University of British Columbia
Math 301, Section 201

Midterm 2

Date: March 15, 2013
Time: 11:00 - 11:50pm

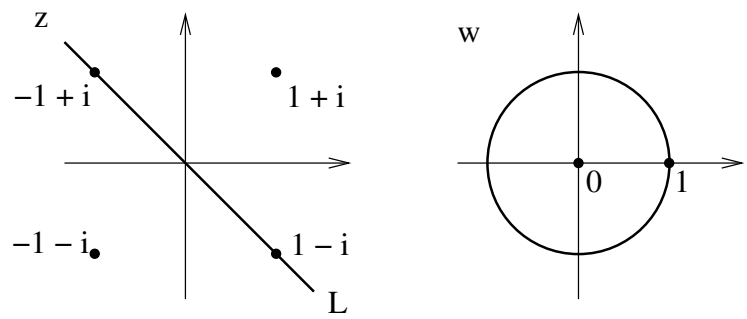
Name (print):
Student ID Number:
Signature:

Instructor: Richard Froese

Instructions:

1. No notes, books or calculators are allowed.
2. Read the questions carefully and make sure you provide all the information that is asked for in the question.
3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.
4. Answer the questions in the space provided. Continue on the back of the page if necessary.

Question	Mark	Maximum
1		12
2		12
3		16
Total		



1.

[4]

(a) Suppose that $f(z)$ is a fractional linear transformation that

i. maps the line L passing through $-1+i$ and $1-i$ to the unit circle,

ii. maps $1+i$ to 0.

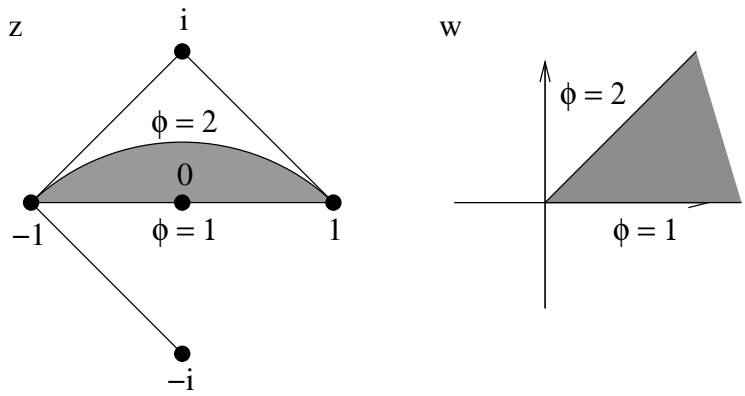
What is $f(-1-i)$? Give a reason.

Solution: Since $1+i$ and$-1-i$ are symmetric with respect to L their images must be symmetric with respect to the unit circle. This implies that $f(-1-i)=\infty$.

[8]

(b) Find all fractional linear transformations satisfying conditions i. and ii. above. If $f(0)=1$ what is $f(1-i)$?

Solution: Linear fractional transformations that map $1+i$ to 0 and $-1-i$ to ∞ have the form $f(z)=a(z-1-i)/(z+1+i)$. Points on the line L be written $-t+it$, $t\in\mathbb{R}$. So to satisfy condition i. we require $|a(-t+it-1-i)/(-t+it+1+i)|=1$. Since $|(-t+it-1-i)/(-t+it+1+i)|=\sqrt{((t+1)^2+(t-1)^2)/((1-t)^2+(1+t)^2)}=1$ the condition becomes $|a|=1$. If $f(0)=1$ then $a(-1-i)/(1+i)=-a=1$ so $a=-1$ and $f(1-i)=i$.



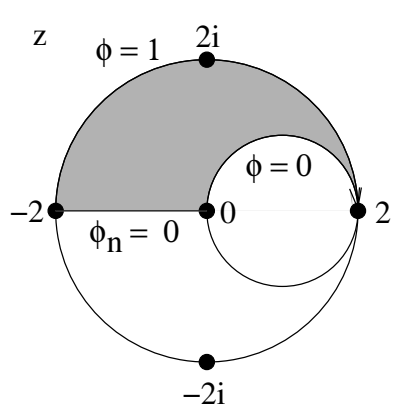
2.

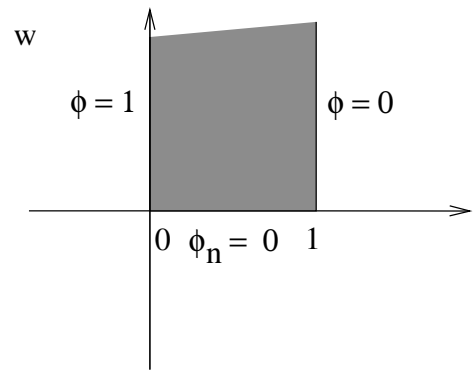
[12]

(a) Solve Laplace's equation $\Delta\phi = 0$ in the shaded region (above the segement $[-1, 1]$ and below the circle $|z + i| = \sqrt{2}$) with the indicated boundary conditions.

Solution: Use the map $f(z) = (z, -1, 0, 1) = -(z + 1)/(z - 1)$. This maps the segment $[-1, 1]$ to $[0, \infty]$. The arc of circle forming the top boundary will then also be mapped to a ray joining 0 to ∞. The tangent to the circle at -1 makes an angle of $\pi/2$ with the segement $[-i, -1]$, and therefore an angle of $\pi/4$ with the real axis. By conformality the second ray must also make an angle of $\pi/4$ with the real axis.

On the target domain, the solution is $\Phi(w) = 1 + (4/\pi)\mathrm{Arg}(w)$ or $\Phi(u + iv) = 1 + (4/\pi)\cot^{-1}(u/v)$. The solution on the original domain is $\phi(z) = \Phi(f(z))$. Since $f(x + iy) = u(x, yt) + iv(x, y) = (1 - x^2 - y^2)/((x - 1)^2 + y^2) + i2y/((x - 1)^2 + y^2)$ we get $\phi(x + iy) = 1 + (4/\pi)\cot^{-1}((1 - x^2 - y^2)/(2y))$.





3.

[4]

(a) Determine the image of the shaded region under $f(z) = -(z+2)/(z-2)$ and draw it on the w plane above

Solution: We have $f(-2) = 0$, $f(0) = 1$ and $f(2) = \infty$. So the segment $[-2, 0]$ maps to $[0, 1]$ and by conformality the other two sides of the boundary are mapped to vertical rays. So the image domain is a vertical half strip.

[8]

(b) Solve Laplace's equation in the shaded region with the indicated boundary conditions. In the diagram, ϕ_n denotes the normal derivative $\partial\phi/\partial n$.

Solution: In the target the solution is $\Phi(u + iv) = 1 - u$. $u(x, y) = \Re f(x + iy) = (2 - x^2 - y^2)/((x - 2)^2 + y^2)$. So the solution is $\phi(x + iy) = 1 - u(x, y)$.

[4] (c) The conformal map $g(w) = \sin(\pi w/2)$ maps the image region from part (a) to the first quadrant. Explain how you could use this additional map to solve Laplace's equation above with the boundary condition on the segment $[-2, 0]$ (i.e., $\phi_n = 0$) replaced by $\phi = 0$. You need not carry out the calculations in detail.

Solution: The map $g(f(z))$ maps the original domain to the first quadrant with boundary conditions $\Phi = 0$ on the positive real axis and $\Phi = 1$ on the positive imaginary axis. Thus the solution on the target domain is $\Phi(w) = (2/\pi)\mathrm{Arg}(u + iv)$. Write $g(f(x + iy)) = u(x, y) + iv(x, y)$. Then the solution is $\phi(x + iy) = \Phi(u(x, y) + iv(x, y))$.