Math 300 Homework 7

4.2: 5
The circle $C$ is parametrized by $z(\theta) = i + 4e^{i\theta}$, $\theta \in [0, 2\pi]$. So

\[
\int_C \frac{6}{z - i}^2 = \int_0^{2\pi} \frac{6}{(4e^{i\theta})^2} 4i e^{i\theta} \, d\theta = \frac{24i}{16} \int_0^{2\pi} e^{-i\theta} \, d\theta = 0, \\
\int_C \frac{2}{z - i} = \int_0^{2\pi} \frac{2}{e^{i\theta}} 4i e^{i\theta} \, d\theta = 2i \int_0^{2\pi} d\theta = 4\pi i, \\
\int_C \frac{1}{z} = \int_0^{2\pi} \frac{4i e^{i\theta} \, d\theta}{e^{4i\theta}} = 4i \int_0^{2\pi} e^{4i\theta} \, d\theta = 0 \\
\int_C -3(z - i)^3 \, dz = \int_0^{2\pi} (4e^{i\theta})^3 4i e^{i\theta} \, d\theta = -192 \int_0^{2\pi} e^{4i\theta} \, d\theta = 0
\]

The desired integral is the sum of these by linearity, so its value is $4\pi i$.

4.2: 6(a)
A parametrization of $\Gamma$ is $z(\theta) = 2e^{i\theta}$, $\theta \in [0, 2\pi]$. So

\[
\int_{\Gamma} z \, dz = \int_0^{2\pi} 2e^{i\theta} \, 2ie^{i\theta} \, d\theta = 4i \int_0^{2\pi} d\theta = 8\pi i.
\]

4.2: 6(b)
A parametrization of $\Gamma$ is $z(\theta) = 2e^{-i\theta}$, $\theta \in [0, 2\pi]$. So

\[
\int_{\Gamma} z \, dz = \int_0^{2\pi} 2e^{-i\theta} \, 2(-i)e^{-i\theta} \, d\theta = -4i \int_0^{2\pi} d\theta = -8\pi i.
\]

4.2: 6(c)
A parametrization of $\Gamma$ is $z(\theta) = 2e^{-i\theta}$, $\theta \in [0, 6\pi]$. So

\[
\int_{\Gamma} z \, dz = \int_0^{6\pi} 2e^{-i\theta} \, 2(-i)e^{-i\theta} \, d\theta = -4i \int_0^{6\pi} d\theta = -24\pi i.
\]
4.2: 7
A parametrization of the segment $\Gamma$ is $z(t) = t(1 + 2i)$, $t \in [0, 1]$. So
\[
\int_{\Gamma} \Re(z)\,dz = \int_0^1 \Re(t(1 + 2i))dt = \int_0^1 t(1 + 2i)dt = \frac{1 + 2i}{2}
\]

4.2: 11(a)
A parametrization of $\Gamma$ is $z(t) = (1-t)(-i) + t$, $t \in [0, 1]$. So
\[
\int_{\Gamma}(2z - 1)\,dz = \int_0^1 2((1-t)(-i) + t)(i + 1)dt = \int_0^1 2 - 2i + 4it dt = \int_0^1 (1 + i)((2i)t + 1 - 2i + 2t)dt = 3 + i.
\]

4.2: 11(b)
A parametrization of $\Gamma_1$ is $z_1(t) = (-i)(1-t) = -i + it$. Then $\dot{z}_1(t) = i$ and
\[
\int_{\Gamma_1}(2z - 1)\,dz = \int_0^1 (-2i + 2it + 1)idt = 1 + i.
\]
A parametrization of $\Gamma_2$ is $z_2(t) = t$. Then $\dot{z}(z_2(t)) = 1$ and
\[
\int_{\Gamma_2}(2z - 1)\,dz = \int_0^1 (2t + 1)dt = 2.
\]
Now
\[
\int_{\Gamma_1 + \Gamma_2}(2z - 1)\,dz = 1 + i + 2 = 3 + i.
\]

4.2: 11(c)
This time we can use $z(t) = e^{it}$, $t \in [-\pi/20]$ to parametrize $\Gamma$. Then $\dot{z}(t) = ie^{it}$ and
\[
\int_{\Gamma}(2z - 1)\,dz = \int_{-\pi/2}^{\pi/2} (2e^{it} + 1)ie^{it} dt = 2i = \int_{-\pi/2}^{\pi/2} e^{2it} dt + i \int_{-\pi/2}^{\pi/2} e^{it} dt = e^{2it}\bigg|_{-\pi/2}^{\pi/2} + e^{it}\bigg|_{-\pi/2}^{0} = 1 - (-1) + 1 - 0 = 3 + i
\]
4.2: 14(a)
\[
\left| \int_C \frac{dz}{z^2 - i} \right| \leq \max_{|z|=3} \frac{1}{|z|^2 - 1} \cdot 3 \cdot 2\pi \leq \frac{1}{|z|^2 - 1} \cdot 6\pi = \frac{3\pi}{4}
\]
Here we used \(|z^2 - i| \geq |z^2| - |i| = |z|^2 - 1|.

4.2: 14(b)
If \(\gamma\) is the vertical line segment from \(R\) to \(R + 2\pi i\), then
\[
\left| \int_{\gamma} e^{3z} \frac{dz}{1 + e^z} \right| \leq \max_{z \in \gamma} \frac{|e^{3z}|}{|e^z| - 1} \cdot 2\pi = \max_{y \in [0, 2\pi]} \frac{|e^{3(R+iy)}|}{|e^{(R+iy)}| - 1} \cdot 2\pi = \frac{e^{3R}}{e^R - 1} \cdot 2\pi.
\]

4.3: 1(b)
Since \(e^z\) has antiderivative \(e^z\) valid everywhere, \(\int e_z = e^{-1} - e^1 = 1/e - e\).

4.3: 1(c) We may use the principal branch \(\text{Log}(z)\) as an antiderivative for \(1/z\), since the contour \(\Gamma\) in the question misses the branch cut on the negative real axis. So
\[
\int_{\Gamma} \frac{1}{z} \, dz = \text{Log}(3i) - \text{Log}(-3i) = \ln |3| + \frac{i\pi}{2} - \left( \ln |3| - \frac{i\pi}{2} \right) = i\pi
\]

4.3: 1(g)
The principal branch of \(z^{1/2}\) has its branch cut on the negative real axis. Since the indicated contour \(\Gamma\) does not cross the cut,
\[
\int_{\Gamma} z^{1/2} \, dz = \frac{2z^{3/2}}{3} \bigg| _{-}^{i} = \frac{2}{3}(i^{3/2} - (-i)^{3/2})
\]
Here \(i^{3/2}\) is the principal branch given by \(e^{(3/2)\text{Log}(i)} = e^{(3/2)i(\pi/2)} = e^{3i\pi/4}\)

4.3: 1(h)
We need to find an antiderivative for \(\text{Log}(z)^2\). We can use \(F(z) = z \text{Log}(z)^2 - 2z \text{Log}(z) + 2z\) since
\[
F'(z) = \text{Log}(z)^2 + 2z \text{Log}(z) \frac{1}{z} - 2 \text{Log}(z) - 2z \frac{1}{z} + 2 = \text{Log}(z)^2.
\]
(I found this by trial and error. Maybe there is a better way.) It is important that the contour does not intersect the branch cut of the \( \text{Log} \). Then

\[
\int_{\Gamma} \text{Log}(z)^2 dz = F(i) - F(1)
\]

\[
= (\pi - 2) + (2 - \pi^2/4)i
\]

Here we used \( \text{Log}(i) = i\pi/2 \) and \( \text{Log}(1) = 0 \)

4.3: 5

This example shows that the integral of \( 1/z \) depends on the contour and not just the endpoints of the contour. If \( 1/z \) had an antiderivative the integral would only depend on the endpoints.

**Derive formula (10) on p 135 for and use it to find the derivative.**

To derive a formula for \( \tan^{-1}(z) \) we must solve \( \tan(w) = z \) for \( w \). Starting with

\[
\tan(w) = \frac{\sin(w)}{\cos(w)} = \frac{e^{iw} - e^{-iw}}{2i} \frac{2}{e^{iw} + e^{-iw}} = \frac{e^{iw} - e^{-iw}z}{ie^{iw} + ie^{-iw}} = z
\]

So

\[
e^{iw} - e^{-iw} = ize^{iw} + ize^{-iw}
\]

\[
(1 - iz)e^{iw} = (1 + iz)e^{-iw}
\]

\[
e^{-2iw} = \frac{1 - iz}{1 + iz}
\]

\[
w = \tan^{-1}(z) = i \frac{2}{\log \left( \frac{1 - iz}{1 + iz} \right)}
\]

Differentiating gives

\[
\frac{d}{dz} \tan^{-1}(z) = \frac{i}{2} \left( \frac{1 + iz}{1 - iz} \right) \left( -i(1 + iz) - (i)(1 - iz) \right)
\]

\[
= \frac{i}{2} \left( \frac{-2i}{(1 + iz)(1 - iz)} \right)
\]

\[
= \frac{1}{1 + z^2}
\]