The wave operators

We have seen that the point spectrum $\sigma_{pp}(H)$ corresponds to bound states in the (weak) sense that $\psi \in \mathcal{H}_{pp}$ implies that for every $\epsilon > 0$ there exists $R > 0$ such that

$$\sup_t \|(1 - \chi_R)e^{-itH}\psi\| < \epsilon.$$ 

Here $\chi_R$ denotes the indicator function for a ball of radius $R$.

For the $\psi$ contained the continuous spectral subspace $\mathcal{H}_c$ we proved the RAGE theorem:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \|\chi_R e^{-itH}\psi\|^2 dt = 0.$$ 

This implies that a state in the continuous spectral subspace must leave a bounded region infinitely often. But the RAGE theorem does not rule out that a state could also return to a bounded set infinitely often. In fact, such behaviour is possible for states in the singular continuous spectral subspace.

The goal of time dependent scattering theory is to give a detailed description of the evolution of $e^{-itH}\psi$ as $t \to \pm \infty$ when $\psi$ is in that absolutely continuous spectral subspace $\mathcal{H}_{ac}$.

One way to describe the long time behaviour of $e^{-itH}\psi$ is to find a simpler operator $H_0$ (called the free Hamiltonian) whose time evolution approximates the real large time behaviour. Ideally, the operator $H_0$ is understood completely. For example, in (short-range) two-body scattering theory $H = -\Delta + V$ and $H_0 = -\Delta$. For the moment, we will assume that $H_0$ has purely absolutely continuous spectrum, but make no further assumptions.

The pair $(H, H_0)$ are said to be asymptotically complete if

1. For large positive and negative times, every orbit $e^{-itH}\psi$ with $\psi \in \mathcal{H}_{ac}(H)$ is approximated by a free orbit. In other words, for every $\psi \in \mathcal{H}_{ac}(H)$ there exists two vectors $\varphi_\pm$ such that

$$\lim_{t \to \pm \infty} \|e^{-itH}\psi - e^{-itH_0}\varphi_\pm\| = 0.$$ 

2. For large positive and negative times, every free orbit $e^{-itH_0}\varphi$ is the asymptotic description of some orbit under $H$. In other words, for every $\varphi \in \mathcal{H}$ there exists two vectors $\psi_\pm \in \mathcal{H}_{ac}(H)$ such that

$$\lim_{t \to \pm \infty} \|e^{-itH_0}\varphi - e^{-itH}\psi_\pm\| = 0.$$
\[ \| e^{-itH} \psi - e^{-itH_0} \psi_\pm \| = \| e^{-itH_0} (e^{itH_0} e^{-itH} \psi - \psi_\pm) \| = \| (e^{itH_0} e^{-itH} \psi - \psi_\pm) \| \]

condition (1) is equivalent to the existence of the strong limit

\[ \Omega_{\pm}(H_0, H) = \text{s-lim}_{t \to \mp \infty} e^{itH_0} e^{-itH} P_{ac} \]

while condition (2) is equivalent to the existence of the strong limit

\[ \Omega_{\pm}(H, H_0) = \text{s-lim}_{t \to \mp \infty} e^{itH} e^{-itH_0} \]

Thus, asymptotic completeness is equivalent to the existence of these four strong limits. It is usually much easier to establish the existence of the wave operators \( \Omega_{\pm}(H, H_0) \). This is because the estimates that are needed require detailed knowledge the “inside” unitary group — in this case \( e^{-itH_0} \).

**Proposition 1.1** Suppose that \( \Omega_{\pm}(H, H_0) \) exist. Then

\[ \text{Ran} \Omega_{\pm}(H, H_0) \subseteq \mathcal{H}_{ac}(H) \]

**Proof:** To begin, we note that if \( d\nu_{\psi,H} \) is the probability measure (spectral measure) associated with the observable \( H \) and the state \( \psi \), then

\[ \langle \psi, e^{-itH} \psi \rangle = \int e^{-it\lambda} d\nu_{\psi,H}(\lambda) \]

is the Fourier transform of the measure.

Secondly, we note the intertwining property

\[ e^{-itH} \Omega_{\pm}(H, H_0) = \text{s-lim}_{s \to \mp \infty} e^{-itH} e^{isH} e^{-isH_0} \]

\[ = \text{s-lim}_{s \to \mp \infty} e^{i(s-t)H} e^{-i(s-t)H_0} e^{-itH_0} \]

\[ = \Omega_{\pm}(H, H_0) e^{-itH_0} \]

Since \( \Omega_{\pm} \) are limits of unitary operators \( \| \Omega_{\pm} \psi \| = \| \psi \| \) for any \( \psi \in \mathcal{H} \). This implies \( \langle \Omega_{\pm} \psi, \Omega_{\pm} \phi \rangle = \langle \psi, \phi \rangle \). Now suppose that \( \psi \in \text{Ran} \Omega_{\pm} \). Then \( \psi = \Omega_{\pm} \phi_\pm \). Therefore

\[ \overline{d\nu_{\psi,H}}(t) = \langle \psi, e^{-itH} \psi \rangle \]

\[ = \langle \Omega_{\pm} \phi_\pm, e^{-itH} \Omega_{\pm} \phi_\pm \rangle \]

\[ = \langle \Omega_{\pm} \phi_\pm, \Omega_{\pm} e^{-itH_0} \phi_\pm \rangle \]

\[ = \langle \phi_\pm, e^{-itH_0} \phi_\pm \rangle \]

\[ = d\nu_{\phi_\pm,H_0}(t) \]

This implies \( d\nu_{\phi,H_0} = d\nu_{\psi,H} \). But \( d\nu_{\phi_\pm,H_0} \) is purely absolutely continuous, since \( H_0 \) has purely absolutely continuous spectrum. Thus \( \psi \in \mathcal{H}_{ac}(\mathcal{H}) \). \( \square \)
Proposition 1.2 Suppose that \( \Omega_{\pm}(H, H_0) \) exist. Then \( \Omega_{\pm}(H_0, H) \) exist (and asymptotic completeness holds) if and only if \( \mathcal{H}_{ac}(H) \subseteq \text{Ran} \Omega_{\pm}(H, H_0) \)

Proof: \( \Omega_{\pm}(H_0, H) \) exists if and only if for every \( \psi \in \mathcal{H}_{ac}(H) \), there exists \( \varphi_{\pm} \) with

\[
\lim_{t \to \pm \infty} \| e^{-itH} \psi - e^{-itH_0} \varphi_{\pm} \| = 0.
\]

This happens if and only if for every \( \psi \in \mathcal{H}_{ac}(H) \), there exists \( \varphi_{\pm} \) with

\[
\lim_{t \to \pm \infty} \| \psi - e^{itH} e^{-itH_0} \varphi_{\pm} \| = 0,
\]

i.e., with \( \psi = \Omega_{\pm}(H, H_0) \varphi_{\pm} \).

These two propositions show that asymptotic completeness is equivalent to the existence of the wave operators \( \Omega_{\pm}(H, H_0) \) and the equalities \( \text{Ran} \Omega_{\pm}(H, H_0) = \mathcal{H}_{ac}(H) \).

Variations on the definition of wave operators

Sometimes the wave operators \( \Omega_{\pm}(H_0, H) \) are defined using \( P_c \) rather than \( P_{ac} \), i.e.,

\[
\Omega_{\pm}(H_0, H) = \text{s-lim}_{t \to \pm \infty} e^{itH_0} e^{-itH} P_c
\]

In this case asymptotic completeness implies \( \mathcal{H}_{ac}(H) = \mathcal{H}_{ac}(H) \) so there is no singular continuous spectrum.

It is often useful to consider the situation where \( H_0 \) does not necessarily have purely absolutely continuous spectrum. In this case we replace \( e^{-itH_0} \) with \( e^{-itH_0} P_{ac}(H_0) \) in the definitions. In this situation the roles of \( H_0 \) and \( H \) are symmetrical and given asymptotic completeness, the wave operators provide a unitary equivalence of \( \mathcal{H}_{ac}(H_0) \) and \( \mathcal{H}_{ac}(H) \). In particular, if the absolutely continuous spectrum of \( H_0 \) is empty, then \( H \) does not have any absolutely continuous spectrum either. This is the situation considered in the section on trace class scattering below.

The scattering operator

If asymptotic completeness holds, the scattering operator, defined by

\[
S = \Omega_{-}(H_0, H) \Omega_{+}(H, H_0)
\]

is a unitary operator. The scattering operator has the following physical interpretation. Consider an orbit \( \psi_t = e^{-itH} \psi \). If in the distant past \( \psi_t \sim e^{-itH_0} \phi \), then in the far future \( \psi_t \sim e^{-itH_0} S \phi \).
An important property of the scattering operator is that it commutes with $e^{itH_0}$. This follows from the intertwining properties of the wave operators, which imply that

$$e^{itH_0}S = e^{itH_0}S e^{itH_0}$$

so that

$$[e^{itH_0}, S] = 0. \quad (1.1)$$

This implies that $S$ commutes with other functions of $H_0$ too. To see this we start with functions $f$ that are the Fourier transforms of nice (say $C_0^\infty$) functions. Then one can establish the formula

$$f(H_0)\psi = (2\pi)^{-1/2} \int_\mathbb{R} \hat{f}(t)e^{itH_0}\psi dt$$

Here $\psi$ is an arbitrary element of the Hilbert space, $f(H_0)$ on the left side denotes the function defined by the functional calculus and the integral on the right side is a Riemann integral of a Hilbert space valued function. From this representation and (1.1) it follows that $[f(H_0), S] = 0$ for these functions, and by a limiting argument for all bounded Borel functions $f$. This is, by definition, what it means to say that $S$ commutes with $H_0$.

Now, we know that $H_0 = -\Delta$ is diagonalized by the Fourier transform. More explicitly, $\mathcal{F}^*H_0\mathcal{F}$ is multiplication by $|k|^2$ on $L^2(\mathbb{R}^n, d^n k)$. Changing to polar co-ordinates, we write $\lambda = |k|$ and $\omega = k/|k|$ and think of $\psi(k) = \psi(\lambda \omega)$ as an $L^2(S^{n-1})$ valued function of $\lambda$. Then

$$L^2(\mathbb{R}^n, d^n k) = L^2(\mathbb{R}^+, L^2(S^{n-1}), \lambda^{n-1} d\lambda)$$

with $\mathcal{F}^*H_0\mathcal{F}$ being multiplication by $\lambda^2$. The fact that $S$ commutes with $H_0$ now implies that there is an operator $S(\lambda) : L^2(S^{n-1}) \rightarrow L^2(S^{n-1})$ such that $(S\psi)(\lambda, \omega) = S(\lambda)\psi(\lambda, \omega)$. The family of operators $S(\lambda)$ defined for almost every $\lambda$ by this procedure, actually turns out to be analytic in $\lambda$ in many cases. $S(\lambda)$ is called the scattering matrix (even though in dimensions greater than one it is actually an operator.)

**Trace class scattering**

There are two main methods in time dependent scattering theory. They are trace class methods and positive commutator methods. Positive commutator methods have led to great advances in the last fifteen years. Perhaps most importantly, positive commutator estimates are the only
known way to prove asymptotic completeness for $N$ body scattering. A good reference is *Scattering Theory of Classical and Quantum N-particle Systems*, by Dereziński and Gérard.

In this lecture I will present a sketch of the proof of the existence of wave operator under a trace condition. This circle of ideas was developed mainly by Birman, Kato, Kuroda and Pearson. The proof is taken from Barry Simon’s book on trace ideals. The advantage of this method is that the hypothesis are symmetric in $H$ and $H_0$ (called $A$ and $B$ below.) so that the proof of the existence of $\Omega_\pm (A, B)$ also proves the existence of $\Omega_\pm (B, A)$ are and thereby established asymptotic completeness. However, when this method is applied to Schrödinger operators, the decay rate required for $V$, namely better than $|x|^{-n}$ for large $|x|$, is too restrictive. Asymptotic completeness for $H = -\Delta + V$ and $H_0 = -\Delta$ actually holds as soon as $V$ decays better than $|x|^{-1}$.

Notice that to show that a strong limit of operators of the form $W_t = e^{itA} J e^{-itB} P_{ac}(B)$ exists, it suffices to show that the limit exists on a dense set $D$. Here $J$ is assumed to be a bounded operator. To see this notice

$$\|W_t\| \leq \|J\|$$

and so is bounded independently of $t$. Thus to show that $W_t \psi$ has a limit we let $\epsilon > 0$ and choose $\phi \in D$ such that $\|\psi - \phi\| < \epsilon$. Then

$$\limsup_{s, t \to \infty} \|W_t \psi - W_s \psi\| \leq \limsup_{s, t \to \infty} \|W_t \phi - W_s \phi\| + 2\|J\| \epsilon = 2\|J\| \epsilon$$

This implies that $W_t \psi$ is Cauchy and thus converges.

We will need the following lemma

**Lemma 1.3** Suppose that $A$ is self-adjoint, $C$ is bounded with $C(A + i)^{-n}$ compact for some $n$. Then

$$\text{s-lim}_{t \to \pm \infty} Ce^{-itA} P_{ac} = 0$$

**Proof:** To begin we assume that $C$ has rank one. Then $C\psi = \langle \varphi_1, \psi \rangle \varphi_2$ and we must show that for every $\phi \in \mathcal{H}$,

$$\lim_{t \to \pm \infty} \langle \varphi_1 e^{-itA} P_{ac} \phi \rangle = 0 \tag{1.2}$$

By the spectral theorem $\langle \varphi_1 e^{-itA} P_{ac} \phi \rangle$ is the Fourier transform of an absolutely continuous measure. This tends to zero for large $|t|$ by the Riemann-Lebesgue lemma. Once we know the theorem for rank one $C$, it follows easily for $C$ finite rank, since these are sums of rank one operators. Then, by an approximation argument, it follows for $C$ compact. If $C(A + i)^{-n}$ is compact, we first note that it suffices to to prove (1.2) for $\phi$ in a dense set. If $\phi \in D(A^n)$ then

$$\text{s-lim}_{t \to \pm \infty} Ce^{-itA} P_{ac} \phi = \text{s-lim}_{t \to \pm \infty} C(A + i)^{-n} e^{-itA} P_{ac} (A + i)^n \phi = 0$$

$\square$
**Theorem 1.4** Let $A$ and $B$ be self-adjoint operators. Suppose there exists a bounded operator $J$ so that $AJ - JB$ is trace class. Then

$$
\Omega_{\pm}(A, B; J) = \lim_{t \to \pm \infty} e^{iA}Je^{-itB}P_{ac}(B)
$$

exist.

**Remark:** To be precise, the condition on $AJ - JB$ means that there exists a trace class operator $C$ such that for every $\psi \in \mathcal{D}(A)$ and $\varphi \in \mathcal{D}(B)$ the equation $\langle A\psi, J\varphi \rangle - \langle \psi, JB\varphi \rangle = \langle \psi, C\varphi \rangle$ holds. It is not hard to show that this implies that if $\varphi \in \mathcal{D}(B)$ then $J\varphi \in \mathcal{D}(A)$ and $AJ\varphi = JB\varphi + C\varphi$.

We will use this below when we rewrite expressions that appear.

If $J = I$ we obtain the so-called Kato-Rosenblum theorem. However, for $H = -\Delta + V$ and $H_0 = -\Delta$ this would require $V$ to be trace class. This never happens for a multiplication operator.

The power of including the operator $J$ is demonstrated by the following corollary.

**Corollary 1.5 (Kuroda-Birman theorem)** If $(A + i)^{-1} - (B + i)^{-1}$ is trace class then the pair $(A, B)$ then $\Omega_{\pm}(A, B)$ exists.

Since the hypotheses are symmetric in $A$ and $B$ we automatically get asymptotic completeness.

**Proof:** Take $J = (A + i)^{-1}(B + i)^{-1}$. Then $AJ - JB = (A + i)^{-1} - (B + i)^{-1}$ and is compact, so we may apply Theorem 1.4 to conclude that $\Omega_{\pm}(A, B; (A + i)^{-1}(B + i)^{-1})$ exist.

Now we have

$$
\Omega_{\pm}(A, B; (A + i)^{-1}(B + i)^{-1})(B + i)\psi = \Omega_{\pm}(A, B; (A + i)^{-1})\psi
$$

so $\Omega_{\pm}(A, B; (A + i)^{-1})$ exists on the dense set $D(B)$ and hence exists.

Since $(A + i)^{-1} - (B + i)^{-1}$ is compact $\Omega_{\pm}(A, B; (A + i)^{-1} - (B + i)^{-1})$ and equals zero by the lemma. Thus

$$
\Omega_{\pm}(A, B; (B + i)^{-1}) = \Omega_{\pm}(A, B; (A + i)^{-1}) - \Omega_{\pm}(A, B; (A + i)^{-1} - (B + i)^{-1})
$$

exists. But $\Omega_{\pm}(A, B; (B + i)^{-1})(B + i)\psi = \Omega_{\pm}(A, B)\psi$ so $\Omega_{\pm}(A, B)$ exists on the dense set $D(B)$ and hence everywhere. \qed

We need the following lemma for the proof of Theorem 1.4.

**Lemma 1.6** There is a dense subset $\mathcal{M}$ of $\mathcal{H}_{ac}(A)$ such that for $\phi \in \mathcal{M}$

$$
\int_{-\infty}^{\infty} |\langle \psi, e^{-itA}\phi \rangle|^2 dt \leq C_\phi \|\psi\|^2
$$

**Proof:** We may consider the restriction of $A$ to $\mathcal{H}_{ac}(A)$. We write $\mathcal{H}_{ac}(A)$ as a direct sum of cyclic subspaces, and it suffices to consider these. So, suppose that $A$ has a cyclic vector. Then $A$
is unitarily equivalent to multiplication by \( \lambda \) on \( L^2(\mathbb{R}, d\mu(\lambda)) \) where \( d\mu \) is absolutely continuous. Then we may write \( d\mu(\lambda) = f(\lambda)^2 d\lambda \) for \( f \geq 0 \) and \( f \in L^2(\mathbb{R}) \).

\[
\langle \psi, e^{-itA}\phi \rangle = \int_{-\infty}^{\infty} e^{-it\lambda} \psi(\lambda) \phi(\lambda) f^2(\lambda) d\lambda
\]

We take \( \mathcal{M} \) to be the dense set \( \{ \phi : |\phi(\lambda) f(\lambda)| \leq C \} \). If \( \phi \) belongs to this set, then by the Plancherel theorem

\[
\int_{-\infty}^{\infty} |\langle \psi, e^{-itA}\phi \rangle|^2 dt = \|\psi(\lambda) \phi(\lambda) f^2(\lambda)\|_{L^2(\mathbb{R}, d\lambda)}^2 
\leq C^2 \|\psi(\lambda)\|_{L^2(\mathbb{R}, d\mu)}^2
\]

\( \square \)

**Proof of Theorem 1.4:** Let \( W_t = e^{itA}J e^{-itB} P_{ac}(B) \). We will show that \( \lim W_t \phi \) exists for \( \phi \in \mathcal{M} \).

Since

\[
\|(W_t - W_s)\phi\|^2 = \langle \phi, W_t^*(W_t - W_s)\phi \rangle - \langle \phi, W_s^*(W_t - W_s)\phi \rangle
\]

it suffices to show \( \lim_{s,t \to \infty} \langle \phi, W_t^*(W_t - W_s)\phi \rangle = 0 \). For convenience we consider the case \( s \leq t \).

Now we do the following calculation, initially for \( \phi \in D(B) \) so that the differentiations are allowed.

\[
\langle \phi, W_t^*(W_t - W_s)\phi \rangle = \langle \phi, e^{iaB} e^{-iaB} W_t^*(W_t - W_s) e^{iaB} e^{-iaB} \phi \rangle 
= \langle \phi, e^{iaB} W_t^*(W_t - W_s) e^{-iaB} \phi \rangle + \int_0^a \frac{d}{dw} \langle \phi, e^{iaB} \left( e^{-iwB} W_t^*(W_t - W_s) e^{iwB} \right) e^{-iaB} \phi \rangle dw 
= \langle \phi, e^{iaB} W_t^*(W_t - W_s) e^{-iaB} \phi \rangle + \int_0^a \langle \phi, e^{iaB} e^{-iwB} [W_t^*(W_t - W_s), iB] e^{iwB} e^{-iaB} \phi \rangle dw 
= \langle \phi, e^{iaB} W_t^*(W_t - W_s) e^{-iaB} \phi \rangle + \int_0^a \langle \phi, e^{iaB} [W_t^*(W_t - W_s), iB] e^{-iaB} \phi \rangle dw
\]

Here we made the change of variable \( u = a-w \).

To rewrite the second term, consider the formal calculation

\[
[W_t^*(W_t - W_s), B] = P_{ac}(B) e^{itB} (J^2 B - B J^2) e^{-itB} P_{ac}(B)
+ P_{ac}(B) e^{itB} (J e^{i(t-s)A} JB - BJ e^{i(t-s)A} J) e^{-itB} P_{ac}(B)
= P_{ac}(B) e^{itB} (J^2 B - JAJ + JAJ - B J^2) e^{-itB} P_{ac}(B)
+ P_{ac}(B) e^{itB} (J e^{i(t-s)A} JB - J e^{i(t-s)A} AJ + J e^{i(t-s)A} AJ - B J e^{i(t-s)A} J) e^{-itB} P_{ac}(B)
= P_{ac}(B) e^{itB} (-JC + CJ) e^{-itB} P_{ac}(B)
+ P_{ac}(B) e^{itB} (-J e^{i(t-s)A} C + C e^{i(t-s)A} J) e^{-itB} P_{ac}(B),
\]

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where $C$ is compact. By the remark following the statement of the theorem, we may insert this calculation into (1.3) for $\phi \in D(B)$. Now only bounded operators are involved on the right and left sides of (1.3) so by a density argument the right and left sides are equal for any $\phi$.

Now we let $a \to \infty$. The first term on the left side can be estimated using

$$\left| \langle \phi, e^{iaB} W_t^*(W_t - W_s) e^{-iaB} \phi \rangle \right| = \left| \int_s^t \langle \phi, e^{iaB} W_t^* (e^{iaA} (A - JB) e^{-iaB} P_{ac}(B)) e^{-iaB} \phi \rangle \, du \right|$$

$$\leq \int_s^t \|\phi\| \|J\| \|C e^{-(a+u)B} P_{ac}(B)\| \|\phi\| \, du,$$

where $C$ is trace class. This tends to zero as $a \to \infty$ by the Lemma and dominated convergence.

To estimate the second term, we use that we have replaced the quantity $[W_t^* (W_t - W_s), iB]$ on the right of (1.3) by a sum of terms of the form $Y(s,t) C e^{-isB}$ or its adjoint, where $Y(s,t)$ is uniformly bounded in $s$ and $t$ and $C$ is trace class. Let

$$C = \sum \mu_n \langle \psi_n, \cdot \rangle \eta_n$$

with $\sum \mu_n < \infty$. Then we must estimate terms like

$$\int_0^\infty \langle \phi, e^{iuB} Y(s,t) C e^{-isB} e^{-iuB} \phi \rangle \, du$$

$$\leq \int_0^\infty \sum \mu_n \langle \phi, e^{iuB} Y(s,t) \eta_n \rangle \langle \psi_n, e^{-isB} e^{-iuB} \phi \rangle \, du$$

$$\leq \int_0^\infty \sum \mu_n |\langle e^{-iuB} \phi, Y(s,t) \eta_n \rangle | \langle \psi_n, e^{-i(s+u)B} \phi \rangle | \, du$$

$$\leq \left\{ \int_0^\infty \sum \mu_n \langle e^{-iuB} \phi, Y(s,t) \eta_n \rangle^2 \, du \right\}^{1/2} \left\{ \int_s^\infty \sum \mu_n |\psi_n, e^{iuB} \phi\rangle|^2 \, du \right\}^{1/2}$$

Since $\phi \in \mathcal{M}$ we have

$$\int_0^\infty \sum \mu_n |\langle e^{-iuB} \phi, Y(s,t) \eta_n \rangle|^2 \, du \leq \sum \mu_n \int_{-\infty}^\infty |\langle e^{-iuB} \phi, Y(s,t) \eta_n \rangle|^2 \, du$$

$$\leq \sum \mu_n \|Y_{s,t}\|^2 C_{\phi}$$

$$\leq C'_\phi$$

Similary we find that $\sum \mu_n \langle \psi_n, e^{iuB} \phi \rangle^2 \in L^1(du)$ Hence

$$\lim_{s \to \infty} \int_s^\infty \sum \mu_n \langle \psi_n, e^{i(u)B} \phi \rangle^2 = 0$$

To conclude lets compute what kind of decay is required for $V$ in dimension 3 to apply this method to $H_0 = -\Delta$ and $H = -\Delta + V$. The difference of resolvents is

$$(H_0 + i)^{-1} - (H + i)^{-1} = (H_0 + i)^{-1} V(H + i)^{-1} = (H_0 + i)^{-1} V(H_0 + i)^{-1} (H_0 + i)(H + i)^{-1}$$
If $V$ is $-\Delta$ bounded with bound less than one, then $(H_0 + i)(H + i)^{-1}$ is bounded. So it suffices to show that $(H_0 + i)^{-1}|V|^{1/2}$ is Hilbert-Schmidt. Since $(p^2 + i)^{-1} \in L^2(\mathbb{R}^3)$ it suffices that $|V|^{1/2} \in L^2(\mathbb{R}^3)$. So if $V(x) \sim |x|^{-\alpha}$ for large $|x|$ we require $|x|^{2-\alpha}$ be integrable, or that $\alpha > 3$.

Dimensions $n \geq 3$ require more sophisticated ways of proving that an operator is trace class. Simon’s trace ideal book shows how to handle potentials decreasing like $|x|^{-n-\epsilon}$ in $n$ dimensions.