6.6: 3

To compute \( I = \int_0^\infty \frac{x^\alpha}{(x+9)^2} \) for \(-1 < \alpha < 1\), \( \alpha \neq 0 \) we use the contours \( \Gamma_+ \), \( \Gamma_- \), \( C_R \) and \( C_\epsilon \) shown below:

We choose a branch of \( z^\alpha \) which equals \( x^\alpha \) on the contour \( \Gamma_+ \) (i.e., when \( z \) approaches \( x \in \mathbb{R}_+ \) from above, \( z^\alpha \to x^\alpha \)) and has its branch cut on \( \mathbb{R}_+ \). Concretely, our branch \( z^\alpha \) can be defined as

\[
z^\alpha = |z|^\alpha e^{ia\theta}
\]

where \( \theta \) is the (unique) element in \([0, 2\pi) \cap \arg z\).

On the contour \( \Gamma_- \) we have \( z^\alpha = |z|^\alpha e^{2\pi i \alpha} = x^\alpha e^{2\pi i \alpha} \) (i.e., when \( z \) approaches \( x \in \mathbb{R}_+ \) from below, \( z^\alpha \to x^\alpha e^{i\alpha \theta} \))

Define \( f(z) = \frac{z^\alpha}{(x-9)^2} \) and integrate \( f(z) \) around the closed contour \( \Gamma_+ + C_R - \Gamma_- + C_\epsilon \). The Cauchy residue formula gives

\[
\oint_{\Gamma_+ + C_R - \Gamma_- + C_\epsilon} f(z)dz = 2\pi i \text{Res} [f(z), z = -9].
\]

We compute

\[
\text{Res} [f(z), z = -9] = \lim_{z \to -9} \left[ \frac{d}{dz} (z+9)^2 f(z) \right] = \alpha(-9)^{\alpha-1}
\]

We compute the limits

\[
\lim_{\epsilon \downarrow 0, R \uparrow \infty} \int_{\Gamma_+} f(z)dz = I,
\]

\[
\lim_{\epsilon \downarrow 0, R \uparrow \infty} \int_{\Gamma_-} f(z)dz = e^{\alpha 2\pi i} I,
\]
Our basic estimate yields
\[
\lim_{\epsilon \downarrow 0} \left| \int_{C_\epsilon} f(z) \, dz \right| \leq \max_{z \in C_\epsilon} \frac{|z|^\alpha}{|z + 9|} \times \text{length } C_\epsilon \leq \frac{\epsilon^\alpha}{(9 - \epsilon)^2} \times 2\pi \epsilon \to 0
\]
as \epsilon \downarrow 0, since \(\alpha + 1 > 0\) implies \(\epsilon^{\alpha+1} \to 0\).

Similarly
\[
\lim_{R \uparrow \infty} \left| \int_{C_R} f(z) \, dz \right| \leq \max_{z \in C_R} \frac{|z|^\alpha}{|z + 9|} \times \text{length } C_R \leq \frac{R^\alpha}{(R - 9)^2} \times 2\pi R \to 0
\]
as \(R \uparrow \infty\), since \(\alpha - 1 < 0\).

Inserting this information into the Cauchy residue formula and solving for \(I\) yields
\[
I = \left( -2i \right) \pi \alpha(-9)^{\alpha-1} \left( e^{i\alpha \pi} - e^{-i\alpha \pi} \right) / \sin(\pi \alpha)
\]
Here we used that \((-9)^{\alpha} = |-9|^{\alpha} e^{i\alpha \pi}\).

Notice that \(\alpha = 0\) is not a problem. We could either take the limit in this formula or calculate the integral directly.

### 6.6: 5

To compute
\[
I = \int_0^\infty \frac{x^{\alpha-1}}{x^2 + x + 1}
\]
we can use the same contours and the same branch of \(z^\alpha\) as 6.6:3.

Factor \(z^2 + z + 1 = (z - \omega_1)(z - \omega_2)\) where
\[
\omega_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2} = e^{2\pi i/3}, \quad \omega_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2} = e^{-2\pi i/3}
\]

We must find the residues at \(z = \omega_1\) and \(z = \omega_2\). To find \(\text{Res} \left[ \frac{z^{\alpha-1}}{z^2 + z + 1}, z = \omega_1 \right]\) we can use the formula for a simple pole, i.e., the residue is the value of \(z^\alpha\) at \(\omega_1\) divided by the derivative of \(z^2 + z + 1\), namely \(2z + 1\) evaluated at \(\omega_1\).

\[
\text{Res} \left[ \frac{z^{\alpha-1}}{z^2 + z + 1}, z = \omega_1 \right] = \frac{z^{\alpha-1}}{2z + 1}
\]
evaluated at \(z = \omega_1\). To compute the numerator we use our range of angle procedure as follows: write \(\omega_1 = |\omega_1| e^{i\theta}\) with \(\theta \in \arg(\omega_1)\) in our range of angles, namely \([0, 2\pi]\). This gives \(\theta = 2\pi/3\) and so the value of our branch of \(z^{\alpha-1}\) at \(z = \omega_1\) is \(e^{i(\alpha-1)2\pi/3}\). Thus
\[
\text{Res} \left[ \frac{z^{\alpha-1}}{z^2 + z + 1}, z = \omega_1 \right] = \frac{e^{i(\alpha-1)2\pi/3}}{1 + i\sqrt{3} + 1} = \frac{e^{i(\alpha-1)2\pi/3}}{i\sqrt{3}}
\]
Similarly, we follow the same procedure with the residue at \( \omega_2 = e^{-2\pi i/3} \). We don’t use the given angle \(-2\pi i/3\) in the our calculation because it is not in the range \([0, 2\pi)\). Instead, we use \( \theta = 4\pi i/3 \) so that

\[
\text{Res} \left[ \frac{z^{\alpha - 1}}{z^2 + z + 1}, z = \omega_1 \right] = -\frac{e^{i(\alpha - 1)4\pi/3}}{i\sqrt{3}}
\]

Then following the same steps as 6.6:3 with \( f(z) = \frac{z^{\alpha - 1}}{x^2 + x + 1} \) we find

\[
I = 1 - e^{2\pi i(\alpha - 1)/3} \frac{2\pi i}{2 \pi} (\text{Res}[f(z), z = \omega_1] + \text{Res}[f(z), z = \omega_2])
\]

Now we write \( A = 2\pi i(\alpha - 1)/3 + i\pi \) and \( B = 4\pi i(\alpha - 1)/3 + i\pi \) and use the symmetrization formula

\[
A = (A + B)/2 + (A - B)/2, \quad B = (A + B)/2 - (A - B)/2.
\]

We compute \((A + B)/2 = i\pi \alpha - i\pi/2\) and \((A - B)/2 = i(-2\pi \alpha - \pi)/6\). Then

\[
\left( e^{2\pi i(\alpha - 1)/3} + e^{4\pi i(\alpha - 1)/3 + i\pi} \right) = e^A + e^B
\]

Substituting into the formula for \( I \) yields the result in the book

\[
I = \frac{2\pi}{\sqrt{3}} \cos \left( \frac{2\alpha \pi + \pi}{6} \right) \frac{1}{\sin(\pi \alpha)}
\]

6.6: 8

We want to show that

\[
\text{p.v.} \int_{-\infty}^{\infty} \frac{\ln(|x|)}{x^2 + 4} \, dx = \frac{\pi}{2} \ln(2)
\]

Since the integrand is even, we can write the left side as \( 2I \) where

\[
I = \int_{0}^{\infty} \frac{\ln(|x|)}{x^2 + 4} \, dx.
\]

This should be understood as an improper Riemann integral

\[
\lim_{\epsilon \to 0, R \to \infty} \int_{\epsilon}^{R} \frac{\ln(|x|)}{x^2 + 4} \, dx
\]
Following the hint, take \( f(z) = \frac{\log(z)}{z^2 + 4} \) and integrate it along the contour \([\epsilon, R] + C_R - [-\epsilon, -R] + C_\epsilon\), where \( C_\epsilon \) is the intersection of the circle \( \{|z| = a\} \) with the upper half plane. Here \( C_R \) is oriented from left to right and \( C_\epsilon \) the other way. Cauchy’s residue theorem implies

\[
\int_{[\epsilon, R]} f(z) \, dz + \int_{C_R} f(z) \, dz - \int_{[-\epsilon, -R]} f(z) \, dz + \int_{C_\epsilon} f(z) \, dz = 2\pi i \text{Res} \{f(z), z = 2i\}
\]

We can estimate

\[
\left| \int_{C_R} f(z) \, dz \right| \leq \max_{|z|=R} \left| \frac{\ln(|z|) + i \text{Arg}(z)}{|z|^2 - 4} \right| \times \pi R
= \frac{\ln(R) + \pi}{R^2 - 4} \times \pi R
\]

\[
\to 0
\]

as \( R \to \infty \). Similarly

\[
\left| \int_{C_\epsilon} f(z) \, dz \right| \leq \max_{|z|=\epsilon} \left| \frac{\ln(|z|) + i \text{Arg}(z)}{4 - \epsilon^2} \right| \times \pi \epsilon
= \frac{\ln(1/\epsilon) + \pi}{4 - \epsilon^2} \times \pi \epsilon
\]

\[
\to 0
\]

as \( \epsilon \to 0 \).

The contour \([-\epsilon, -R]\) can be parametrized with \( z(r) = r \), \( r \in [\epsilon, R] \). Then \( z'(r) = -1 \) and since the contour lies above the cut, on the contour we have \( \log(-r) = \ln(r) + i\pi \). Thus

\[
\int_{[-\epsilon, -R]} f(z) \, dz = \int_{[\epsilon, R]} \frac{\log(-r)}{(-r)^2 + 4} (-1) \, dr = -\int_{[\epsilon, R]} \frac{\ln(r) + i\pi}{r^2 + 4} \, dr \to -I - i\pi J
\]

as \( R \to \infty \) and \( \epsilon \to 0 \), where

\[
J = \int_0^{\infty} \frac{1}{r^2 + 4} \, dr
\]

The residue of \( f(z) \) at \( z = 2i \) is given by

\[
\text{Res} \left[ \frac{\log(z)}{z^2 + 4}, z = 2i \right] = \frac{\log(2i)}{4i} = \frac{\ln 2 + \frac{\pi}{2}}{4i}
\]

The residue theorem now gives

\[
2I + i\pi J = 2\pi i \frac{\ln 2 + \frac{\pi}{2}}{4i} = \frac{\pi}{2} \ln 2 + i \frac{\pi^2}{4}
\]

Taking the real part yields

\[
2I = \frac{\pi}{2} \ln 2.
\]

as required.

Note: we can find the value of \( J \) by taking the imaginary part. This gives \( J = \frac{\pi}{4} \).
Following essentially the same steps as 6.6:8, except that now

\[ I = \int_0^\infty \frac{\ln(|x|)}{x^2 + 1} \, dx \]

yields

\[ 2I = \pi \ln 1 = 0 \]

Let \( z^{\alpha - 1} \) with \( 0 < \alpha < 1 \) be a branch that is equal to \( x^{\alpha - 1} \) when \( z = x + i0 \) and has a cut that is disjoint from the first quadrant. For example take \( z^{\alpha - 1} \) to be the principal branch. If we want to describe this via a range of angles we could write

\[ z^{\alpha - 1} = |z|^{\alpha - 1} e^{(\alpha - 1)\theta} \]

where \( \theta \) is the unique element of \( \text{arg}(z) \cap (-\pi, \pi) \).

Now define \( f(z) = z^{\alpha - 1} e^{iz} \) and integrate it around the contour \( [\epsilon, R] + C_R - [i\epsilon, iR] + C_\epsilon \)

For every \( \epsilon \) and \( R \) the sum around the contour is 0 by Cauchy's theorem.

We can estimate

\[ \left| \int_{C_\epsilon} f(z) \, dz \right| \leq \max_{z \in C_\epsilon} |z|^{\alpha - 1} |e^{iz}| \times \text{length } C_\epsilon \leq \epsilon^{\alpha - 1} \times \pi \epsilon / 2 = \text{const.} \epsilon^\alpha \rightarrow 0 \]

as \( \epsilon \downarrow 0 \), since \( \alpha > 0 \). Here we used that \( |e^{iz}| = |e^{i(x+iy)}| = e^{-y} \leq 1 \) for \( z \) in the upper half plane.

The integral over \( C_R \) is estimated as in the proof of Jordan's lemma. We use that \( \sin(\theta) \geq 2\theta / \pi \) to estimate

\[ \left| \int_{C_R} f(z) \, dz \right| \leq \int_0^{\pi/2} (R e^{i\theta})^{\alpha - 1} e^{iRe^{i\theta}} iRe^{i\theta} \, d\theta \]

\[ \leq R^\alpha \int_0^{\pi/2} e^{-R \sin(\theta)} \, d\theta \]

\[ \leq R^\alpha \int_0^{\pi/2} e^{-2R \theta / \pi} \, d\theta \]

\[ \leq R^\alpha \text{const.} R^{-1} \rightarrow 0 \]
as $R \to \infty$ because $\alpha - 1 < 0$.

Thus we are left with the integrals on the real line and the imaginary axis:

$$\lim_{\epsilon \downarrow 0, R \uparrow \infty} \int_{\epsilon}^{R} x^{\alpha - 1} e^{ix} \, dx = \lim_{\epsilon \downarrow 0, R \uparrow \infty} \int_{\epsilon}^{R} (iy)^{\alpha - 1} e^{-y} \, idy$$

When computing $(iy)^{\alpha - 1}$ for our branch, we choose the argument of $iy$ to be in our range $(-\pi, \pi]$. Thus $(iy)^{\alpha - 1} = (ye^{i\pi/2})^{\alpha - 1} = y^{\alpha - 1} e^{i\pi(\alpha - 1)/2}$ Taking into account the factor of $i = e^{i\pi/2}$ in the idy term, we get

$$\int_{0}^{\infty} x^{\alpha - 1} e^{ix} \, dx = e^{i\pi\alpha/2} \int_{0}^{\infty} y^{\alpha - 1} e^{-y} \, dy = e^{i\pi\alpha/2} \Gamma(\alpha).$$

Then we may take the imaginary part to get

$$\int_{0}^{\infty} x^{\alpha - 1} \sin(x) \, dx = \sin(\pi\alpha/2) \Gamma(\alpha).$$

Here $\Gamma(\alpha) = \int_{0}^{\infty} y^{\alpha - 1} e^{-y} \, dy$ is the $\Gamma$ function.