MA 101/209 MIDTERM EXAM 2 SOLUTIONS

Marks

1. Evaluate these integrals:

[4] (a) $\int xe^{3x}dx$ $= \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx$

Parts: $u = x$ $\quad du = dx$

$dv = e^{3x}dx$ $\quad v = \frac{1}{3}e^{3x}$ $\quad = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

Answer $\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

[4] (b) $\int_0^{\pi/2} \cos^5\theta \, d\theta$

$\int \cos^4\theta \cos\theta \, d\theta = \int (1-\sin^2\theta)^2 \cos\theta \, d\theta$ $\qquad$ let $u = \sin\theta$

$du = \cos\theta \, d\theta$

$= \int (1-u^2)^2 du = \int (1-2u^2+u^4) du$

$= u - \frac{2}{3}u^3 + \frac{u^5}{5} + C = \sin\theta - \frac{2}{3}\sin^3\theta + \frac{\sin^5\theta}{5} + C$

$\int_0^{\pi/2} \cos^5\theta \, d\theta = \left[\sin\theta - \frac{2}{3}\sin^3\theta + \frac{\sin^5\theta}{5}\right]_0^{\pi/2} = 1 - \frac{2}{3} + \frac{1}{5} = \frac{15-10+3}{15}$

Answer $1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$ =

2. Evaluate these integrals:

[4] (a) $\int \frac{dx}{4-x^2}$

$$\frac{1}{4-x^2} = \frac{1}{(2+x)(2-x)} = \frac{A}{2+x} + \frac{B}{2-x}$$

$$= \frac{(2-x)A + (2+x)B}{(2-x)(2+x)}$$

$$= \frac{(B-A)x + 2(A+B)}{(2-x)(2+x)}$$

$$\Rightarrow B-A=0$$
$$2(A+B)=1$$
$$\Rightarrow A=B=1/4$$

$$= \frac{1}{4}\int \frac{dx}{2+x} + \frac{1}{4}\int\frac{dx}{2-x}$$

$$= \frac{1}{4}\ln|2+x| - \frac{1}{4}\ln|2-x| + C$$

OR $\frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| + C$

Answer
$\frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| + C$

[4] (b) $\int \frac{dx}{\sqrt{4-x^2}}$

let $x = 2\sin\theta$
$dx = 2\cos\theta\, d\theta$
$\sqrt{4-x^2} = 2\cos\theta$

$$= \int \frac{2\cos\theta\, d\theta}{2\cos\theta} = \int d\theta$$

$$= \theta + C$$

$$= \sin^{-1}\left(\frac{x}{2}\right) + C$$

Answer
$\sin^{-1}\left(\frac{x}{2}\right) + C$

3. Determine convergence of these improper integrals: explain but do not evaluate.

[4] (a) $\int_0^\infty \frac{x}{x^2+1}dx$

$u = x^2+1$

$du = 2x\,dx$ so

$\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|+C$

$= \frac{1}{2}\ln(x^2+1)+C$

$= \lim_{T\to\infty} \frac{1}{2}\ln(T^2+1)$

$= \infty$

Answer: diverges

[4] (b) $\int_1^3 \frac{dx}{(x-1)^{1/3}}$

$= \lim_{t\to 1^+} \int_t^3 \frac{dx}{(x-1)^{1/3}} = \lim_{t\to 1^+} \frac{3(x-1)^{2/3}}{2}\Big|_t^3 = \frac{3\cdot 2^{2/3}}{2}.$

Answer: converges

[4] (c) $\int_0^\infty (1+\sin x)e^{-x}dx$

$0 \le (1+\sin x)e^{-x} \le 2e^{-x}$

$\int_0^\infty 2e^{-x} = \lim_{T\to\infty}\left[-2e^{-x}\right]_0^T$

$= 2e^0 = 2$

converges

By comparison

Answer: converges

[6] 4. Recalling the identity $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots$ valid for $|t| < 1 \ldots$

(a) Give the power series expansion for $\int \frac{dx}{1+x^3}$

substitute $-t = x^3$ or $t = -x^3$

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \cdots$$

$$\int \frac{dx}{1+x^3} = x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots$$

(b) Estimate $\int_0^{1/2} \frac{dx}{1+x^3}$ using the first three terms of this series (do not simplify).

$$1/2 - \frac{1}{4 \cdot 2^4} + \frac{1}{7 \cdot 2^7}$$

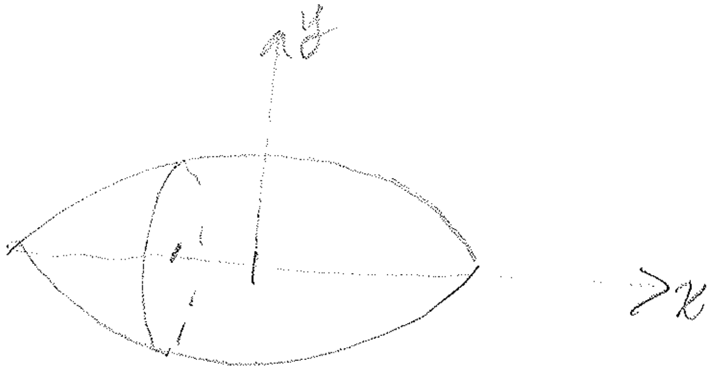
(c) Give a good upper bound to the error in that estimate, with explanation.

$$\text{Error} < \left| \frac{(1/2)^{10}}{10} \right| = \frac{1}{10{,}240}$$

Answer

$|\text{error}| < \frac{1}{10240}$

[6] 5. Find the volume enclosed by rotating the curve $y = \cos x$, $-\pi/2 \le x \le \pi/2$, about the x-axis.



The cross-section $\perp$ to x-axis at x is a circle of radius $\cos x$, area $A(x) = \pi \cos^2 x$

$$V = \int_{-\pi/2}^{\pi/2} \pi \cos^2 x \, dx = \pi \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2x}{2} \, dx$$

$$= \frac{\pi}{2} \left[x + \frac{\sin 2x}{2} \right]_{-\pi/2}^{\pi/2} = \frac{\pi}{2} [\pi]$$

Answer $\pi^2/2$