Part I. Differential forms. Let U be an open subset of $\mathbb{R}^n$. The *differential forms* are an algebra over C^∞ functions $f : U \to \mathbb{R}$. The algebra contains all the functions f (of degree 0), together with abstract elements $dx^1, dx^2, \ldots, dx^n$ and their linear combinations $f_1\, dx^1 + f_2\, dx^2 + \cdots + f_n\, dx^n$. These are the elements of degree 1. The rest of the algebra is generated by products of elements of this form.

In general, a differential form of homogenous degree d looks like this.

$$\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_d \le n} a_{i_1, i_2, \ldots i_d} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_d}$$

The elements dx^i in this sum are stuck together by an associative product, the *wedge product*, which acts on differential forms as follows.

Let ω and η be two differential forms, and let f be a smooth function from U to $\mathbb{R}$. Then

$$f \wedge \omega = \omega \wedge f = f\omega.$$

Here $f\omega$ is the differential form obtained from ω by multiplying all the coefficients by f. In particular, if g is another smooth function, $f \wedge g = g \wedge f$ is the same as fg.

Also, functions pass through the wedge product.

$$f\omega \wedge \eta = \omega \wedge f\eta.$$

For all elements dx^i, dx^j, the product is antisymmetric.

$$dx^i \wedge dx^j = -dx^j \wedge dx^i.$$

In particular,

$$dx^i \wedge dx^i = 0.$$

So, if we have two forms $\alpha = \sum a_j\, dx^j$ and $\beta = \sum b_j\, dx^j$, where a_j and b_j are smooth functions $U \to \mathbb{R}$, then

$$\alpha \wedge \beta = \sum_{j<l} (a_j b_l - a_l b_j)\, dx^j \wedge dx^l = -\beta \wedge \alpha.$$

In general, we can permute the elements of a monomial form like

$$dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_d}$$

in any way we like by successive transpositions of adjacent elements, provided that we reverse the sign if the permutation is odd.

Convince yourself that if α and β are differential forms of homogenous degree a and b in the generating elements dx^j, then

$$\alpha \wedge \beta = (-1)^{ab} \beta \wedge \alpha.$$

A general form is the sum of forms of degree 0, 1, $\ldots$, n. No form can have degree more than n, since any monomial in a wedge of $n+1$ terms has to look like $(\cdots \wedge dx^i \wedge \cdots \wedge dx^i \wedge \cdots)$, which is zero by the antisymmetry rules.

For example, in $\mathbb{R}^3$, the differential forms are linear combinations of these forms.

degree 0	f
degree 1	$f\,dx^1 + g\,dx^2 + h\,dx^3$
degree 2	$f\,dx^2 \wedge dx^3 + g\,dx^3 \wedge dx^1 + h\,dx^1 \wedge dx^2$
degree 3	$f\,dx^1 \wedge dx^2 \wedge dx^3,$

where $f, g, h \in C^\infty(U)$.

In dimension n, the $C^\infty(U)$-vector space of all the differential forms of degree d has dimension $\binom{n}{d}$.

A digression about the Maxwell equations. For another example, in $\mathbb{R}^4$, the forms of degree 2 all look like

$$-E_x\; dt \wedge dx - E_y\; dt \wedge dy - E_z\; dt \wedge dz + \\ B_x\; dy \wedge dz + B_y\; dz \wedge dx + B_z\; dx \wedge dy.$$

Here $E_x, E_y, E_z, B_x, B_y, B_z \in C^\infty(U)$. Then Maxwell's equations look like

$$^*d\omega = ic\nabla \cdot \mathbf{B}\; dt + \frac{1}{ic}\left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times E\right) = 0$$

$$^*d^*\omega = \nabla \cdot \mathbf{E}\; dt + \left(-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B}\right) = \frac{q}{\varepsilon_0}\, dt + \frac{\mathbf{j}}{\varepsilon_0 c^2}$$

which is pretty cool.

Here the '$*$' operator is defined in terms of the Riemannian metric, which is where 'c' comes in. It works as follows: Pick an orientation on $\mathbb{R}^n$, or just pick some form ω of degree n. Let $\omega_1, \omega_2, \ldots, \omega_n$ be any orthonormal basis in $\mathbb{R}^n$ that has that orientation, or any orthonormal basis for which the wedge $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$, which is always equal to $J\omega$ for some constant $J \neq 0$, is set up so that $J > 0$. Then

$$*\,(\omega_j \wedge \omega_{j+1} \wedge \cdots \wedge \omega_n) = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{j-1}$$

and for any permutation $\sigma = \sigma_1 \cdots \sigma_{j-1}\sigma_j \cdots \sigma_n$,

$$^*(\omega_{\sigma(j)} \wedge \omega_{\sigma(j+1)} \wedge \cdots \wedge \omega_{\sigma(n)}) = \operatorname{sign}(\sigma)\, \omega_{\sigma(1)} \wedge \omega_{\sigma(2)} \wedge \cdots \wedge \omega_{\sigma(j-1)}$$

and we can extend by linearity.

We made a choice of basis. The definition only depends on the orientation and not on the basis, but this isn't obvious (well, it may be obvious to you).

For example, an orthonormal basis in Minkowski $\mathbb{R}^4$ is $dx, dy, dz, ic\; dt$. Ordinarily our differential forms have only real components, but since we are hanging out in Minkowski space we have to make some concessions. We have $^*(dz \wedge dt) = (1/ic)^*(dz \wedge ic\; dt) = (1/ic)\, dx \wedge dy$, for example.

The operator 'd'. Let f be a function. Then we say

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}\, dx^j.$$

Notice especially that $d(x^j) = dx^j$.

Let $\omega = f\, dx^I$ be a monomial differential form. Here I is supposed to be a sequence $i_1, \cdots, i_d$, so dx^I means $dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_d}$. Then we define d by

$$d\omega = df \wedge dx^I,$$

extending by linearity to all differential forms.

This operation increases the degree of a differential form by one. Forms of degree n vanish under the operation d, since any form of degree $n+1$ has to be zero.

It also has the property that

$$d^2\omega = 0.$$

This is because

$$d^2 f\, dx^I = d\left(\frac{\partial f}{\partial x^j}\, dx^j \wedge dx^I\right) = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\, dx^i \wedge dx^j \wedge dx^I\right) = 0$$

by antisymmetry and the fact that partial derivatives commute.

Also, if ω and η are two differential forms of degree d and e, then

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^d\, \omega \wedge d(\eta).$$

First, make the observation that

$$d(fg) = g\, df + f\, dg.$$

If we consider monomials $\omega = f\, dx^I$ and $\eta = f\, dx^J$, then

$$\begin{aligned} d(\omega \wedge \eta) &= d(fg\, dx^I \wedge dx^J) \\ &= (g\, df + f\, dg) \wedge dx^I \wedge dx^J \\ &= (df \wedge dx^I) \wedge (g\, dx^J) + (-1)^d (f\, dx^I) \wedge (dg \wedge dx^J) \\ &= d(\omega) \wedge \eta + (-1)^d \omega \wedge d(\eta). \end{aligned}$$

The factor of $(-1)^d$ comes from exchanging dg with dx^I.

The pullback map. Let $F : U \to U'$ be a smooth map between two open sets, $U \subset \mathbb{R}^n$, and $U' \subset \mathbb{R}^m$; then we define the *pullback map* F^* as follows. If $\omega = f\, dx'^{i_1} \wedge dx'^{i_2} \wedge \cdots \wedge dx'^{i_d}$ is a differential form on U', then $F^*(\omega)$ is

$$F^*(\omega) = (f \circ F)\, d(x'^{i_1}) \wedge d(x'^{i_2}) \wedge \cdots \wedge d(x'^{i_d})$$

We are thinking of x'^j as functions on U, so

$$d(x'^j) = \frac{\partial F_j}{\partial x^i}\, dx^i.$$

We now have the really nice property that $dF^* = F^*d$. Let's see why. Let $\omega = f\,dx'^I$ be a monomial differential form. We compare the value of $F^*d\omega$ and $dF^*\omega$.

$$F^*d\omega = F^*\left(\frac{\partial f}{\partial x'^j}\,dx'^j \wedge dx'^I\right) = \frac{\partial f}{\partial x'^j}\frac{\partial x'^j}{\partial x^i}dx^i \wedge dx'^I = d(f\circ F)\wedge dx'^I$$

and

$$dF^*\omega = d[(f\circ F)\,dx'^I] = d(f\circ F)\wedge dx'^I$$

since $d(dx'^j) = d^2x'^j = 0$. The general case follows by linearity.

Another important property is that, when we have two maps

$$U \xrightarrow{F} U' \xrightarrow{G} U'',$$

then $(G\circ F)^* = F^*\circ G^*$. This map takes differential forms on U'' to differential forms on U.

This is easy to check for functions, and for elements dx''^i we write

$$(G\circ F)^*(dx''^i) = \frac{\partial x''^i}{\partial x^j}\,dx^j = \frac{\partial x''^i}{\partial x'^m}\frac{\partial x'^m}{\partial x^j}\,dx^j = F^*\left(\frac{\partial x''^i}{\partial x'^m}\,dx^m\right) = F^*(G^*(dx''^i)).$$

A similar computation works for all monomial differential forms, and so it's true for all forms by linearity.

The definition of a differentiable manifold. A topological manifold of dimension m is a topological space M where each point $x\in M$ has a neighbourhood U_x which is homeomorphic to an open set in $\mathbb{R}^m$. These homeomorphisms $\xi_x : U\subset \mathbb{R}^m \to U_x$ are called coordinate charts.

A differentiable manifold of dimension m is a manifold where we have fixed a collection of coordinate charts, and where they match up smoothly. This means that if $U_x\cap U_y$ isn't empty, and $U'_x = \xi_x^{-1}(U_x)$, then

$$\xi_y^{-1}\circ\xi_x : U'_x\cap\xi_x^{-1}(\xi_y(U'_y)) \to U'_y\cap\xi_y^{-1}(\xi_x(U'_x))$$

is a diffeomorphism between open subsets of $\mathbb{R}^n$.

We say that a map $F : M\to N$ between two manifolds is smooth if for any two points $x\in M$ and $y\in N$, the composite map $\xi_y^{-1}\circ F\circ\xi_x$, a map from a subset of $\mathbb{R}^m$ to a subset of $\mathbb{R}^n$, is smooth on the open set $U'_x\cap(\xi_y^{-1}\circ F\circ\xi_x)^{-1}(U'_y)$ where it's defined.

The important thing that the coordinate charts give us is the concept of a differentiable function, a smooth function, and so on, on this topological space.

We now need to give some examples of manifolds, and say what it means to have a differential form on a manifold.

Examples of manifolds. We can consider $\mathbb{R}^n$ or any open subset of $\mathbb{R}^n$ to be a differentiable manifold with one coordinate chart, the identity map.

There's only one other differentiable manifold that we use in the proof of the Brouwer fixed point theorem, namely the n-dimensional sphere, S^n.

The underlying space of S^n is the set

$$S^n = \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1 \right\} \subset \mathbb{R}^n$$

given the relative topology.

Okay, so what should constitute a smooth function on a small piece of this surface? The manifold looks locally like $\mathbb{R}^n$ if we project it onto a plane. So we might as well take this sort of projection for coordinate charts.

Let $\vec{n} = e_{n+1} \in \mathbb{R}^{n+1}$ be the point at the top of the sphere, the "north pole." Let $\vec{s} = -\vec{n}$ be the antipodal point, the south pole.

We set up two charts on the sets

$$\mathit{North} = S^n \setminus \{\vec{s}\}$$

and

$$\mathit{South} = S^n \setminus \{\vec{n}\}.$$

The charts are from the *stereographic projection*, which sends a point x in S^n to the point where the ray from $\vec{s}$ to x (or from $\vec{n}$ to x) hits the plane $x^{n+1} = 0$.

The map f_N is the inverse of that projection.

$$\begin{aligned} f_N &: \mathbb{R}^n \to \mathit{North} \\ f_N(x) &= \left[\frac{2x^1}{1+r^2}, \frac{2x^2}{1+r^2}, \ldots, \frac{2x^n}{1+r^2}, \frac{1-r^2}{1+r^2} \right]. \\ f_N^{-1}(y) &= \left[\frac{y^1}{1+y^{n+1}}, \ldots, \frac{y^n}{1+y^{n+1}} \right]. \end{aligned}$$

Here $r^2 = (x^1)^2 + (x^2)^2 + \cdots + (x^n)^2$ is the radius.

These are inverses because

$$r^2 = \frac{(y^1)^2 + \cdots + (y^n)^2}{(1+y^{n+1})^2} = \frac{1-(y^{n+1})^2}{(1+y^{n+1})^2} = \frac{1-y^{n+1}}{1+y^{n+1}}, \text{ so } \frac{2}{1+r^2} = (1+y^{n+1}).$$

The map $f_S : \mathbb{R}^n \to \mathit{South}$ is the same except with the sign of y^{n+1} reversed. For example,

$$f_S^{-1}(y) = \left[\frac{y^1}{1-y^{n+1}}, \ldots, \frac{y^n}{1-y^{n+1}} \right].$$

It's obvious from the definitions that f_N and f_S match up smoothly.

Orientable manifolds. A manifold M is *oriented* if the maps $\xi_y^{-1} \circ \xi_x$ preserve the orientation of the coordinate system, in the sense that their Jacobian is always greater than zero.

$$\det \left[\frac{\partial y^i}{\partial x^j} \right] > 0.$$

The Jacobian isn't ever equal to zero, since $\xi_y^{-1} \circ \xi_x$ is a diffeomorphism. So on any connected set, the sign of the Jacobian has to be constant. (Otherwise the two open sets $U = \{J(x) > 0\}$ and $V = \{J(x) < 0\}$ would partition the set.)

A manifold M is *orientable* if there's an oriented manifold M' on the same topological space as M so that the identity map $\mathrm{id} : M \cong M'$ is a diffeomorphism.

The manifold S^n that we just defined isn't oriented; in fact, $f_S^{-1} \circ f_N$ has a negative Jacobian, since

$$f_S^{-1} \circ f_N = \frac{x}{r^2}.$$

$$\frac{\partial y^i}{\partial x^j} = \delta_{ij}\frac{1}{r^2} - 2\frac{x^i x^j}{r^4} = \frac{1}{r^2}\Big(I - 2(x/r)(x/r)^T\Big).$$

Since the matrix in parentheses is a reflection, it has determinant -1, so the Jacobian is $J = -(1/r^2)^n < 0$.

However, we can orient it by changing the sign of one of the coordinates in the map f_S, so that the Jacobian changes sign; the new manifold is compatible with the old manifold, and it's oriented. Therefore, S^n is orientable.

Differential forms on a manifold. A differential form on a manifold is defined by giving a differential form ω_x for every coordinate chart in such a way that, when two coordinate neighbourhoods U_x and U_y overlap, the form ω_x agrees with the pullback of ω_y on the set $U_x' \cap \big(\xi_x^{-1} \circ \xi_y\big)(U_y')$.

$$\begin{array}{ccc} U_x' & \xrightarrow{\xi_y^{-1}\circ\xi_x} & U_y' \\ \omega_x = \omega_x' & \leftarrow & \omega_y \end{array}$$

Then if we have a map $F : M \to N$ between two manifolds and a differential form ω on N, we can define the pullback of ω along F as follows. On every point x' in the set U_x', the point $F(x')$ is in some coordinate neighbourhood U_y. Then ω_x around x' is set equal to $(\xi_y^{-1} \circ F \circ \xi_x)^*(\omega_y)$ in a neighbourhood of the point $\xi_y^{-1}(F(x'))$.

This doesn't depend on the choice of the coordinate neighbourhood U_x, and it results in a smooth differential form defined everywhere in U_x'.

Let's make a technical observation for later. If we have a Hausdorff manifold M, a coordinate neighbourhood U_x, and a differential form ω on U_x', and the differential form is zero outside a compact set $C \subset U_x'$, then we can extend ω to the whole manifold. We write down its values on other sets U_y' as follows.

- On the open set $U_y' \cap \xi_y^{-1}(U_x)$, we set ω_y to the pullback $(\xi_y^{-1} \circ \xi_x)^*(\omega)$.
- The set $\xi_x(C)$ is closed, since $\xi_x(C)$ is compact and M is Hausdorff. On the open set $U_y' \cap \xi_y^{-1}(M \setminus \xi_x(C))$, we set ω_y to zero.

If a point y' is in both sets, then the definitions agree, since ω is zero outside C.

The fact that we can make a form this way is used implicitly in "A closed form which isn't exact".

Cohomology.

We saw that the operator d increases the degree of a homogenous form by one. Fix a manifold M. Consider the sequence

$$0 \longrightarrow \begin{Bmatrix}\text{forms of}\\ \text{degree } 0\end{Bmatrix} \xrightarrow{d} \begin{Bmatrix}\text{forms of}\\ \text{degree } 1\end{Bmatrix} \xrightarrow{d} \quad \cdots \quad \xrightarrow{d} \begin{Bmatrix}\text{forms of}\\ \text{degree } n\end{Bmatrix} \longrightarrow 0.$$

These sets have an $\mathbb{R}$-vector space structure, and d is $\mathbb{R}$-linear on them. So even without any further information, the following questions already make sense to ask.

- Which forms are in the kernel of d? That is, when is $d\omega = 0$? These are called the *closed forms*, and the set of closed forms of degree j will be denoted by $Z^j(M)$.
- Which forms are in the image of d? That is, when does there exist ω' so that we have $d\omega' = \omega$? These are the *exact forms*, which we denote by $B^j(M)$.

For one thing, we know that $d^2 = 0$, so every exact form $\omega = d\omega'$ is closed since $d\omega = d^2\omega' = 0$. So, the vector space $B^j(M)$ is contained in the vector space $Z^j(M)$, and we can take the quotient.

Let the *j-th cohomology group* of M be

$$H^j(M) = \text{the closed forms of degree } j \text{ modulo the exact forms} = \frac{Z^j}{B^j}.$$

This group measures how much the statement "closed forms are exact" is false. In particular, if it's trivial, then every closed form of degree j on M is exact.

A map $F : M \to N$ induces a map $F^* : H^j(N) \to H^j(M)$. This is because an element of $H^j(N)$ is an equivalence class of forms $\omega + d\omega'$, where ω' is a form of degree $j-1$. Therefore,

$$F^*(\omega + d\omega') = F^*(\omega) + F^*(d\omega') = F^*(\omega) + dF^*(\omega') \in H^j(M)$$

is well-defined; furthermore, this map is linear, and $(G \circ F)^* = F^* \circ G^*$.

Poincaré lemma. Now we can find the cohomology of $\mathbb{R}^n$. We claim that

$$H^j(\mathbb{R}^n) \cong \begin{cases} 0 & \text{if } j \neq 0 \\ \mathbb{R} & \text{if } j = 0. \end{cases}$$

Here by 0 we mean the zero vector space.

We do this by induction. We start with $\mathbb{R}^1$. Consider $H^0(\mathbb{R}^1)$. The only exact form of degree 0 is the zero form. What are the closed forms of degree 0? These are the functions f for which

$$df = \frac{\partial f}{\partial x^1} dx^1 = 0,$$

but for the derivative to be zero everywhere, f must be constant. So the quotient space $H^0(\mathbb{R}^1)$ is just the set of constant functions, which is a 1-dimensional $\mathbb{R}$-vector space.

All forms of degree 1 on $\mathbb{R}^1$ are closed; what forms are exact? For $\omega = f\,dx^1$ to be an exact form, we have to have $\omega = dF$ for some function F. But if we choose

$$F = \int_0^x f(\xi)\,d\xi,$$

then $dF = f\,dx^1$, so all 1-forms are exact too. It follows that $H^1(\mathbb{R}^1) = 0$.

Now we say that $H^j(\mathbb{R}^n) \cong H^j(\mathbb{R}^n \times \mathbb{R})$ for every nonnegative integer j; in fact, for any manifold, it's true that $H^j(M) \cong \mathbb{H}^j(M \times \mathbb{R})$.

How come? Let's call the extra coordinate t. Consider the sequence of functions

$$\begin{array}{ccccc} \mathbb{R}^n \times \mathbb{R} & \xrightarrow{r} & \mathbb{R}^n & \xrightarrow{i} & \mathbb{R}^n \times \mathbb{R}, \\ H^j(\mathbb{R}^n \times \mathbb{R}) & \xleftarrow{r^*} & H^j(\mathbb{R}^n) & \xleftarrow{i^*} & H^j(\mathbb{R}^n \times \mathbb{R}) \end{array}$$

where i is the inclusion map $x \mapsto (x, 0)$, and r is the retraction, $(x,t) \mapsto x$.

We have $r \circ i = id$, so $(r \circ i)^* = i^* \circ r^*$ is the identity on forms, and so also on every space $H^j(\mathbb{R}^n)$.

The map $(i \circ r)^*$ doesn't act like the identity on forms; in fact, the map $i \circ r$ sends (x,t) to $(x,0)$.

Each monomial form either has dt in it or it doesn't; if it has dt in it, then we write it as $f\,dt \wedge dx^I$, where $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_j}$, and none of the dx^i is dt. We might have to change the sign to rearrange it so that dt is first. If it doesn't contain dt, then we write it $f\,dx^I$.

Then by looking at the definition of the pullback, we can write down the action of $(i \circ r)^*$. On functions, it acts like $(i \circ r)^*\big(f(x,t)\big) = f(x,0)$; i.e., the value of the new function at (x,t) is the value of the old function at $(x,0)$. In general the action is like this.

$$\begin{aligned} (i \circ r)^*\big(f(x,t)\,dx^I\big) &= f(x,0)\,dx^I \\ (i \circ r)^*\big(f(x,t)\,dt \wedge dx^I\big) &= 0 \end{aligned}$$

Here $(i \circ r)^*(dx^i) = d(x^i) = dx^i$, but $(i \circ r)^*(dt) = d(0) = 0$. This isn't the identity map on forms, but as a map between cohomology groups it is! To see this, we make up a map T, which *decreases* the degree of forms by 1, so that

$$id - (i \circ r)^* = dT + Td$$

on forms. If we can find such a map, the difference $id - (i \circ r)^*$ takes closed forms to exact forms, so on cohomology $id - (i \circ r)^* = 0$, which means that $id = (i \circ r)^*$.

Here is T.

$$\begin{aligned} T\big(f(x,t)\,dt \wedge dx^I\big) &= \left(\int_0^t f(x,t')\,dt'\right) dx^I \\ T(f(x,t)\,dx^I) &= 0. \end{aligned}$$

Now we just work out $dT + Td$.

$$\begin{aligned} dT(f\,dx^I) + Td(f\,dx^I) = T(df)\,dx^I &= \left(\int_0^t \frac{\partial f}{\partial t}\,dt'\right)dx^I \\ &= \big[f(x,t) - f(x,0)\big]\,dx^I \end{aligned}$$

This is the same as $id - (i \circ r)^*$ on forms $f\,dx^I$.

On forms $f\,dt \wedge dx^I$,

$$\begin{aligned} dT(f\,dt \wedge dx^I) &= \left(\frac{\partial}{\partial x^j}\int_0^t f(x,t')\,dt'\right)dx^j \wedge dx^I \\ &\quad + \left(\frac{\partial}{\partial t}\int_0^t f(x,t')\,dt'\right)dt \wedge dx^I \\ Td(f\,dt \wedge dx^I) &= T\left[\frac{\partial}{\partial x^j} f(x,t)\,dx^j \wedge dt \wedge dx^I\right] \\ &= -\left(\int_0^t \frac{\partial}{\partial x^j} f(x,t')\,dt'\right)dx^j \wedge dx^I \end{aligned}$$

The minus sign comes from reversing dx^j and dt to apply T. Most of the terms of the sum $(dT + Td)\big(f\,dt \wedge dx^I\big)$ cancel, and the result is just $f(x,t)\,dt \wedge dx^I = id = id - (i \circ r)^*$, which is exactly right.

So $id = (i \circ r)^*$ on cohomology. Since $r^* \circ i^* = id$ and $i^* \circ r^* = id$, these two maps $i^* : H^j(\mathbb{R}^n \times \mathbb{R}) \to H^j(\mathbb{R}^n)$ and $r^* : H^j(\mathbb{R}^n) \to H^j(\mathbb{R}^n \times \mathbb{R})$ must be isomorphisms.

By induction, $H^j(\mathbb{R}^n) \cong H^j(\mathbb{R})$, which proves the lemma.

So,

$$H^j(\mathbb{R}^n) \cong \begin{cases} 0 & \text{if } j > 0 \\ \mathbb{R} & \text{if } j = 0. \end{cases}$$

This means that the cohomology group $H^n(\mathbb{R}^{n+1})$ is trivial when $n \geq 1$.

But $H^n(S^n)$ isn't trivial.

We construct a form ω on S^n of degree n so that there exists no ω' for which we have $d\omega' = \omega$. To do this, we introduce the concept of integration of a differential form.

Let M be a compact oriented n-dimensional manifold. Let ω be a differential form of degree n. The integral

$$\int_M \omega$$

is defined as follows. Let $U_1, U_2, \ldots, U_m$ be a finite cover of M by coordinate neighbourhoods, and let $\varphi_1, \varphi_2, \ldots, \varphi_m \in C^\infty(M)$ be a subordinate *partition of unity*. What this means is that

$$0 \le \varphi_j(x) \le 1 \text{ for } j = 1 \ldots m \text{ and all } x \in M.$$

The closure of the set $\{\varphi_j(x) > 0\}$ is contained in U_j.

$$\varphi_1(x) + \varphi_2(x) + \cdots + \varphi_m(x) \equiv 1.$$

We won't prove that $\varphi_1, \varphi_2, \ldots, \varphi_m$ always exist, because it would take too long, but it's not very hard and it's proven all over the place.

Integrals on coordinate charts. On each coordinate neighbourhood U, an n-form ω has an associated differential form ω_U on $U' \subset \mathbb{R}^n$. Since ω_U is an n-form on a subset of $\mathbb{R}^n$, it's a monomial,

$$\omega_U = f_U\, dx^1 \wedge \cdots \wedge dx^n,$$

where f_U is a function on U'.

Let the integral of ω over the open set U be

$$\int_U \omega = \int_{U'} f_U\, dx^1\, dx^2 \cdots dx^n.$$

Changes of coordinates don't affect the integral. Suppose that we have two coordinate neighbourhoods U and V, and also suppose that $\omega \equiv 0$ outside $U \cap V$. (It makes sense to say ω is or isn't zero at a point x, because the pullback of zero is zero, so if $\omega_U(\xi^{-1}(x)) = 0$ in one chart then it's zero for every other chart too.)

Let the coordinates in U be called $x^1, \ldots, x^n$, and the coordinates in V be called $y^1, \ldots, y^n$. By the definition of a differential form on a manifold, on the sets in U' and V' corresponding to the intersection $U \cap V$,

$$f_V\, dy^1 \wedge \cdots \wedge dy^n = (\xi_V^{-1} \circ \xi_U)^*(f_U\, dx^1 \wedge \cdots \wedge dx^n) = f_U\, d(x^1) \wedge \cdots \wedge d(x^n).$$

But

$$\begin{aligned} &d(x^1) \wedge \cdots \wedge d(x^n) = \\ &\left(\frac{\partial x^1}{\partial y^1} dy^1 + \cdots + \frac{\partial x^1}{\partial y^n} dy^n\right) \wedge \cdots \wedge \left(\frac{\partial x^n}{\partial y^1} dy^1 + \cdots + \frac{\partial x^n}{\partial y^n} dy^n\right) = \\ &\qquad \sum_{\substack{\sigma \text{ permutation} \\ \text{of } 1 \ldots n}} \operatorname{sign}(\sigma) \frac{\partial x^{\sigma(1)}}{\partial y^1} \cdots \frac{\partial x^{\sigma(n)}}{\partial y^n}\, dy^1 \wedge \cdots \wedge dy^n = \\ &\qquad\qquad \det\left[\frac{\partial x^i}{\partial y^j}\right] dy^1 \wedge \cdots \wedge dy^n. \end{aligned}$$

This is just the Jacobian, J, times $dy^1 \wedge \cdots \wedge dy^n$.

By the change of variables formula,

$$\int_U \omega = \int_{U' \cap V} f_U\, dx^1\, dx^2 \cdots dx^n = \int_{U \cap V'} f_U\, |J|\, dy^1\, dy^2 \cdots dy^n.$$

The manifold M is oriented, which is just the statement that $J > 0$. So, the absolute value doesn't do anything, and

$$\int_U \omega = \int_{U' \cap V} f_U = \int_{U \cap V'} f_V J = \int_V \omega.$$

The whole manifold. Define the integral of an arbitrary form ω by partitioning it into integrals on the coordinate charts and adding them up.

$$\int_M \omega = \sum_{j=1}^{m} \int_{U_j} \varphi_j \, \omega.$$

The value of the integral doesn't depend on the partition, since if U_j, φ_j is one partition and V_i, ψ_i is another partition,

$$\int_M \omega = \sum_j \int_{U_j} \varphi_j \, \omega = \sum_j \sum_i \int_{U_j \cap V_i} \varphi_j \, \psi_i \, \omega = \sum_i \int_{V_i} \psi_i \, \omega.$$

Stokes's theorem on manifolds. There is a version of Stokes's theorem on manifolds, which we quote without proof. It states that, on an oriented n-manifold M, then for any $(n-1)$-form ω on M,

$$\int_M d\omega = \int_{\partial M} \omega,$$

where ∂M is the "boundary" of M, an $(n-1)$-dimensional manifold. The manifold S^n has no boundary, so Stokes's theorem just looks like

$$\int_M d\omega = 0,$$

for any $(n-1)$-form.

A closed form which isn't exact. Now we can easily pick a closed form which isn't exact. We take a form which is zero outside a coordinate neighbourhood, and whose integral is positive on that neighbourhood; this is easy to do.

Then such a form has a positive integral taken over the whole manifold, and so it can't be exact since every exact form has integral zero. It follows that $H^n(S^n)$ isn't trivial (in fact, it's isomorphic to $\mathbb{R}$).

No-retraction theorem. We can now prove the no-retraction theorem, which implies Brouwer's fixed point theorem.

MAIN THEOREM. *There is no retraction $r : D^{n+1} \to S^n$ from the unit $(n+1)$-disk onto the unit n-sphere.*

Assume that such a retraction exists. Then we have the following lemma.

SMOOTHING LEMMA. *If there is a retraction $r : D^{n+1} \to S^n$, then there's a smooth retraction $\tilde{r}$ from $\mathbb{R}^{n+1}$ onto S^n.*

We start with the retraction r. Then we go through some steps.

Step 1. Let $r_1 : \mathbb{R}^{n+1} \to S^n$ be the map

$$r_1(x) = \begin{cases} x/|x| & \text{if } |x| > 1/2, \text{ and} \\ r(2x) & \text{if } |x| \leq 1/2. \end{cases}$$

which is the radial projection when $|x| > 1/2$, and a small copy of r when $|x| \leq 1/2$.

This function is continuous, since r is a retraction.

Step 2. Now r_1 is smooth outside the disk of radius 1/2, since the radial projection is smooth.

Consider the disk of radius 1. We have a function $r_1\big|_{D^{n+1}} : D^{n+1} \to S^n$, and we think of it at the moment as a function onto $\mathbb{R}^{n+1}$.

Let R be a C^∞ function $D^{n+1} \to \mathbb{R}^{n+1}$ which approximates the continuous map r_1 accurately enough that $|R(x) - r_1(x)| < 1$ when $x \in D^{n+1}$. Such a function exists since e.g. by applying the Stone-Weierstrass theorem we can approximate any continuous function by polynomials on a compact subset of $\mathbb{R}^{n+1}$.

Now let $\lambda(r)$ be a C^∞ function which is 1 when $r < 3/5$ and zero when $r > 4/5$.

Finally, let $r_2 : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be

$$r_2(x) = r_1(x) + \lambda(|x|)(R(x) - r_1(x)).$$

Here $R(x)$ is only defined on the disk of radius 1, but that's okay because λ is identically zero when $r \geq 1$ anyway.

This function is not quite a retraction. We have $r_2(x) = r_1(x) = x$ on the unit sphere, but a retraction would have to be onto S^n, and this function is only near S^n, not actually on it.

However, $|R(x) - r_1(x)| < 1$, so $r_2(x) \neq 0$ for any x. Therefore, the final step is a radial projection.

Step 3. Compose r_2 with the radial projection $x \mapsto x/|x|$. We get a map $\tilde{r}$ from $\mathbb{R}^{n+1}$ onto the sphere. This makes sense because r_2 is never zero, and since the radial projection is smooth away from zero, the composition is still smooth. Furthermore, $r_2(x) = x$ when x is on the sphere, so the result is a retraction. We call this map $\tilde{r}$.

The result is a smooth retraction. □

The proof is anticlimactic. Consider the maps

$$S^n \xrightarrow{i} \mathbb{R}^{n+1} \xrightarrow{r} S^n.$$

The composition of the retraction with the inclusion is $r \circ i$, which is the identity, so the pullback of this, $(r \circ i)^* = i^* \circ r^*$, is the identity too.

The cohomology group $H^n(\mathbb{R}^{n+1})$ is trivial, so the pullback $r^* : H^n(S^n) \to H^n(\mathbb{R}^{n+1})$ has to be zero. That means that the composition of the pullback of the inclusion, i^*, with r^* is zero; but the composition is also the identity map on $H^n(S^n)$, a nontrivial vector space.

Here is a picture.

$$H^n(S^n) \xleftarrow{i^*} 0 \xleftarrow{r^*} H^n(S^n).$$

But on the other hand,

$$i^* \circ r^* = (r \circ i)^* \neq 0.$$

This is a contradiction. Therefore, no such retraction exists, and the Brouwer fixed point theorem is true.

$\square$