• Indented contours

• Principal value integrals

\textbf{Goal: Compute } \int_0^\infty \frac{\sin(x)}{x} \, dx

\textbf{Step 1} \quad I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(x)}{x} \, dx

= \lim_{R \to \infty} \frac{1}{2} \int_{-R}^{R} \frac{\sin(x)}{x} \, dx

\textbf{Aside: This looks like the Fourier type integrals } \int_{-\infty}^{\infty} e^{ikx} \frac{p(x)}{q(x)} \, dx \text{ we considered last time. New complication is that } \frac{1}{x} \text{ has a singularity on the contour of integration. We would like to write } \sin(x) = \text{Im}(e^{ix}) \quad (\text{or } e^{ix} - e^{-ix})

\text{but } \frac{e^{ix}}{x} \text{ is not integrable at } 0 \]

\[ \int_{-R}^{R} \frac{\sin(x)}{x} \, dx = \]

\[ \text{as } R \to \infty \]
Strategy: Cut away a small interval \((-\epsilon, \epsilon)\) around the singularity and take a limit \(\epsilon \to 0\). This is called the principal value integral.

\[
\int_{[-R, R]} \frac{\sin(x)}{x} \, dx = \lim_{\epsilon \to 0} \int_{[-R, -\epsilon] \cup [\epsilon, R]} \frac{\sin(x)}{x} \, dx
\]

\[
= \lim_{\epsilon \to 0} \int_{[-R, -\epsilon] \cup [\epsilon, R]} \text{Im} \left( \frac{e^{ix}}{x} \right) \, dx
\]

Now we can introduce \(\frac{e^{ix}}{x}\):

\[
= \text{Im} \left[ \lim_{\epsilon \to 0} \int_{[-R, -\epsilon] \cup [\epsilon, R]} \frac{e^{ix}}{x} \, dx \right]
\]

Add and subtract 2 contours \(C_R^+\) and \(C_\epsilon^+\):

\[
\int_{[-R, -\epsilon] \cup [\epsilon, R]} \frac{e^{ix}}{x} \, dx = \int_{[-R, -\epsilon]} + \int_{C_\epsilon^+} + \int_{\epsilon R} + \int_{C_R^+} + \int_{\epsilon} + \int_{C_\epsilon^+}.
\]
Claim: (Semicircular contours)

If \( f(z) \) has a simple pole at \( z_0 \), then

\[
\lim_{\gamma \to 0} \int_{\text{semi-ccl}} f(z) \frac{e^{iz}}{z} \, dz = \pi i \text{ Res}[f; z_0]
\]

For our examples:

\[
\int_{C_6} \frac{e^{iz}}{z} \, dz \to \frac{\pi i}{1} \quad \text{by Cauchy's Residue Theorem}
\]

\[
\int_{C_p} \frac{e^{iz}}{z} \to 0 \quad \text{by Jordan's Lemma}
\]
Residue calculus can be used to compute

\[
\sum_{n=-\infty}^{\infty} f(n)
\]

where

\[ f(z) = \frac{P(z)}{q(z)} \]

\[ \text{deg } q > \text{deg } p + 2 \]

\[ q(n) \neq 0 \text{ for } n \in \mathbb{Z} \]

Let

\[ g(z) = \frac{\pi f(z) \cos(\pi z)}{\sin(\pi z)} = \frac{\pi f(z) \cos(\pi z)}{\pi \cos(\pi z)} \]

Singularity of \( g(z) \) at

- Zeros of \( q; \ z, z_2, z_3, \ldots, z_k \)
- Integers \( n = 0, \pm 1, \pm 2 \)

\[ \text{Res}_z \left[ g; n \right] = \frac{\pi f(n) \cos(\pi n)}{\pi \cos(\pi n)} = f(n) \]
\[ \oint_{\Gamma_N} \pi f(z) \cot(\pi z) \, dz \]

\[ = 2\pi i \left[ \sum_{n = -N}^{N} \text{Res} \left[ g : n \right] + \sum_{z_1, \ldots, z_k} \text{Res} \left[ g : z_i \right] \right] \]

\[ = 2\pi i \left[ \sum_{n = -N}^{N} f(n) + \sum_{z_1, \ldots, z_k} \text{Res} \left[ g : z_i \right] \right] \]

\[ \left| \oint_{\Gamma_N} \pi f(z) \cot(\pi z) \, dz \right| \]

\[ \leq \max_{z \in \Gamma_N} \pi |f(z)| \left| \cot(\pi z) \right| \cdot \text{length} \left( \Gamma_N \right) \]

for \( z \in \Gamma_N \):

\[ |f(z)| \leq \frac{c}{n^2} \]

\[ \left| \cot(\pi z) \right| \leq M \quad \text{(independent of } N) \]

\[ \text{length} \left( \Gamma_N \right) = 4 \cdot (2N + 1) \]

So \( \oint_{\Gamma_N} \pi f(z) \cot(\pi z) \, dz \to 0 \) as \( N \to \infty \)

\[ \sum_{n = -\infty}^{\infty} f(n) = -\sum_{z_1, \ldots, z_k} \text{Res} \left[ g : z_i \right] \]

Example \( \sum_{n = -\infty}^{\infty} \frac{1}{n^2 + 1} = -\left\{ \text{Res} \left[ \frac{\pi \cot(\pi z)}{z^2 + 1} ; i \right] + \text{Res} \left[ \frac{\pi \cot(\pi z)}{z^2 + 1} ; -i \right] \right\} \]

\[ = -\pi \frac{\cos(i\pi)}{\sin(i\pi) \cdot 2i} + \frac{\pi \cos(-i\pi)}{\sin(-i\pi) \cdot (-2i)} = \pi \coth(\pi) \]
Next we need to show that

\[ \sup_{z \in \Gamma_n} |\cot(\pi z)| \leq C_2. \]

Let \( z = x + iy \) with \( y > 0 \). Then

\[
|\cot(\pi z)| = \left| \frac{e^{i\pi x} + e^{-i\pi x}}{e^{i\pi x} + e^{-i\pi x}} \right| = \left| e^{i\pi x} \right| + \left| e^{-i\pi x} \right| - \left| e^{i\pi x} - e^{-i\pi x} \right| = \frac{e^{\pi y} + e^{-\pi y}}{e^{\pi y} - e^{-\pi y}} = 1 + e^{-2\pi y}. 
\]

So if \( y \) is large enough to ensure \( e^{-2y} \leq 1/2 \) then the right side is bounded by \( (1 + 1/2)/(1 - 1/2) = 3 \). Similarly, when \( y < 0 \) we get and

\[ |\cot(\pi z)| \leq \frac{1 + e^{2\pi y}}{1 - e^{2\pi y}}. \]

which is also bounded for \( y \) negative and large. This implies that \( |\cot(\pi z)| \) is bounded by a constant independent of \( n \) on the top and bottom of \( \Gamma_n \). We could also compute a bound for the left and right sides of \( \Gamma_n \) explicitly. Alternatively we can argue that \( |\cot(\pi (n+1/2) + iy)| \) is a continuous function that tends to \( 1 \) for \( y \) tending to \( \pm \infty \). Thus there must be a maximum value. But \( |\cot(\pi z)| \) is a periodic function so the value of \( |\cot(\pi (n+1/2) + iy)| \) is independent of \( n \). This implies that we have a uniform bound for \( |\cot(\pi z)| \) on the left and right sides of \( \Gamma_n \). Altogether, we have \( \sup_{z \in \Gamma_n} |\cot(\pi z)| \leq C_2 \) as required. Finally

\[ \text{length}(\Gamma_n) \leq C_3 n. \]

Thus

\[ \left| \int_{\Gamma_n} f(z)dz \right| = \left| \int_{\Gamma_n} \frac{p(z) \cot(\pi z)}{q(z)} dz \right| \leq \frac{C_1C_2C_3}{n} \rightarrow 0 \]

as \( n \to \infty \). This implies that

\[ \lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{k \in \mathbb{Z} \setminus \{0\},|k| \leq n} \frac{p(k)}{q(k)} + \sum_{w \in Q} \text{Res}[f; w] \right\} = 0. \]

which gives the formula we wish to prove.

**Example**

\[
\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = -\pi \text{Res} \left[ \frac{\cot(\pi z)}{1+z^2}; z = i \right] = -\pi \text{Res} \left[ \frac{\cot(\pi z)}{1+z^2}; z = -i \right] 
\]

\[ = -\pi \frac{\cot(-\pi i)}{1-2i} = \pi \coth(\pi) \approx 3.153348095 \]

**Example**

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^2} = \frac{1}{2\pi} \int_{\mathbb{R}} \text{Res}[\frac{\cot(\pi z)}{z^2}; z = 0] = \frac{1}{6} \frac{\pi^2}{6} 
\]

**Remark** This method doesn’t work for \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) but we can get a formula for \( \sum_{n=1}^{\infty} \frac{(-1)^n p(n)}{q(n)} \) using \( f(z) = \frac{p(z)}{q(z)\sin(\pi z)} \).