6.7:6

Let \( q(z) = z^6 + 4z^2 \) and \( p(z) = z^6 + 4z^2 - 1 = q(z) - 1 \). On the contour \(|z| = 1\) we have

\[
|q(z)| = |z|^2|z^4 + 4| = |z^4 + 4| \geq 4 - |z|^4 = 4 - 1 = 3 > 1.
\]

Thus Rouché’s theorem says that \( q \) and \( p \) have the same number of zeros in the unit disk. The polynomial \( q(z) \) has a zero of order 2 at \( z = 0 \). The other zeros of \( q \) lie on the circle \(|z| = \sqrt{2}\). So \( q \) and therefore also \( p \) have 2 zeros (counted with multiplicity) in the unit disk.

6.7:7

When \(|z| = 2\) we have

\[
|z^3 + 27| \geq 27 - 8 = 19 > 18 = |9z|
\]

So Rouché’s theorem says \( z^3 + 27 \) and \( z^3 + 9z + 27 \) have the same number of zeros in the disk of radius 2. But all the zeros of \( z^3 + 27 \) lie on the circle \(|z| = 3\). Thus both \( z^3 + 27 \) and \( z^3 + 9z + 27 \) have no zeros in the disk of radius 2.

6.7:8

We wish to show that all the roots of \( p(z) = z^6 - 5z^2 + 10 \) lie in the annulus \( 1 < |z| < 2 \).

When \(|z| = 2\) we have

\[
|z^6 + 10| \geq |z^6| - 10 = 64 - 10 = 54 > 20 = |5z^2|.
\]

This shows that \( z^6 + 10 \) and \( z^6 - 5z^2 + 10 \) have the same number of zeros in \( \{ z : |z| < 2 \} \) and \( \{ z : |z| < 2 \} \). The zeros of \( z^6 + 10 \) satisfy \(|z| = (10)^{1/6}\) which is \( < 2 \). So \( z^6 - 5z^2 + 10 \) has 6 (i.e., all) zeros in \( \{ z : |z| < 2 \} \).

On the other hand, when \(|z| = 1\), then \(|z^6 - 5z^2| < |z|^6 + 5|z|^2 = 6 < 10\). So \( z^6 - 5z^2 + 10 \) has the same number of zeros in \( \{ z : |z| = 1 \} \) as the constant polynomial 10, i.e., none.

Thus the zeros must all lie in the annulus \( 1 < |z| < 2 \).
6.7:9

When $|z| = 1$ we have

$$|z^3 - 2z^2 + z - 1| \leq |z|^3 + 2|z|^2 + |z| + 1 = 5 < 6 = |6z^4|.$$ 

This implies that $6z^4$ and $6z^4 + z^3 - 2z^2 + z - 1$

6.7:10

We want to find the winding number of $f(z) = 2 - e^z - z$ around a D shaped contour consisting of a semi-circle in the right half plane of radius $R$ and centre 0 together with the segment $[iR, -iR]$ on the imaginary axis.

When $z = iy$ with $y \in \mathbb{R}$ we have $\text{Re} f(iy) = 2 - \cos(y) \geq 1$ so the direction vector $\hat{f}(iy) = f(iy)/|f(iy)|$ lies in the right half plane. This means that the winding can be at most $\pi$. Since the direction of $\hat{f}$ approaches $-i$ when $y \to \infty$ and $i$ when $y \to -\infty$ we see that the change in argument when $y$ travels down the imaginary axis is $\pi$. On the semicircle part we have $\hat{f}(Re^{-\theta}) \to -e^{i\theta}$ as $R \to \infty$. So the semi circle also contributes $\pi$. So the total winding is $2\pi$ and we conclude that the number of zeros is 1.

If $z$ is a zero, i.e., $2 - e^z - z = 0$ then if taking the conjugate yields $2 - e^{-z} - \overline{z} = 0$ so that $\overline{z}$ is a solution whenever $z$ is. If $z$ were not real then we would have two distinct solutions $z$ and $\overline{z}$ contradicting our counting result.

6.7:11

To use the Nyquist criterion we must determine the change in argument in $p(iy)$ as $y$ goes from $\infty$ to $-\infty$. Following the hint we write

$$p(iy) = (y^2 - 2)(y^2 - 1) + i(y(1 - 2y^2)).$$

and find where $p$ hits the axes. This will only occur if $\text{Re} p(iy) = 0$ (at $y \in \{\pm 1, \pm \sqrt{2}\}$) or $\text{Im} p(iy) = 0$ (at $y \in \{0, \pm 1/\sqrt{2}\}$). We collect the information about the crossings:
This can be viewed schematically as:

I haven’t checked to make sure the line is on the right side of the axis e.g., between D and E because it doesn’t affect the winding. From the picture we see there is no winding along the imaginary axis. So the total winding is $4\pi$ (from the semicircle) +0. So the number of zeros in the right half plane is $4\pi/(2\pi) = 2$. 