Math 301 Lecture 20

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Goal: Given \( z_1, z_2, z_3 \), \( w_1, w_2, w_3 \), find a FLT that maps
\( z_i \rightarrow w_i \), \( i = 1, 2, 3 \)

Subgoal: Find FLT that maps \( z_1 \rightarrow 0 \), \( z_2 \rightarrow 1 \), \( z_3 \rightarrow \infty \)

\[
f(z) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}
\]

Equilateral

\[
= \left( z_1, z_2, z_3 \right)
\]

\[
\lim_{z_1 \to \infty} \frac{(z_2-z_3)}{(z_2-z_1)} = \lim_{z_1 \to \infty} \frac{(z_2-z_3)}{(z_2-z_1)}
\]

Note:
\[
(2 \infty, z_2, z_3) = \frac{(z_2-z_3)}{(z-z_3)} \quad (z_2, z_1, z_3) = \frac{(z_2-z_1)}{(z-z_1)}
\]

\[
(z, z_1, z_2, z_3) = \frac{(z_2-z_1)}{(z_2-z_3)}
\]

Back to main goal
If \( f(z) \) is a FLT mapping \( z_1 \rightarrow 0 \), \( z_2 \rightarrow 1 \), \( z_3 \rightarrow \infty \)

\[
f(w) = \left( w_1, w_2, w_3 \right)
\]

Then \( g \circ f(z) \) is the desired transfer. To compute

\[
\begin{align*}
\text{equate} \quad (w, w_1, w_2, w_3) &= (z, z_1, z_2, z_3) \\
\text{and solve for } w
\end{align*}
\]

Example: Find a FLT mapping \( \{|z| = 1\} \) to \( \{|z-(1+i)| = 2\} \)

\[
pick 3 points on each circle eq\]
\[
z_1 = 1, \quad z_2 = i, \quad z_3 = -1
\]

\[
w_1 = 3 + i, \quad w_2 = |1 + 3i|, \quad w_3 = -1 + i
\]
Example: Find a FLT that maps the unit circle to the half plane \( \{ \text{Re} z \geq 1 \} \).

- Pick points \( z_1, z_2, z_3 \) on the boundary of the unit circle, e.g., \( 1, i, -1 \).
- Pick points \( w_1, w_2, w_3 \) on the boundary of the half plane with matching orientation (i.e., circle is to left when going \( z_1 \to z_2 \to z_3 \), so we want half plane on left when we go \( w_1 \to w_2 \to w_3 \)), e.g., \( \infty, i+1, 1 \).

Solve \( (w, \infty, i+1) = (z_1, i, -1) \) for \( w \):

\[
\frac{i}{w-1} = \left( \frac{z-1}{z+1} \right) \left( \frac{i+1}{i-1} \right) = \frac{-i \ z + i}{z + 1}
\]

\[
f(0 \ i \ (-1) \ (-1 \ i \ 1 \ 1 \ 1) (z) = f(0 \ -2i \ 2i) (z) = \frac{-2}{z-1}
\]

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Example: Find a FLT that maps the unit circle to the exterior of \( \{ \text{Re} z = 1 \} \).

Find \( w \):

\[
\begin{align*}
&z = -1 \\
&i = 2i \\
&z_1 = 1 \\
&z_2 = 2i \\
\end{align*}
\]

Send
\[
\begin{align*}
&1 \to i-1 \\
&z \to 2i \\
&-1 \to 1+i \\
\end{align*}
\]

Solve
\[
\begin{pmatrix}
-1 + i & 2i & i+1 \\
1 & i-1 & -1 \\
2i & i+1 & 1+i
\end{pmatrix}
\]

\[
\frac{z-1}{z+1} \cdot \frac{i}{w-1} = \frac{-2}{z-1}
\]
\[
A'B = \begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix} \rightarrow \frac{2 - i}{-i} = \frac{i + 1}{2} = \begin{bmatrix} i & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} i & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} \frac{w - i + 1}{w - i - 1} \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \frac{1}{2} + 1
\]

\[
f \left( \begin{bmatrix} i & 1+i \\ 1 & -(i+1) \end{bmatrix} \right) (w) = f \left( \begin{bmatrix} -1 & i \\ 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} f \left( \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \right) \right) \begin{bmatrix} -1 & 1+i \\ 1 & 1 \end{bmatrix}
\]

\[
f \left( \begin{bmatrix} i & 1+i \\ 1 & -(i+1) \end{bmatrix} \right) \left( \frac{2 - i}{-i} \right) = \frac{-2 + 2i}{2i} = \frac{-2 - i}{i} = \frac{i + 1}{2}
\]

\[
f \left( \begin{bmatrix} i & 1+i \\ 1 & -(i+1) \end{bmatrix} \right) \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \end{bmatrix}
\]

Lecture 21

**Goal**: Find a FLT that maps the region between 2 circles to an annulus.

![Diagram showing circles and annulus]

Note: we can't specify 3 to; three pts on each circle.

**Symmetric points**

Define \( z_1 \) and \( z_2 \) are symmetric w.r.t. a circle \( C \) if every line or circle connecting \( z_1, z_2 \) meets \( C \) orthogonally.

Specifically, if \( C = \mathbb{R} \), then \( z_1 \) and \( \overline{z}_1 \) are symmetric.

Specify one if \( z_1 \in \mathbb{R} \), center of \( C \) then \( z_2 = \overline{z}_1 \)
Mapping of symmetric points: If \( z_1, z_2 \) are symmetric w.r.t. \( C \)

and \( f \) is a FLT then \( f(z_1) \) and \( f(z_2) \) are symmetric w.r.t. \( f(C) \).

Following from the fact that FLT map circles/lines \( \rightarrow \) circles/lines

and is conformal.

Finding symmetric points:

Given a circle \( C = \{ z : |z-a| = R \} \) and a point \( a \)

what point \( a' \) is symmetric to \( a \) w.r.t. \( C \)?

Strategy: find a FLT mapping \( C \to \mathbb{R} \)

Then \( f(a') \) and \( f(a) \) are symmetric w.r.t. \( f(C) = \mathbb{R} \). Thus

\( f^{-1}(f(a')) \) is and \( f^{-1}(f(a)) \) are symmetric w.r.t. \( f^{-1}(\mathbb{R}) \).

To find \( f \) we pick 3 pts on \( C \) : \( a-R, a+iR, a+R \)

Then \( (z, a-R, a+iR, a+R) \) maps these pts to \( 0, 1, \infty \), so it maps \( C \) to \( \mathbb{R} \).

\[
    f(z) = \begin{pmatrix} 2 & (a-R) \\ 2 & (a+R) \end{pmatrix} \begin{pmatrix} a+iR-(a+R) \\ a+iR-(a-R) \end{pmatrix} 
\]

\[
    = \frac{2-(a-R)}{2-(a+R)} \frac{1}{(iR-k)} 
    = \frac{2-(a-R)}{2-(a+R)} \begin{pmatrix} f(i-(a-R)) \\ f(1-(a+R)) \end{pmatrix} 
\]
\[ f(\alpha) = \frac{i\alpha - i(a-k)}{\alpha - (a+k)} \]

\[ \bar{f}(\alpha) = \frac{-i\bar{\alpha} + i(\bar{a}-k)}{\bar{\alpha} - (\bar{a}+k)} = i \left[ \frac{i(\bar{a}-k)}{1 - (\bar{a}+k)} \right] (\bar{\alpha}) \]

\[ \alpha^x = f^{-1}(\bar{f}(\alpha)) = f\left( \frac{-i(a-k)}{-1} \right) \left( \frac{f(\alpha)}{i} \right) = f\left( \frac{-i(a+k)}{-1} \right) \left( \frac{i(\bar{a}-k)}{1 - (\bar{a}+k)} \right) \]

\[ = \frac{-i\bar{\alpha} + i(\bar{a}-k)}{\bar{\alpha} - (\bar{a}+k)} + i(a-k) \]

\[ = \frac{i\bar{\alpha} - i(\bar{a}-k)}{\bar{\alpha} - (\bar{a}+k)} + i(a-k) \]

\[ \frac{(a+k)\bar{\alpha} - (a+k)(\bar{a}-k) + (a-k)\bar{\alpha} - (a-k)(\bar{a}+k)}{\bar{\alpha} - (\bar{a}-k) + \bar{\alpha} - (\bar{a}+k)} \]

\[ = \frac{2\alpha \bar{\alpha} - |a|^2 + aR - \bar{a}R + R^2 - |a|^2 - aR + \bar{a}R + R^2}{2 \bar{\alpha} - 2 \bar{a}} \]

\[ = \frac{-a|\bar{\alpha}|^2 + R^2}{\bar{\alpha} - \bar{a}} = \frac{a(\bar{\alpha} - \bar{a}) + R^2}{\bar{\alpha} - \bar{a}} \]

\[ = \frac{a + R^2}{\bar{\alpha} - \bar{a}} = \alpha^x \]

**Interpretation**

\[ \alpha^x - a = \frac{R^2(a - \bar{a})}{|a - \bar{a}|^2} \Rightarrow \arg(\alpha^x - a) = \arg(a - \bar{a}) \]

\[ |\alpha^x - a| = \frac{R^2}{|a - \bar{a}|} \]
Find a FLT mapping \( D \) onto an annulus

**Step 1** Find \( \alpha, \alpha^* \) that are symmetric w.r.t. both \( C, L \).

**Step 2** Find a FLT mapping \( \alpha \to 0 \) and \( \alpha^* \to \infty \).

Then \( f(c) \) and \( f(l) \) must be symmetric w.r.t. \( D \) and \( \infty \), i.e., they are even about \( 0 \).

**Step 1:** \( \alpha^*, \alpha \) are symm. w.r.t. \( L \) if \( \alpha^2 = \alpha \)

\[ \alpha^2, \alpha \text{ are symm. w.r.t. } C \] \( (a=2i, \ R=1) \) \( \Rightarrow \alpha^2 = a + \frac{R^2}{\alpha - \bar{a}} \)

\[ = 2i + \frac{1}{\alpha + 2i} \]

So solve \( \alpha = 2i + \frac{1}{\alpha + 2i} \), \( (\alpha - 2i)(\alpha + 2i) = 1 \), \( (\alpha)^2 + 4 = 1 \)

\[ \alpha^2 = -3 \quad \alpha = \pm i\sqrt{3} \quad \alpha = i\sqrt{3} \quad \alpha^* = -i\sqrt{3} \]

**Step 2** We want \( \alpha \to 0 \), \( \alpha^* \to \infty \). One more choice say \( 0 \to 1 \)

\[ f(z) = (z, i\sqrt{3}, 0, -i\sqrt{3}) \]

\[ z = \frac{z - i\sqrt{3}}{z + i\sqrt{3}} \]

is the desired mapping
Then \( f(L) = \{ |w| = 1 \} \)

\[
    f(c) = \text{well, } f(i) = \frac{i - i\sqrt{3}}{1 + i\sqrt{3}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3} - 1)^2}{2} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}
\]

So it must be \( \{ |w| = 2 - \sqrt{3} \} \)

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**Lecture 2.2**

**Problem:** Calculate the steady state temperature distribution for a pipe with boiling water traveling through it. What is \( T(1+i) \)?

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Needed to solve \( \Delta t = 0 \) with BC as shown.

**Solu:** map via \( f(z) = \frac{-2 + i\sqrt{3}}{2 + i\sqrt{3}} \) to

\[
    T(u,v) = A \ln \left( \sqrt{u^2 + v^2} \right) + B
\]

\( (= Re \left( A \log(u) + B \right) ) \) - explain: even though \( \log \) has a branch cut

\[
    T(2-\sqrt{3},0) = 100 \\
    A \ln(2-\sqrt{3}) + B = 100 \\
    T(0,0) = 0 \\
    A \ln(1) + B = 0
\]

\[
    \implies B = 0 \\
    A = \frac{100}{\ln(2-\sqrt{3})}
\]

\[
    T(u,v) = \frac{100}{\ln(2-\sqrt{3})} \ln \left( \sqrt{u^2 + v^2} \right) = \frac{50}{\ln(2-\sqrt{3})} \ln \left( \frac{u^2 + v^2}{u^2 + (v + \sqrt{3})^2} \right)
\]

\[
    f(z) = \frac{-x - iy + i\sqrt{3}}{x + i(y + \sqrt{3})} = \frac{(-x + i(-y + \sqrt{3}))(x - iy + \sqrt{3})}{x^2 + (y + \sqrt{3})^2}
\]

\[
    = \frac{-x^2 + 3 - y^2 + i \left( x \left( y + \sqrt{3} \right) + x \left( -y + \sqrt{3} \right) \right)}{u \left( v + \sqrt{3} \right) ^2}
\]
\begin{align*}
\phi &= \frac{3 - x^2 - y^2}{x^2 + (y + \sqrt{3})^2} - i \frac{2\sqrt{3}x}{x^2 + (y + \sqrt{3})^2} \\
\psi &= \frac{3 - x^2 - y^2}{x^2 + (y - \sqrt{3})^2} - i \frac{2\sqrt{3}x}{x^2 + (y - \sqrt{3})^2} \\
\tan \varphi &= \frac{\psi}{\phi} \quad \text{where} \quad \varphi = \arctan \frac{\psi}{\phi}
\end{align*}

\begin{align*}
t(x, y) &= T(\phi(x, y), \psi(x, y)) = \frac{100}{\ln(2 - \sqrt{3})} \frac{1}{2} \ln \left( u^2 + v^2 \right) \\
&= \frac{100}{\ln(2 - \sqrt{3})} \ln \left( \frac{x^2 + (y + \sqrt{3})^2}{x^2 + (y + \sqrt{3})^2} \right) \\
t(1, 1) &= \frac{100}{\ln(2 - \sqrt{3})} \ln \left( \frac{1 + (1 - \sqrt{3})^2}{1 + (1 + \sqrt{3})^2} \right) \approx 64.797^\circ
\end{align*}

Temp profile along \( y = 0 \) for \( x \in [0, 1] \):
\begin{align*}
u(0, y) &= \frac{2 - y^2}{(y + \sqrt{3})^2} \\
t &= \frac{100}{\ln(2 - \sqrt{3})} \ln \left( \frac{\sqrt{3} + y}{\sqrt{3} + y} \right)
\end{align*}