Math 300: notes on uniform convergence
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Let $F_n(z)$, $n = 0, 1, 2, \ldots$ be a sequence of complex functions defined for $z \in K \subseteq \mathbb{C}$.

The sequence $F_n(z)$ converges to $F(z)$ pointwise if for each fixed $z \in K$

$$\lim_{n \to \infty} |F_n(z) - F(z)| = 0$$

The sequence $F_n(z)$ converges to $F(z)$ uniformly on $K$ if

$$\lim_{n \to \infty} \sup_{z \in K} |F_n(z) - F(z)| = 0$$

(Here sup denotes the supremum or least upper bound. If $K$ is closed and bounded and $F(z)$ is continuous then you may replace sup with max in the definition.)

The series $\sum_{j=0}^{\infty} f_j(z)$ converges to $F(z)$ uniformly on $K$ if the partial sums $F_n(z) = \sum_{j=0}^{n} f_j(z)$ converge to $F(z)$ uniformly on $K$.

**Proposition** Suppose the power series $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ converges at $z = z_1$. Let $R = |z_1 - z_0|$. Then

(i) The series converges pointwise for any $z$ with $|z - z_0| < R$.

(ii) If $F(z)$ denotes the (pointwise) limit in (i) and $R_1 < R$, then the series converges to $F(z)$ uniformly on $\{z : |z - z_0| \leq R_1\}$.

**Proof:**

(i) Since $\sum_{j=0}^{\infty} a_j (z_1 - z_0)^j$ converges its terms must tend to zero, i.e., $|a_j(z_1 - z_0)^j| = |a_j|R^j \to 0$. This implies that they are bounded, i.e., there is a positive number $M$ such that $|a_j|R^j \leq M$. Now let $z$ satisfy $R_1 = |z - z_0| < R$. Then

$$|a_j(z - z_0)^j| = |a_j|R_1^j = |a_j|R^j \left( \frac{R}{R_1} \right)^j \leq M \left( \frac{R}{R_1} \right)^j$$

By the comparison test with the geometric series on the right we conclude that $\sum_{j=0}^{\infty} a_j (z - z_0)^j$ converges. Call the limit $F(z)$.

(ii) To show that the convergence to $F(z)$ is uniform we let $F_n(z)$ denote the partial sum and
estimate

\[
\max_{z:|z-z_0|\leq R_1} |F_n(z) - F(z)| = \max_{z:|z-z_0|\leq R_1} \left| \sum_{j=0}^{n} a_j (z - z_0)^j - \sum_{j=0}^{\infty} a_j (z - z_0)^j \right|
\]

\[
= \max_{z:|z-z_0|\leq R_1} \left| \sum_{j=n+1}^{\infty} a_j (z - z_0)^j \right|
\]

\[
\leq \max_{z:|z-z_0|\leq R_1} \sum_{j=n+1}^{\infty} |a_j||z - z_0|^j
\]

\[
\leq \sum_{j=n+1}^{\infty} |a_j|R_1^j
\]

\[
\leq M \sum_{j=n+1}^{\infty} \left( \frac{R_1}{R} \right)^j
\]

The right side \(\to 0\) as \(n \to \infty\) which shows the uniform convergence.

**Theorem** If \(F_n(z) \to F(z)\) uniformly on \(K\) and each \(F_n(z)\) is continuous, then \(F(z)\) is also continuous.

**Proof:** Let \(z_0 \in K\). We must show that given \(\epsilon > 0\) there exists \(\delta > 0\) such that \(|z - z_0| < \delta \Rightarrow |F(z) - F(z_0)| < \epsilon\).

By the uniform convergence of \(F_n\) there exists \(N\) such that \(\sup_{z \in K} |F_N(z) - F(z)| < \epsilon/3\).

By the continuity of \(F_N\) there exists \(\delta\) such that \(|z - z_0| < \delta \Rightarrow |F_N(z) - F_N(z_0)| < \epsilon/3\).

Then \(|z - z_0| < \delta \Rightarrow |F(z) - F(z_0)| = |F(z) - F_N(z) + F_N(z) - F_N(z_0) + F_N(z_0) - F(z_0)|\)

\[
\leq |F(z) - F_N(z)| + |F_N(z) - F_N(z_0)| + |F_N(z_0) - F(z_0)|
\]

\[
= \epsilon/3 + \epsilon/3 + \epsilon/3
\]

\[
= \epsilon
\]

**Theorem** If \(F_n(z)\) is a sequence of continuous functions and \(F_n(z) \to F(z)\) uniformly on \(K\), then for every contour \(\gamma\) in \(K\)

\[
\lim_{n \to \infty} \int_{\gamma} F_n(z)dz = \int_{\gamma} F(z)dz
\]

**Proof:**

\[
\left| \int_{\gamma} F_n(z)dz - \int_{\gamma} F(z)dz \right| \leq \int_{\gamma} |F_n(z) - F(z)|dz
\]

\[
\leq \max_{z \in \gamma} |F_n(z) - F(z)| \times \text{length}(\gamma)
\]

\[
\leq \max_{z \in K} |F_n(z) - F(z)| \times \text{length}(\gamma) \to 0
\]

as \(n \to \infty\).
**Theorem**  (Morera’s theorem) Let \( f(z) \) be continuous in a domain \( D \). Then \( \int_{\gamma} f(z)dz = 0 \) for every closed contour \( \gamma \) in \( D \) \iff \( f \) is analytic in \( D \).

**Proof:** We showed that if \( f \) is analytic then \( \int_{\gamma} f(z)dz = 0 \) for every closed contour \( \gamma \) in \( D \). In the other direction we showed that if \( \int_{\gamma} f(z)dz = 0 \) for every closed contour \( \gamma \) in \( D \) then \( f \) has an antiderivative \( F(z) \) in \( D \). The function \( F \) is analytic with \( F' = f \) so it remains to show that the derivative of an analytic function is analytic. This can be done using the Cauchy integral formula (see the textbook). \( \Box \)

**Theorem**  Suppose that \( f_n(z) \) is a sequence of analytic functions converging uniformly to \( f(z) \) in a domain \( D \). Then \( f \) is analytic in \( D \).

**Proof:** Let \( \gamma \) be a closed contour in \( D \). Since each \( f_n(z) \) is analytic we have \( \int_{\gamma} f_n(z)dz = 0 \) for each \( n \). Thus

\[
\int_{\gamma} f(z)dz = \lim_{n \to \infty} \int_{\gamma} f_n(z)dz = 0
\]

This implies \( f \) is analytic by Morera’s theorem. \( \Box \)