ 0 & \text{if } x \in M - U_i \end{cases}$$

put $g_i = (f_i, \lambda_i) : M \to \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ and

$$g = (g_1, g_2, \ldots, g_k)$$

The f_i's are clearly C^r, and so the g_i's are as well, and finally g is C^r.

claim 2: if $x \in B_i$, then g_i is immersive.

pf

It suffices to prove that if $\gamma(t)$ and $\tilde{\gamma}(t)$ are representatives of unequal tangent vectors in TM_x, then $g_i \circ \gamma(t)$ and $g_i \circ \tilde{\gamma}(t)$ are not equivalent curves at $g_i(x) \in \mathbb{R}^{n+1}$.

This is easy since:

$$g_i \circ \gamma(t) = [\varphi_i(\gamma(t)), \lambda(\gamma(t))]$$

$$\Rightarrow \frac{d}{dt} g_i \circ \gamma(t)\Big|_{t=0} = \left[\frac{d}{dt}\varphi_i(\gamma(t))\Big|_{t=0}, \frac{d}{dt}\lambda(\gamma(t))\Big|_{t=0}\right]$$

$\#$

$$\frac{d}{dt} g_i \circ \tilde{\gamma}(t)\Big|_{t=0} = \left[\frac{d}{dt}\varphi_i(\tilde{\gamma}(t))\Big|_{t=0}, \frac{d}{dt}\lambda(\tilde{\gamma}(t))\Big|_{t=0}\right]$$

But since $\gamma, \tilde{\gamma}$ are not equivalent curves, the first terms are unequal.

Hence g_i induces a $1 \to 1$ map on the Tangent space at each point in B_i. So g_i is immersive ∎

But, the B_i's cover $M \Rightarrow g$ is immersive at each point in M.

$\Rightarrow g$ is an immersion.

g is also injective since:

(1) $x \in B_i$, $x \neq y$, $y \in B_i$ then:

$$g_i(x) = (\varphi_i(x), 1) \neq (\varphi_i(y), 1) = g_i(y)$$

(2) $x \in B_i$, $x \neq y$, $y \notin B_i$ then:

$$g_i(x) = (\_\_, 1) \neq (\_\_, \lambda_i(y)) = g_i(y)$$

since $y \notin B_i \Rightarrow \lambda_i(y) \neq 1$.

$\Rightarrow$ g is an injective C^r immersion

$\Rightarrow$ g is a continuous bijection from M, compact, to $g(M) \subseteq \mathbb{R}^{(n+1)K}$, Hausdorff

$\Rightarrow$ g is a homeomorphism onto its image.
$\Rightarrow$ g is an embedding.

∎

The "Medium" Whitney Embedding Theorem.

Theorem: Let M be a compact Hausdorff C^r n-dimensional manifold $2 \le r \le \infty$. Then there is a C^r embedding of M in $\mathbb{R}^{2n+1}$.

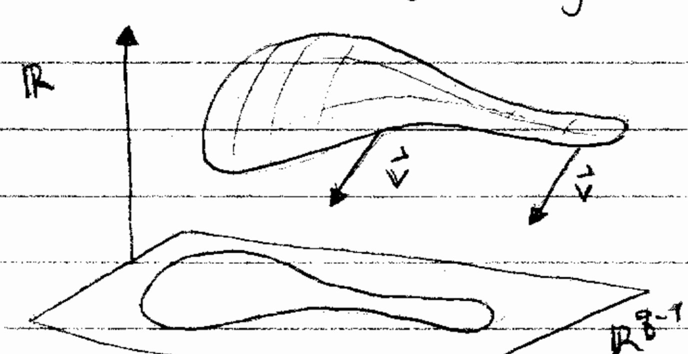
Proof: By the "easy" W.E.T, M embeds in $\mathbb{R}^q$ for some q. If $q \le 2n+1$, we're done. So, we assume $q > 2n+1$.

Replace M by its image under an embedding. It suffices to prove we can embed M in $\mathbb{R}^{q-1}$, since iteration of the argument eventually embeds M in $\mathbb{R}^{2n+1}$.

So

$M \subseteq \mathbb{R}^q$, $q \ge 2n+1$. Identify $\mathbb{R}^{q-1}$ with the set $\{x \in \mathbb{R}^q : x_q = 0\}$. If $\vec{v} \in \mathbb{R}^q - \mathbb{R}^{q-1}$, denote by $P_{\vec{v}} : \mathbb{R}^q \to \mathbb{R}^{q-1}$ the projection parallel to $\vec{v}$.

Goal: Find a vector $\vec{v}$ such that $P_{\vec{v}}|M : M \to \mathbb{R}^{q-1}$ is a C^r embedding.



For $P_{\vec{v}}|M$ to be an embedding it must be:

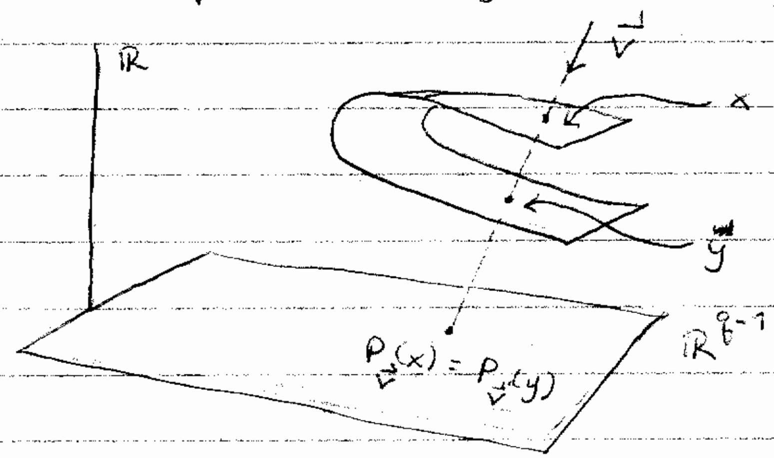
(i) injective on M

(ii) An immersion on TM.

Conditions (i) and (ii) restrict our choices of $\vec{v}$. Clearly we need only consider unit vectors.

Notice that if $x, y \in M$, $x \neq y$, then $\vec{v} \neq \frac{x-y}{|x-y|}$ for otherwise $P_{\vec{v}}(x) = P_{\vec{v}}(y)$.

ie:



Also, (ii) $\Rightarrow$ that the Kernel of the linear map induced by P_v must be trivial on TM_x for every $x \in M$. That is, for every $(x, \vec{w}) \in TM$, with $\vec{w} \neq 0$ we require:

$$\vec{v} \neq \frac{\vec{\omega}}{|\vec{\omega}|}$$

(Note: we are considering $\vec{\omega}$ as a vector in $\mathbb{R}^q$, so the norm makes sense.)

Thus, we have 2 conditions:

(1) $\vec{v} \neq \frac{x-y}{|x-y|}$ for $x, y \in M$, $x \neq y$

(2) $\vec{v} \neq \frac{\vec{\omega}}{|\vec{\omega}|}$ for any $(x, \vec{\omega})$, $\vec{\omega} \neq 0$, in TM.

Consider the map: $\sigma : M \times M - \Delta \longrightarrow S^{q-1}$
$$(x,y) \longmapsto \frac{x-y}{|x-y|}$$

where $\Delta = \{(a,b) \in M \times M : a = b\}$

Then, $\vec{v}$ satisfies (1) iff $\vec{v}$ is not in the image of σ. Note that $M \times M - \Delta$ is a C^r manifold of dimension $2n$, since:

$$\varphi_\alpha = (\varphi_i, \varphi_j) \qquad U_\alpha = U_i \times U_j$$

Form an atlas for $M \times M$, then by restricting to $M \times M - \Delta$, and using the subspace topology we get that $M \times M - \Delta$ is a C^r manifold. Hence σ is a C^r map, as is easily seen by moving into local coordinates. Note that since $M \subseteq \mathbb{R}^q$ $x-y$ is a well defined object. However, we need local coordinates to check differentiability as M need not be affine.

Lemma 2: Let $g: P \to Q$ be a C^1 map. If $\dim Q > \dim P$ then $[g(P)]^c$ is dense in Q.

pf: See M.W. Hirsch, Differential Topology p. 69.

By taking $Q = S^{q-1}$, $M \times M - \Delta = P$ so that:

$$\dim(M \times M - \Delta) = 2n < q-1 = \dim S^{q-1} \quad (*)$$

we have that every non-empty open set in S^{q-1} contains a vector $\vec{u} \notin \sigma(M \times M - \Delta)$.

Note that $(*)$ is the limiting factor in our application of this process, and will prevent this argument from embedding M in $\mathbb{R}^{2n}$, or anything "smaller" (i.e. $\mathbb{R}^{2n-1}$, --).

Now, note that condition (2) holds for every $(x, \vec{z})$ in TM provided it holds when $|\vec{z}| = 1$.

Let $T_1 M = \{ (x, \vec{z}) \in TM : |\vec{z}| = 1 \}$

Claim 3: $T_1 M$ is a C^{r-1} submanifold of TM.
pf: just take the restriction of the atlas for TM to $T_1 M$. Note, that there is some subtlety to deal with here, depending on whether or not one requires the topology on $T_1 M$

To be the subspace topology inherited from TM. For a full discussion see Hirsch, Differential Topology p. 21 → 27.

<u>Claim 4</u>: T_1M is compact.

pf: It suffices to show that for any sequence $\{(x_n, \vec{w}_n)\} \subseteq T_1M$, there is a convergent subsequence. Let $\{(x_n, \vec{w}_n)\} \subseteq T_1M$ be a sequence. then $\{x_n\}$ is a sequence in M. Since M is compact $\exists \{x_{n_k}\}$ s.t. $x_{n_k} \to x$ for some $x \in M$. Now consider the sequence $\{(x_{n_k}, \vec{w}_{n_k})\}$. The sequence $\{\vec{w}_{n_k}\}$ is just a sequence in S^{q-1}, which is compact. Thus $\exists \{\vec{w}_{n_{k_j}}\}$ s.t. $w_{n_{k_j}} \to w \in S^{q-1}$. This follows since $|\vec{w}_n| = 1$ for each $n \in \mathbb{N}$.
Then, clearly $\{(x_{n_{k_j}}, \vec{w}_{n_{k_j}})\}$ is the desired sequence. ∎

Define: $\tau : T_1M \to S^{q-1}$
$$(x, \vec{w}) \mapsto \vec{w}.$$

τ is clearly C^{r-1} since if $(\varphi_\alpha, T\varphi_\alpha)$ are coordinates on T_1M, then τ is just the map

$\tau \circ (\varphi_\alpha, T\varphi_\alpha)^{-1} = T\varphi_\alpha^{-1}$ which is C^{r-1} since M is C^r.

Now, since: $\dim T_1 M = 2n-1 < \dim S^{q-1}$

we can use Lemma 2 to see that $\tau(T_1 M)^c$ is dense in S^{q-1}.

Since $T_1 M$ is compact ~~[illegible]~~ $\tau(T_1 M)$ is compact $\Rightarrow \tau(T_1 M)$ is closed $\Rightarrow \tau(T_1 M)^c$ is open, and dense in S^{q-1}.

Then $\tau(T_1 M)^c \cap S^{q-1} \cap \mathbb{R}^q - \mathbb{R}^{q-1} = E \neq \emptyset$.

Since $\mathbb{R}^q - \mathbb{R}^{q-1}$ is open in $\mathbb{R}^q$
$\Rightarrow S^{q-1} \cap \mathbb{R}^q - \mathbb{R}^{q-1}$ is open in the relative topology, and it is clearly non-empty.

<u>But</u> $\tau(T_1 M)^c \cap S^{q-1} \cap \mathbb{R}^q - \mathbb{R}^{q-1}$ is open, as it is the intersection of two open sets, and it is non-empty. <u>So</u>, since $\sigma(M \times M - \Delta)^c$ is also dense in S^{q-1}, we have:

$$E \cap \sigma(M \times M - \Delta)^c \neq \emptyset.$$

So, choose $\vec{v} \in E \cap \sigma(M \times M - \Delta)^c$.

Then $P_{\vec{v}}|M : M \to \mathbb{R}^{q-1}$ is an injective immersion.

Since M is compact, and $P_{\vec{v}}(M) \subseteq \mathbb{R}^{q-1}$ is Hausdorff (since it inherits the subspace topology from $\mathbb{R}^{q-1}$), $P_{\vec{v}}|M$ is a continuous bijection from M compact to $P_{\vec{v}}(M)$ Hausdorff, and so it is a homeomorphism onto $P_{\vec{v}}(M)$.

$\Rightarrow P_{\vec{v}}|M$ is an embedding.