Describing sets using complex notation

filename: setdescriptions.tex

January 8, 2016

For each equation determine the set of $z$ values that satisfy the equation. Sketch the set. Show that the equation is equivalent to a “standard” equation for that set.

Example 1

\[ |z - z_0| = r \quad \text{for } z_0 \in \mathbb{C} \text{ and } r > 0 \]

This says that $z$ is a distance $r$ from $z_0$, So the set is a circle centred at $z_0$ with radius $r$. Let $z = x + iy$ and $z_0 = a + ib$. Then

\[ |z - z_0| = r \iff |z - z_0|^2 = r^2 \quad \text{(because both sides are positive)} \]
\[ \iff |(x - a) + i(y - b)|^2 = r^2 \]
\[ \iff (x - a)^2 + (y - b)^2 = r^2 \]

This is the standard equation for a circle.
Example 2

\[|z - z_0| = |z - z_1| \quad \text{for} \quad z_0, z_1 \in \mathbb{C}\]

This says that \(z\) is equidistant from \(z_0\) and \(z_1\). These points lie on the bisecting line perpendicular to the segment joining \(z_0\) and \(z_1\).

The equation can be rewritten

\[
|z - z_0| = |z - z_1| \iff |z - z_0|^2 = |z - z_1|^2 \quad \text{(both sides positive)}
\]

\[
\iff |z|^2 - 2 \text{Re}(z \overline{z}_0) + |z_0|^2 = |z|^2 - 2 \text{Re}(z \overline{z}_1) + |z_1|^2
\]

\[
\iff 2 \text{Re}(z(\overline{z}_1 - \overline{z}_0)) + |z_0|^2 - |z_1|^2 = 0
\]

If \(z = x + iy\) this has the form \(ax + by + c = 0\) so it is the equation of some line. To see that it is the line we want, notice that \(z\) lies on the bisecting line if \(z - (z_0 + z_1)/2 \perp z_1 - z_0\) (as vectors). The dot product of the vectors corresponding to two complex numbers \(w_1\) and
$w_2$ is $\text{Re}(w_1\bar{w}_2)$ (check!). So the equation of the bisecting line is

$$\text{Re}((z - (z_0 + z_1)/2)(\bar{z}_1 - \bar{z}_0)) = 0$$

$$\iff 2\text{Re}(z(\bar{z}_1 - \bar{z}_0)) + \text{Re}(|z_0|^2 + z_0\bar{z}_1 - z_1\bar{z}_0 - |z_1|^2) = 0$$

$$\iff 2\text{Re}(z(\bar{z}_1 - \bar{z}_0)) + |z_0|^2 - |z_1|^2 = 0$$

In the last step we used that $z_0\bar{z}_1$ and $z_1\bar{z}_0$ have the same real part, as they are complex conjugates.
Example 3

\[ |z - 1| = \text{Re}(z) \]

This says that the distance of \( z \) to the \( y \) axis is equal to the distance to 1. The set of such points form a parabola. The equation is equivalent to two conditions: \( \text{Re}(z) \geq 0 \) and \( |z - 1|^2 = (\text{Re}(z))^2 \). If \( z = x + iy \) these conditions are

\[
x \geq 0 \quad \text{and} \quad (x - 1)^2 + y^2 = x^2 \iff x \geq 0 \quad \text{and} \quad x = (1 + y^2)/2.
\]

The first condition is redundant since the second condition implies that \( x \) is positive. So the original equation is equivalent to the equation \( x = (1 + y^2)/2 \) for a parabola.
**Example 4**

\[ |z - 1| = - \text{Re}(z) \]

By the reasoning in the previous example, if \( z = x + iy \) this is equivalent to the two conditions

\[ x \leq 0 \quad \text{and} \quad x = (1 + y^2)/2. \]

These are contradictory, so the original equation has no solutions.

**Example 5**

\[ z^2 + \overline{z}^2 = 2 \]

If \( z = x + iy \) then this equation becomes

\[ x^2 - y^2 + 2ixy + x^2 - y^2 - 2ixy = 2 \iff x^2 - y^2 = 1 \]

This is a hyperbola.
Example 6

\[ |z| = 3|z - 1| \]

This describes the set up points whose distance to 0 is three times the distance to 1. It is not immediately clear what set this describes, but the equation is equivalent to

\[
|z|^2 = 9|z - 1|^2 \iff x^2 + y^2 = 9(x - 1)^2 + 9y^2 \\
\iff (x - 9/8)^2 + y^2 = (3/8)^2.
\]

so the set is a circle centred at 9/8 with radius 3/8.

In fact, the theory of linear fractional transformations says that the equation \(|(az + b)/(cz + d)| = r\) describes either a circle or a line. But this is not obvious.
Example 7

\[ |z + 3| + |z - 3| = 10 \]

This is the set of points where the sum of the distances to 3 and \(-3\) is constant. This describes an ellipse with foci \(\pm 3\) on the real axis. Such an ellipse has equation \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\). Given this, we can find \(a\) and \(b\).

The ellipse intersects the \(x\) axis when \(x = a\) and \(y = 0\). We may assume \(a > 0\) so the equation reads \(a + 3 + |a - 3| = 10\). In the region where \(a \leq 3\) we \(|a - 3| = -a + 3\) so the equation reads \(a + 3 - a + 3 - 10\), which has no solutions. In the region where \(a \geq 3\) we have \(|a - 3| = a - 3\) so the equation reads \(a + 3 + a - 3 - 10\), which has solution \(a = 5\).

The ellipse intersects the \(y\) axis when \(x = 0\) and \(y = b\). In this case the equation reads \(|ib+3| + |ib-3| = 10\). But \(|ib-3| = |−ib+3| = |ib + 3|\). So the equation becomes \(|ib - 3| = 5\) which is equivalent to \(b^2 + 9 = 25\). The positive solution is \(b = 4\).

The calculation showing directly that the equation \(|z+3|+|z-3| = 10\) is equivalent to \(x^2/25 + y^2/16 = 1\) is somewhat brutal. Let us show more generally that for \(a > b > 0\) the equation \(|z - \sqrt{a^2 - b^2}| + |z + \sqrt{a^2 - b^2}| = 2a\) with \(z = x + iy\) is equivalent to \(x^2/a^2 + y^2/b^2 = 1\). We
have
\[ |z - \sqrt{a^2 - b^2}| + |z + \sqrt{a^2 - b^2}| = 2a \]
\[ \iff |z - \sqrt{a^2 - b^2}|^2 + |z + \sqrt{a^2 - b^2}|^2 \]
\[ + 2|z - \sqrt{a^2 - b^2}||z + \sqrt{a^2 - b^2}| = 4a^2 \]
\[ \iff |z|^2 - 2x\sqrt{a^2 - b^2} + a^2 - b^2 + |z|^2 + 2x\sqrt{a^2 - b^2} + a^2 - b^2 \]
\[ + 2(|z - \sqrt{a^2 - b^2})(z + \sqrt{a^2 - b^2})| = 4a^2 \]
\[ \iff |z|^2 + a^2 - b^2 + |z|^2 - a^2 + b^2| = 2a^2 \]
\[ \iff |z|^2 - a^2 + b^2| = a^2 + b^2 - |z|^2 \]

This holds if the right side is positive, i.e., \(|z|^2 \leq a^2 + b^2\) (a condition that will turn out to be redundant) and
\[ |z^2 - a^2 + b^2|^2 = (a^2 + b^2 - |z|^2)^2 \]
\[ \iff |z|^4 + 2\text{Re}(z^2)(-a^2 + b^2) + (-a^2 + b^2)^2 \]
\[ = (a^2 + b^2)^2 - 2(a^2 + b^2)|z|^2 + |z|^4 \]
\[ \iff 2(x^2 - y^2)(-a^2 + b^2) - 2a^2b^2 = 2a^2b^2 - 2(a^2 + b^2)(x^2 + y^2) \]
\[ \iff x^2b^2 + y^2a^2 = a^2b^2 \]
\[ \iff x^2/a^2 + y^2/b^2 = 1, \]

which is the equation for the ellipse we are aiming for.

To see that the first condition is redundant, note that since \(1/b^2 \geq 1/a^2\) the equation for the ellipse implies \((x^2 + y^2)/a^2 \leq 1\) which in turn implies \(|z|^2 \leq |a|^2 \leq a^2 + b^2\). This is the first condition.