4. Let \( f \) have an isolated singularity at \( z_0 \) (\( f \) analytic in a punctured neighborhood of \( z_0 \)). Show that the residue of the derivative \( f' \) at \( z_0 \) is equal to zero.

5. Is there a function \( f \) having a simple pole at \( z_0 \) with \( \text{Res} (f; z_0) = 0 \)? How about a function with a pole of order 2 at \( z_0 \) and \( \text{Res}(f; z_0) = 0 \)?

6. Suppose that \( f \) is analytic and has a zero of order \( m \) at the point \( z_0 \). Show that the function \( g(z) = f'(z)/f(z) \) has a simple pole at \( z_0 \) with \( \text{Res}(g; z_0) = m \).

7. Evaluate
\[
\oint_{|z|=1} e^{1/z} \sin(1/z) \, dz.
\]

### 6.2 Trigonometric Integrals over \([0, 2\pi]\)

Our goal in this section is to apply the residue theorem to evaluate real integrals of the form
\[
\int_0^{2\pi} U(\cos \theta, \sin \theta) \, d\theta,
\]
where \( U(\cos \theta, \sin \theta) \) is a rational function (with real coefficients) of \( \cos \theta \) and \( \sin \theta \) and is finite over \([0, 2\pi]\). An example of such an integral is
\[
\int_0^{2\pi} \frac{\sin^2 \theta}{5 + 4 \cos \theta} \, d\theta.
\]

We shall show that (1) can be identified as the parametrized form of a contour integral, \( \int_C F(z) \, dz \), of some complex function \( F \) around the positively oriented unit circle \( C : |z| = 1 \). To establish this identification we parametrize \( C \) by
\[
z = e^{i\theta} \quad (0 \leq \theta \leq 2\pi).
\]
For such \( z \) we have
\[
\frac{1}{z} = \frac{1}{e^{i\theta}} = e^{-i\theta},
\]
and since
\[
\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},
\]
we have the identities\(^\dagger\)
\[
\cos \theta = \frac{1}{2} \left( z + \frac{1}{z} \right), \quad \sin \theta = \frac{1}{2i} \left( z - \frac{1}{z} \right),
\]
Furthermore, when integrating along \( C \),
\[
dz = ie^{i\theta} \, d\theta = iz \, d\theta,
\]
\(^\dagger\)Of course we could use \( \bar{z} \) instead of \( 1/z \), but this would forfeit analyticity.