1. Let $D$ be the first quadrant $\{x + iy : x > 0, y > 0\}$ with the unit (quarter) circle removed. The boundary of $D$ is $D_1 \cup D_2 \cup D_3$ where $D_1 = \{iy : 1 \leq y < \infty\}$, $D_2 = \{e^{i\theta} : 0 \leq \theta \leq \pi/2\}$ and $D_3 = \{x : 1 \leq x < \infty\}$. Solve $\Delta \varphi = 0$ in $D$ with $\varphi = 0$ on $D_1$, $\varphi = 1$ on $D_2$ and $\varphi = 0$ on $D_3$. (Hint: use the Joukowski map composed with another map.)

**Solution:** We can compose the Joukowski map $J(z) = (z + 1/z)/2$ with $z^2$ in either order.

If we apply $z^2$ first, then $D$ is mapped to the upper half plane with the unit (half) circle removed. This is then mapped to the upper half plane under $J$. In this case the composite map is $f_1(z) = J(z^2)$. Under this map, $C_1$ is mapped to $(-\infty, -1)$, $C_2$ is mapped to $[-1..1]$ and $C_3$ is mapped to $[1, \infty)$. We must first solve $\Delta \Phi_1(u, v) = 0$ with boundary conditions

$$\Phi_1(u, 0) = \begin{cases} 
0 & \text{if } u < -1 \\
1 & \text{if } -1 < u < 1 \\
0 & \text{if } 1 < u 
\end{cases}$$

The solution is

$$\Phi_1(u, v) = \frac{1}{\pi} \cot^{-1} \left( \frac{u - 1}{v} \right) - \frac{1}{\pi} \cot^{-1} \left( \frac{u + 1}{v} \right)$$

Now we write

$$f_1(x + iy) = u_1(x, y) + iv_1(x, y) = \frac{1}{2} \left( x^2 - y^2 + (x^2 - y^2)/(x^2 + y^2)^2 + i(xy - xy/(x^2 + y^2)^2) \right)$$

which results in the solution

$$\varphi_1(x, y) = \Phi_1(u_1(x, y), v_1(x, y))$$

$$= \frac{1}{\pi} \cot^{-1} (A_1(x, y)) - \frac{1}{\pi} \cot^{-1} (A_2(x, y))$$

where

$$A_1(x, y) = \frac{u_1(x, y) - 1}{v_1(x, y)} = \frac{(x^3 - x^2y + xy^2 - y^3 - x - y)(x^3 + x^2y + xy^2 + y^3 - x + y)}{2xy(x^2 + y^2 + 1)(x^2 + y^2 - 1)}$$

and

$$A_2(x, y) = \frac{u_1(x, y) + 1}{v_1(x, y)} = \frac{(x^3 + x^2y + xy^2 + y^3 + x + y)(x^3 - x^2y + xy^2 - y^3 + x + y)}{2xy(x^2 + y^2 + 1)(x^2 + y^2 - 1)}$$

If we apply $J(z)$ first, then $D$ is mapped to the first quadrant, which is then mapped to the upper half plane under $z^2$. In this case the composite map is $f_2(z) = J(z)^2$. Under this map, $C_1$ is mapped to $(-\infty, 0]$, $C_2$ is mapped to $[0..1]$ and $C_3$ is mapped to $[1, \infty)$. In this case we must first solve $\Delta \Phi_1(u, v) = 0$ with boundary conditions

$$\Phi_1(u, 0) = \begin{cases} 
0 & \text{if } u < 0 \\
1 & \text{if } 0 < u < 1 \\
0 & \text{if } 1 < u 
\end{cases}$$
The solution is
\[ \Phi_2(u, v) = \frac{1}{\pi} \cot^{-1} \left( \frac{u - 1}{v} \right) - \frac{1}{\pi} \cot^{-1} \left( \frac{u}{v} \right) \]

Now we write
\[ f_2(x + iy) = u_2(x, y) + iv_2(x, y) \]
\[ = \frac{1}{4} \left( x^2 + 2x^2/(x^2 + y^2) + x^2/(x^2 + y^2)^2 - y^2 + 2y^2/(x^2 + y^2)y^2/(x^2 + y^2)^2 \right. \]
\[ + i(2xy - 2xy/(x^2 + y^2)^2) \]

which results in the solution
\[ \varphi_2(x, y) = \Phi_2(u_2(x, y), v_2(x, y)) \]
\[ = \frac{1}{\pi} \cot^{-1} (B_1(x, y)) - \frac{1}{\pi} \cot^{-1} (B_2(x, y)) \]

where
\[ B_1(x, y) = \frac{u_2(x, y) - 1}{v_2(x, y)} = \frac{(x^3 - x^2y + xy^2 - y^3 - x - y)(x^3 + x^2y + xy^2 + y^3 - x + y)}{2xy(x^2 + y^2 + 1)(x^2 + y^2 - 1)} \]

and
\[ B_2(x, y) = \frac{u_2(x, y)}{v_2(x, y)} = \frac{(x^3 + x^2y + xy^2 + y^3 + x - y)(x^3 - x^2y + xy^2 - y^3 + x + y)}{2xy(x^2 + y^2 + 1)(x^2 + y^2 - 1)} \]

Notice that \( A_1 = B_1 \) and \( A_2 = B_2 \). This shows that \( \varphi_2 = \varphi_1 \) so that both methods give the same answer.