### Fourier Transform Summary

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$\widehat{f}(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Definition and inversion</strong></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \widehat{f}(k) dk$</td>
<td>$\int_{-\infty}^{\infty} e^{-ikx} f(x) dx$</td>
</tr>
<tr>
<td><strong>Examples</strong></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{1 + a^2}$</td>
</tr>
<tr>
<td>2</td>
<td>$e^{-</td>
</tr>
<tr>
<td>3</td>
<td>$\begin{cases} 1 &amp;</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{\sin(x)}{x}$</td>
</tr>
<tr>
<td>5</td>
<td>$e^{-x^2/(2a^2)}$</td>
</tr>
<tr>
<td><strong>Properties</strong></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$c_1 f_1(x) + c_2 f_2(x)$</td>
</tr>
<tr>
<td>1</td>
<td>$f(x + a)$</td>
</tr>
<tr>
<td>2</td>
<td>$e^{iak} f(x)$</td>
</tr>
<tr>
<td>3</td>
<td>$f(ax)$</td>
</tr>
<tr>
<td>4</td>
<td>$f'(x)$</td>
</tr>
<tr>
<td>5</td>
<td>$ixf(x)$</td>
</tr>
<tr>
<td>6</td>
<td>$f \ast g(x)$</td>
</tr>
<tr>
<td>7</td>
<td>$f(x)g(x)$</td>
</tr>
</tbody>
</table>

### Laplace Transform Summary

<table>
<thead>
<tr>
<th>$y(t)$</th>
<th>$Y(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Definition and inversion</strong></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{st} Y(s) ds$</td>
<td>$\int_0^\infty e^{-st} y(t) dt$</td>
</tr>
<tr>
<td><strong>Examples</strong></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$e^{-at}$</td>
</tr>
<tr>
<td>2</td>
<td>$\sin(\omega t)$</td>
</tr>
<tr>
<td>3</td>
<td>$\cos(\omega t)$</td>
</tr>
<tr>
<td><strong>Properties</strong></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$c_1 y_1(t) + c_2 y_2(t)$</td>
</tr>
<tr>
<td>1</td>
<td>$y'(t)$</td>
</tr>
<tr>
<td>2</td>
<td>$ty(t)$</td>
</tr>
<tr>
<td>3</td>
<td>$e^{at} y(t)$</td>
</tr>
<tr>
<td>4</td>
<td>$u(t-a)y(t-a)$</td>
</tr>
<tr>
<td>5</td>
<td>$y_1 \ast y_2(t)$</td>
</tr>
</tbody>
</table>

In property 4: $u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$. In property 5:

$$y_1 \ast y_2(t) = \int_0^t y_1(t-\tau) y_2(\tau) d\tau$$

Note about the examples 3 and 4: The value of a Fourier or inverse Fourier transform is insensitive to changes of the input function at a single point. In property 6: $f \ast g(x) = \int_{-\infty}^{\infty} f(x-y)g(y) dy$
University of British Columbia
Math 301, Section 201

Final exam

Date: April 18, 2012
Time: 8:30 - 11:00am

Name (print):
Student ID Number:
Signature:

Instructor: Richard Froese

Instructions:

1. No notes, books or calculators are allowed. A summary sheet with properties of Fourier and Laplace transforms is provided.

2. Read the questions carefully and make sure you provide all the information that is asked for in the question.

3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.

4. Answer the questions in the space provided. Continue on the back of the page if necessary.

<table>
<thead>
<tr>
<th>Question</th>
<th>Mark</th>
<th>Maximum</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
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<tr>
<td>2</td>
<td>14</td>
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<tr>
<td>3</td>
<td>15</td>
<td></td>
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<tr>
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<td>14</td>
<td></td>
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</table>
1. Evaluate

\[ f(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{(k^2 + 1)^2} dk. \quad \text{(for } x \in \mathbb{R}) \]

For \( x > 0 \) close the contour in the upper half plane. There is a pole of order 2 at \( k = i \). Thus

\[ f(x) = 2\pi i \lim_{k \to i} \frac{d}{dk} \left( \frac{e^{ikx}}{(k-i)^2(k+i)^2} \right) \]

\[ = 2\pi i \lim_{k \to i} \left[ \frac{i x e^{ikx} (k+i)^2 - 2(k+i) e^{ikx}}{(k+i)^4} \right] \]

\[ = 2\pi i \left[ \frac{i x e^{-x} (2i)^2 - 2(2i) e^{-x}}{(2i)^4} \right] = \frac{\pi i e^{-x} (-4ix - 4i)}{16} \]

\[ = \frac{\pi}{2} e^{-x} (x+1) \]

For \( x < 0 \) we can notice \( f(x) = \overline{f(-x)} = \frac{\pi}{2} e^{x} (-x+1) \)

\[ = \frac{\pi}{2} e^{x} (-x+1) = \frac{\pi}{2} e^{-|x|} (|x|+1). \]

So \( f(x) = \frac{\pi}{2} e^{-|x|} (|x|+1) \) for all \( x \in \mathbb{R} \).
2. Evaluate

\[ I = \int_0^\infty \frac{x^{\alpha-1}}{1 + x} \, dx, \quad 0 < \alpha < 1. \]

explaining all your steps.

Choose the branch cut on the positive real axis.

We have

\[ \int_{-\infty}^{\infty} \frac{x^{\alpha-1}}{1 + x} \, dx = \int_{-\infty}^{\infty} e^{2\pi i (\alpha - 1)} \frac{x^{\alpha-1}}{1 + x} \, dx + \int_{C_\infty} + \int_{C_R} = 
\]

\[ 2\pi i \left( C_R \circ \text{Res} \left( \frac{e^{-x}}{1 + x} \right) \right) = 2\pi i \cdot e^{2\pi i (\alpha - 1)} \]

\[ \int_{C_\infty} \frac{2^{\alpha-1}}{1 + x} \, dx \leq \frac{R^{\alpha-1}}{1 - \epsilon} \cdot 2\pi R \sim R^{\alpha-1} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty \]

\[ \int_{C_R} \frac{2^{\alpha-1}}{1 + x} \, dx \leq \frac{2^{\alpha-1}}{R - 1} \cdot 2\pi R \sim R^{\alpha-1} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty \]

So

\[ (1 - e^{2\pi i (\alpha - 1)}) \, I = 2\pi i \cdot e^{2\pi i (\alpha - 1)} \]

\[ I = \frac{2\pi i}{-e^{\pi i (\alpha - 1)} - e^{-\pi i (\alpha - 1)}} = \frac{\pi}{\sin(\pi (1 - \alpha))} \]
2. (a) What are the branch points of the multivalued function \( f(z) = (z^2 + 1)^{1/2} \)? Is infinity a branch point?

The branch points are \( +i \) and \( -i \). Infinity is not a branch point since we can write

\[
f(z) = z \left( 1 + \frac{1}{z^2} \right)^{1/2} \quad \text{(as a multivalued function)}.
\]

This returns to the same value when \( z \) goes around a large circle.

(b) Using the range of angles method, define a branch of \( f(z) \) that is continuous and positive for \( z \in \mathbb{R} \).

Let \( \Theta_1(z), \Theta_2(z) \) be the angles shown.

where \( \Theta_1 \in \left( -\frac{3\pi}{2}, \frac{\pi}{2} \right), \Theta_2 \in \left[ -\frac{\pi}{2}, \frac{3\pi}{2} \right) \)

Then \( f(z) = \sqrt{z^2 + 1} \ e^{i(\Theta_1 + \Theta_2)/2} \) is the required branch.

(c) Using the range of angles method, define a branch of \( f(z) \) that is analytic outside the unit circle and negative for large positive \( z \in \mathbb{R} \).

Change the range to \( \Theta_1 \in \left( -\frac{3\pi}{2}, \frac{\pi}{2} \right), \Theta_2 \in \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right) \)

For \( z \) on the positive real axis we have \( \Theta_1 = -\Theta_2 + 2\pi k \) for some \( k \).

Determine if \( \Theta \in \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right) \) then for \( z \) on the positive real axis

\( \Theta_2 \in \left( 2\pi, \frac{3\pi}{2} \right], -\Theta_2 \in \left[ -\frac{3\pi}{2}, -2\pi \right), -\Theta_2 + 2\pi \in \left[ -\frac{\pi}{2}, 0 \right), \) which is in the range of \( \Theta_1 \), so \( k = 1 \). So for \( z \) on pos. real axis

\[
f(z) = \sqrt{|z^2 + 1|} \ e^{i(\Theta_1 + \Theta_2)/2} = \sqrt{|z^2 + 1|} \ e^{i\pi} = -\sqrt{|z^2 + 1|}.
\]
3. In this question we evaluate
\[ I = \int_{0}^{\infty} \frac{\sqrt{x}}{x^2 + 1} \, dx. \]

(a) Draw a diagram that shows the contour you are using, and the branch cut for \( \sqrt{x} \).

(b) What does the Cauchy residue theorem say for your contour? Compute the residues that appear.

Let \( f(z) = \frac{\sqrt{z}}{z^2 + 1} \). Then
\[
\int_{[e, R]} f(z) \, dz + \int_{C_R} f(z) \, dz - \int_{[e, -r]} f(z) \, dz + \int_{C_e} f(z) \, dz = 2\pi i \text{Res} \left[ f(z); i \right]
\]
\[
= \frac{2\pi i}{2i} = \pi e^{i\pi/2}.
\]
To evaluate \( \int_{[e, -r]} f(z) \, dz \) let \( u(z) = -r \), \( dz = -dr \)
\[
\int_{[-e, -r]} f(z) \, dz = -\int_{e}^{0} f(-r) \, dr = -\int_{e}^{0} \frac{i\sqrt{r}}{r^2 + 1} \, dr.
\]
(c) Which integrals (over portions of your contour) tend to zero in the limit? Give estimates that show this.

\[
\left| \int_{C_K} f(z) \, dz \right| \leq \max_{|z| = R} \left| \frac{\sqrt{z^2}}{z^2 + 1} \right| \cdot \pi R \leq \frac{R^{1/2}}{R^2 - 1} \pi R \to 0 \quad \text{as } R \to \infty
\]

\[
\left| \int_{C_\varepsilon} f(z) \, dz \right| \leq \max_{|z| = \varepsilon} \left| \frac{\sqrt{z^2}}{z^2 + 1} \right| \cdot \pi \varepsilon \leq \frac{\varepsilon^{1/2}}{1 - \varepsilon^2} \pi \varepsilon \to 0 \quad \text{as } \varepsilon \to 0
\]

(d) Evaluate \( I \).

\[
\lim_{\varepsilon \to 0} \lim_{R \to 0} \int_{C_{\varepsilon, R}} f(z) \, dz = \int_{\varepsilon}^{R} f(z) \, dz = -i \int_{\varepsilon}^{R} \frac{\sqrt{x^2}}{x^2 + 1} \, dx \to -i \int_{\varepsilon}^{R} \, dx
\]

\[
S_0 \quad (1 + i) I = \pi e^{i \pi/4}
\]

\[
I = \frac{\pi e^{i \pi/4}}{1 + e^{i \pi/2}} = \frac{\pi}{2 \cos(\pi/4)} = \frac{\sqrt{2} \pi}{2}
\]
4. Find a fractional linear transformation that maps the shaded region \( \{ z : \text{Im}z \geq 0, |z - 4i| \geq 1 \} \) to the annulus \( \{ z : 1 \leq |z| \leq A \} \) with the real line mapping to the unit circle. What is the outer radius \( A \)? (Recall that \( \alpha \) and \( \alpha^* \) are symmetric with respect to a circle centred at \( a \) with radius \( R \) if \( \alpha^* = a + R^2/(\overline{\alpha} - \overline{a}) \).)

\[
\begin{align*}
\text{Find } \alpha, \alpha^* \text{ that are symmetric with the circle and the real axis.} \quad \text{We need } \alpha^* = \overline{\alpha} \text{ and } \alpha^* = 4i + \frac{1}{\overline{\alpha} + 4i} \quad \therefore \\
\overline{\alpha} &= 4i + \frac{1}{\alpha + 4i} \quad (\overline{\alpha} - 4i)(\overline{\alpha} + 4i) = 1 \quad \overline{\alpha}^2 + 16 = 1 \\
\overline{\alpha}^2 &= -15 \quad \overline{\alpha} = \pm i \sqrt{15} \quad \text{Map } -i \sqrt{15} \to 0 \quad 0 \to 1 \quad i \sqrt{15} \to \infty \\
f(z) &= \frac{z + i \sqrt{15}}{z - i \sqrt{15}} \quad (-1) \quad = \frac{-z - i \sqrt{15}}{z - i \sqrt{15}} \\
\text{When } z = 3i \quad f(z) &= \frac{-3i - i \sqrt{15}}{3i - i \sqrt{15}} = \frac{\sqrt{15} + 3}{\sqrt{15} - 3} = A
\end{align*}
\]
5. Consider the complex velocity potentials for ideal inviscid flow given by \( \Omega_1(z) = iV_0(z - 1/z) \) and \( \Omega_2(z) = i\gamma \log(z) \). Here \( V_0 > 0 \) and \( \gamma > 0 \).

(a) Show that \( \Omega_1(z) \) describes a flow around the unit circle \( \{ z : |z| = 1 \} \).

We must show \( \text{Im} \Omega_1(e^{i\theta}) = \text{const} \).

\[ \Omega_1(e^{i\theta}) = iV_0 \left( e^{i\theta} - \frac{1}{e^{i\theta}} \right) = -2V_0 \sin(\theta) \]

So

\[ \text{Im} \Omega_1(e^{i\theta}) = 0. \]

(b) What are the streamlines for the flow described by \( \Omega_2(z) \)? Plot several of these, and indicate with arrows the direction of the flow.

\[ \Omega_2(z) = i\gamma \log(z) = i\gamma |\log(z)| + i\gamma i \arg(z) \]

\[ \text{Im} \Omega_2 = \gamma \log |z| = c \quad \text{when } |z| = \text{const}. \quad \text{So} \]

streamlines are circles. The complex velocity is

\[ \Omega'_2(z) = -i \frac{\gamma}{z}, \quad \text{so } \Omega'_2(z) = \Re e^{i\theta} \quad \Omega'_{2}(z) = -i \frac{e^{i\theta}}{z} \]
(c) Explain why \( \Omega(z) = \Omega_1(z) + \Omega_2(z) \) also describes a flow around the unit circle \( \{ z : |z| = 1 \} \). What is the asymptotic velocity of this flow as \( |z| \to \infty \)?

Both \( \text{Im} \, \Omega_1 \) and \( \text{Im} \, \Omega_2 \) are constant on the unit circle so \( \text{Im} (\Omega_1 + \Omega_2) = \text{const} \). Hence

\[
\lim_{|z| \to \infty} \frac{\Omega'(z)}{z} = -i \, V_o
\]

(d) Where are the stagnation points for the flow described by \( \Omega(z) \)? For what values of \( \gamma > 0 \) are the stagnation points on the boundary of the unit circle?

\[
N_c = 1, \quad i \, V_o \left( 1 + \frac{1}{z^2} \right) + i \frac{\gamma}{z} = 0, \quad z^2 + \frac{\gamma}{V_o} \, z + 1 = 0
\]

\[
z = -\frac{\gamma}{2 \, V_o} \pm \sqrt{\frac{\gamma^2}{4 \, V_o^2} - 1}, \quad \text{If} \quad \frac{\gamma}{2 \, V_o} < 1, \quad \text{then, lies on the unit circle.}
6. Solve

\[ u_t(x,t) = u_{xx}(x,t) + u(x,t), \quad -\infty < x < \infty, \quad t > 0 \]
\[ u(x,0) = e^{-x^2} \]

Take Fourier transform:

\[ \hat{u}_t(k,t) = -k^2 \hat{u}(k,t) + \hat{u}(k,t) = (1-k^2) \hat{u}(k,t) \]
\[ \hat{u}(k,0) = \sqrt{n} e^{-k^2/4} \]
\[ \hat{u}(k,t) = C(k) e^{(1-k^2)t} \]
\[ \hat{u}(k,0) = C(k) = \sqrt{n} e^{-k^2/4} \]
\[ \hat{u}(k,t) = \sqrt{n} e^{-k^2/4} (1-k^2)t = \sqrt{n} e^t e^{-k^2(\frac{1}{4}t+t)} \]
\[ = \sqrt{n} e^t e^{-k^2(\frac{1}{2}+\frac{3}{2}t)} / 2 \]
\[ = \sqrt{2n} e^t \frac{\sqrt{\frac{1}{2}+\frac{3}{2}t}}{\sqrt{\frac{1}{2}+\frac{3}{2}t}} \frac{1}{\sqrt{1+4t}} e^{-k^2(\frac{1}{4}t+t)} / 2 \]
\[ \text{let } \sigma = \sqrt{\frac{1}{2}+\frac{3}{2}t} \]
\[ = \frac{1}{\sqrt{1+4t}} e^{t/\sigma} e^{-k^2\sigma^2/2} \]
\[ u(x,t) = \frac{e^t}{\sqrt{1+4t}} e^{-x^2/(1+4t)} \]
7. (a) What is the Laplace transform $Y(s)$ of the solution $y(t)$ to the initial value problem

$$y'''(t) - 6y''(t) + 11y'(t) - 6y(t) = e^{-2t}, \quad y(0) = y'(0) = 0, \quad y''(0) = 1?$$

$$s^2 Y(s) - s Y(0) - y'(0) - 6s Y(s) + b s y'(0) + 6 y'(0) + 11 s Y(s) - 11 y(0)$$

$$- 6 Y(s) = \frac{1}{s+2},$$

$$\gamma(s) = 1 + \frac{1}{s+2}, \quad Y(s) = \frac{s+3}{(s^2 - 6s^2 + 11s - 6)(s+2)}.$$

7. (b) Use the Nyquist criterion to determine whether the solution $y(t)$ is bounded as $t$ tends to infinity.

The poles of $\gamma$ are at $s = -2$ and $s$ at the zeros of $p(s) = s^2 - 6s^2 + 11s - 6$. Use Nyquist to determine if $p(s)$ has poles in the right half plane.

$$p(iy) = -i y^3 + 6 y^2 + i (11y - 6)$$

$$\text{Re}(p(iy)) = 6 y^2 - 6, \quad \text{Im}(p(iy)) = -y^3 + 11 y$$

$$\text{Re} p = 0 \text{ when } y = \pm 1, \quad \text{Im} p = 0 \text{ when } y = 0, \pm \sqrt{11}$$

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<thead>
<tr>
<th>$\gamma$</th>
<th>$\text{Re}$</th>
<th>$\text{Im}$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$-b y^2$</td>
<td>$-y^3$</td>
</tr>
<tr>
<td>$\sqrt{11}$</td>
<td>$&gt; 0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$-\sqrt{11}$</td>
<td>$&gt; 0$</td>
<td>$0$</td>
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<td>$0$</td>
<td>$&lt; 0$</td>
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<tr>
<td>$\infty$</td>
<td>$&gt; 0$</td>
<td>$t$</td>
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</table>

All the zeros of $p$ are in the RHP $\rightarrow$ So $\gamma(t)$ is not bounded.

(in fact $p(s) = (s-1)(s-2)(s-3)$ !)
1. Evaluate

\[ I = \int_0^\infty \frac{\cos(\pi x)}{1 - 4x^2} \, dx, \]

explaining all your steps.

The denominator vanishes at \( x = \pm \frac{1}{2} \), but so does \( \cos(\pi x) \) so the singularity is removable at \( x = \pm \frac{1}{2} \).

\[
I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos(n \pi x)}{1 - 4x^2} \, dx
= \frac{1}{2} \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{in\pi x}}{1 - 4x^2} \, dx
= \lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{-\frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}}{k + \epsilon}
= \text{Re} \left[ \frac{1}{2} \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{in\pi x}}{1 - 4x^2} \, dx \right]
= I_{k\epsilon}
\]

\[
I_{k\epsilon} = \int_{C_R} - \int_{C_{\frac{1}{2}}} + \int_{C_{\epsilon}} = 0
\]

\[
\lim_{\epsilon \to 0} \int_{C_{\epsilon}} = \pi i \text{Res} \left[ \frac{e^{in\pi x}}{1 - 4x^2} \right] = \frac{in e^{-in\pi}}{4}
\]

\[
\lim_{\epsilon \to 0} \int_{C_{\epsilon}} = i \pi \text{Res} \left[ \frac{e^{in\pi x}}{1 - 4x^2} \right] = \frac{in e^{in\pi}}{4}
\]

\[
\lim_{R \to \infty} \int_{C_R} = 0 \text{ (Jordan's lemma)} \quad \text{so} \quad I = \frac{1}{2} \left( \frac{in e^{-in\pi}}{4} - \frac{in e^{in\pi}}{4} \right)
\]

\[
= \frac{\pi}{4}
\]
3. Let \( g(z) \) be the branch of \((z - 1)^{1/2}\) defined by

\[
g(z) = e^{\frac{1}{2} \left( \text{Log}(z) + \text{Log}(z-1) \right)},
\]

when \( z \to -1 \) from \( \text{UHP} \).

(a) Compute \( \lim_{z \to -1} g(z) \) when \( z \) approaches \(-1\) from the upper half plane, and from the lower half plane. Is \( g(z) \) continuous at \(-1\)?

\[
g(z) = e^{\frac{1}{2} \left( \text{Log}(z) + \text{Log}(z-1) \right)},
\]

when \( z \to -1 \) from \( \text{UHP} \),

\[
\text{Log}(z) \to i\pi \quad \text{and} \quad \text{Log}(z-1) \to \ln 2 + i\pi
\]

so

\[
g(z) \to e^{\frac{1}{2} \left( i\pi + i\pi + \ln 2 \right)} = -\sqrt{2},
\]

when \( z \to -1 \) from \( \text{LHP} \),

\[
\text{Log}(z) \to -i\pi \quad \text{and} \quad \text{Log}(z-1) \to \ln 2 - i\pi
\]

so

\[
g(z) \to e^{\frac{1}{2} \left( -i\pi - i\pi + \ln 2 \right)} = -\sqrt{2}
\]

Yes, \( g(z) \) is continuous at \(-1\).

(b) Where are the branch cuts for \( g(z) \)?

Branch cuts on \([0, 1]\), on \((\infty, 0)\) the branch cuts cancel.
(c) Show how to construct $g(z)$ using the 'range of angles' method.

To use the range of angles method we write

$$g(z) = \sqrt{2} (z-1)^{\frac{1}{2}} e^{i(\theta_1 + \theta_2)/2}$$

with $\theta_1 \in (-\pi, \pi]$ and $\theta_2 \in (-\pi, \pi]$. This has the same branch cut on $[0,1]$ since the double branch cut on $(-\infty, 0]$ cancels.

Also when $x > 1$ then $g(x)$ is positive for both the original defn and the range of angles defn (where $\theta_1 = \theta_2 = 0$).

Same branch cut & same values at $x > 0 \implies$ same branch.

(d) What is the value of $g(i)$ and $g(-i)$?

\[
\begin{align*}
\text{When } z &= i \quad \theta_1 = \frac{3\pi}{4} \quad \theta_2 = \frac{\pi}{2} \quad g(z) = |i| (i-1)^{\frac{1}{2}} e^{\frac{i\pi}{4} \frac{\pi}{2}} \quad &\text{or} \\
&= 2^{\frac{1}{2}} e^{i\pi/8} \quad \text{when } z = -i \quad \theta_1 = \frac{3\pi}{4} \quad \theta_2 = \frac{-\pi}{2} \quad &\text{so} \\
g(z) &= 2^{\frac{1}{2}} e^{-i\pi/8} \\
\text{Compare with the original defn: When } z &= i \quad \log(z) = \frac{i\pi}{2} \\
\log(i-1) &= \ln\sqrt{2} + i \frac{3\pi}{4} \quad e^{\frac{1}{2} \left( \frac{\ln\sqrt{2}}{2} + \ln\sqrt{2} + i \frac{3\pi}{4} \right)} = 2^{\frac{1}{2}} e^{i\pi/8} \\
z &= -i \text{ similar.}
\end{align*}
\]
4. Solve Laplace's equation $\Delta \phi = 0$ in the region between the circles \( \{ z : |z| = 2 \} \) and \( \{ z : |z-1| = 1 \} \) with boundary condition $\phi = 1$ on the inner circle and $\phi = 0$ on the outer circle.

Find the FLT that maps $\mathbb{C} \ni 2 \rightarrow \infty$, $z \rightarrow 0$ and $-z \rightarrow 1$

$$f(z) = \frac{z - 2i}{z - 2} - \frac{4}{-2 - 2i} = \frac{(z-2i)(1-i)}{z-2}$$

$$f(1+i) = \frac{(1-i)(1-i)}{-1+i} = i-1$$

So region maps to \( \text{(check e.g. } f(-1) \text{)} \).

Write \( f(x+iy) = u(x, y) + i\,v(x, y) \) , \( v(x, y) = \frac{y-x^2-y^2}{x^2+y^2-yx+y} \)

is the solution.
5. Consider the following region with boundary data.

(a) Show that the Joukowsk map \( J(z) = (1/2)(z + 1/z) \) maps the shaded region onto the upper half plane.

\[
\phi = 1
\]

\[
\phi = 0
\]

-1 1

\[
\text{So the boundary maps to the boundary of the upper half plane. To check that the map is to the upper (not lower) half plane, check orientation, for an interior pt. } x = 2i
\]

\[
z \mapsto \frac{1}{2} \left( 2i + \frac{1}{2i} \right) = \frac{1}{2} \left( 2i - \frac{1}{2}i \right) = \frac{1}{2} \cdot \frac{3}{2}i = \frac{3i}{2} \in \text{U.H.P.}
\]
(b) Solve Laplaces equation $\Delta \phi = 0$ in the shaded region, with the indicated boundary conditions.

First solve the problem in $\mu \epsilon U, H, \hat{P}$. ($\omega = \nu + iv$, plane)

$$\begin{align*}
\Phi_0 &= 0, & \Phi_1 &= 1, & \Phi_0 &= 0 \\
-1 & & 1
\end{align*}$$

$$\Phi(w) = A \text{Arg} (w+1) + B \text{Arg} (v-1) + \zeta$$

Boundary condition $C = 0, \pi B = 1, \pi A + \pi B = 0$

$$B = \frac{\pi}{\zeta}, \quad A = -\frac{1}{\pi}$$

$$\Phi(u, v) = -\frac{1}{\pi} \cot^{-1} \left( \frac{u+1}{v} \right) + \frac{1}{\pi} \cot^{-1} \left( \frac{u-1}{v} \right).$$

Write $J(x+iy) = \frac{1}{2} \left( x+iy + \frac{x-iy}{x^2+y^2} \right)$

$$= \frac{1}{2} \left( \frac{x}{x^2+y^2} \right) + i \left( \frac{1}{2} \left( \frac{y}{x^2+y^2} \right) \right).$$

$$\Phi(x, y) = \Phi(u(x, y), v(x, y)).$$
6. (a) What is the Fourier transform of $x^2 e^{-x^2/2}$?

$$\mathcal{F}(e^{-x^2/2}) = \sqrt{2\pi} e^{-k^2/2} \quad \text{(from table with } \sigma = 1)$$

$$\mathcal{F}(x^2 e^{-x^2/2}) = -\frac{d^2}{dk^2} \sqrt{2\pi} e^{-k^2/2}$$

$$= -\sqrt{2\pi} \frac{d}{dk} - k e^{-k^2/2}$$

$$= \sqrt{2\pi} \left( e^{-k^2/2} - k^2 e^{-k^2/2} \right) .$$

(b) Use the Fourier transform to compute $\int_{-\infty}^{\infty} e^{-|x-y|} e^{-|y|} dy$. (Hint: you can use the result from an earlier problem for the last step.)

$$\mathcal{F} \left[ e^{-|x|} \right] = \frac{2}{1 + k^2} \quad \therefore \quad \mathcal{F} \left[ e^{-|x|} \cdot e^{-|y|} \right]$$

$$= \frac{2}{1 + k^2} \cdot \frac{2}{1 + k^2} = \frac{4}{(1 + k^2)^2} . \quad \therefore \quad \text{Thus} \quad e^{-|x|} \cdot e^{-|y|} =$$

$$\int_{-\infty}^{\infty} e^{-|x-y|} e^{-|y|} dy = \mathcal{F}^{-1} \left[ \frac{4}{(1 + k^2)^2} \right] = \frac{4}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \frac{1}{(1 + k^2)^2} dk$$

$$= \frac{\pi}{2} \cdot e^{-x^2} (|x|+1) = e^{-|x|} (|x|+1) .$$
7. (a) Compute the Laplace transform \( Y(s) \) of the solution \( y(t) \) to

\[
y'''(t) + y'(t) - y(t) = \cos(t),
\]

\[
y'''(0) = 1, \quad y''(0) = y'(0) = y(0) = 0,
\]

\[
 s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - 5 Y(s) - y(0) - Y(s) = \frac{s}{s^2 + 1}
\]

\[
 (s^4 + s - 1) Y(s) - 1 = \frac{s}{s^2 + 1}
\]

\[
 Y(s) = \frac{1}{s^4 + s - 1} \left( \frac{s}{s^2 + 1} + 1 \right)
\]

7. (b) Use the Nyquist criterion to decide whether the solution \( y(t) \) grows exponentially as \( t \to \infty \) or not.

\[
p(s) = \frac{s^4 + s - 1}{s + 1}
\]

Check whether \( s^4 + s - 1 \) has any zeros in \( \text{RHP} \).

\[
p(i \gamma) = \gamma^4 + i \gamma - 1 \quad \Re(p(i \gamma)) = \gamma^4 - 1 \quad \Im(p(i \gamma)) = \gamma.
\]

The real \( s \)-axis is traversed \( \gamma = \pm 1 \)

\[
\gamma \quad \text{Re} \quad \text{Im} \quad p
\]

\[
\infty \quad \gamma^4 \quad -\gamma
\]

\[
1 \quad 0 \quad 1
\]

\[
0 \quad -1 \quad 0
\]

\[
-1 \quad 0 \quad -1
\]

\[
-\infty \quad -\gamma \quad -\gamma
\]

\[
\Delta \arg p = 2 \pi
\]

\[
N_+ (p) = \frac{1}{2 \pi} \left( 2 \pi + 4 \pi \right) = 3
\]

Yes, solution grows exponentially since there are \( 3 \) zeros in \( \text{RHP} \).