Full Name:  
Student No:  

Grade:  

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1. (4 points) Determine whether each of the following statements is TRUE or FALSE. [No explanation is needed.]

(a) $\int_{|z|=1} \overline{z} \, dz = \int_{|z|=1} \frac{1}{z} \, dz$.

**Answer:** TRUE

**Solution:** Note that for $z$ on the unit circle, $\overline{z} = \frac{1}{z}$.

(b) If $f(z)$ is analytic at every point in the closed disk $\overline{D} := \{z : |z| \leq 1\}$ and $|f(z)|$ takes a maximum value somewhere in $\overline{D}$, then $f$ is necessarily constant in $\overline{D}$.

**Answer:** FALSE

**Solution:** For example, the function $f(z) := z$ is analytic at every point in $\overline{D}$ and the maximum of $|f(z)|$ over $\overline{D}$ is achieved at every point on the boundary (the maximum is 1!) but $f(z)$ is not constant.

(c) If $f(z)$ has an anti-derivative in a domain $D$, then $f(z)$ is analytic in $D$.

**Answer:** TRUE

**Solution:** The anti-derivative of $f$ is analytic in $D$, and the derivative of an analytic function is also analytic.

(d) $\int_{\gamma} \frac{z^{16} - 2016}{z^{2017}} \, dz = 0$ for every closed contour $\gamma$ that does not pass through the origin.

**Answer:** TRUE

**Solution:** The function $f(z) := \frac{z^{16} - 2016}{z^{2017}} = z^{-2001} - 2016z^{-2017}$ has an anti-derivative $F(z) := \frac{1}{2000}z^{-2000} + z^{-2016}$ in the domain $\mathbb{C} \setminus \{0\}$. Therefore, its integrals over closed contours within this domain vanish.
2. (4 points) Find the value of the following integrals.

(a) \[ \int_{-3i}^{3i} \frac{\text{Re}(z)}{z} \, dz \] from \(-3i\) to \(3i\) along the circle \(|z| = 3\) traversed counter-clockwise.

**Solution:** We can use the parametrization \(z(t) := 3e^{it}\) where \(t \in [-\pi/2, \pi/2]\). Then, \(dz = 3ie^{it} \, dt\) and \(\frac{\text{Re}(z(t))}{z(t)} = \frac{\cos t}{e^{it}}\). Therefore,

\[
\int \frac{\text{Re}(z)}{z} \, dz = \int_{-\pi/2}^{\pi/2} \frac{\cos t}{e^{it}} \cdot 3i e^{it} \, dt = 3i \int_{-\pi/2}^{\pi/2} \cos t \, dt = 3i \sin t \bigg|_{t=-\pi/2}^{\pi/2} = 6i.
\]

(b) \[ \oint_{|z|=5} \frac{2 \cos z}{z^2(2z - \pi)} \, dz \]

**Solution:** Let \(\gamma\) denote the contour \(|z| = 5\) oriented counter-clockwise. Note that \(\gamma\) has both poles \(z = 0\) and \(z = \pi/2\) in its interior. We can either *split the contour* or *split the function*. Let us do the latter. Using partial fraction decomposition, we have

\[
\frac{1}{z^2(z - \pi/2)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z - \pi/2},
\]

which gives

\[
\int_{\gamma} \frac{2 \cos z}{z^2(2z - \pi)} \, dz = A \int_{\gamma} \frac{\cos z}{z} \, dz + B \int_{\gamma} \frac{\cos z}{z^2} \, dz + C \int_{\gamma} \frac{\cos z}{z - \pi/2} \, dz.
\]

Since \(\cos z\) is analytic everywhere, in particular on and inside \(\gamma\), we can apply Cauchy’s formula to find each integral, namely

\[
\int_{\gamma} \frac{\cos z}{z} \, dz = 2\pi i \cos 0 = 2\pi i,
\]

\[
\int_{\gamma} \frac{\cos z}{z^2} \, dz = \frac{2\pi i}{1!} (-\sin 0) = 0,
\]

\[
\int_{\gamma} \frac{\cos z}{z - \pi/2} \, dz = 2\pi i \cos(\pi/2) = 0.
\]

Put together, we get

\[
\int_{\gamma} \frac{2 \cos z}{z^2(2z - \pi)} \, dz = A \times 2\pi i.
\]

We now calculate \(A\) and find \(A = -4/\pi^2\). Therefore,

\[
\int_{\gamma} \frac{2 \cos z}{z^2(2z - \pi)} \, dz = -\frac{8}{\pi}.
\]
3. (2 points) Let $f(z)$ be an entire function and suppose that $|f(z)| \leq 1000|e^z|$ whenever $|z| \geq 10$. Show that $f(z) = ce^z$ for some constant $c \in \mathbb{C}$. [HINT: consider $f(z)/e^z$.]

**Solution:** Let $g(z) := f(z)/e^z$ and note that $g(z)$ is entire, because both $f(z)$ and $e^z$ are entire and $e^z$ is everywhere non-zero. By the assumption, $|g(z)| \leq 1000$ whenever $|z| \geq 10$. On the other hand, $|g(z)|$ is continuous and hence has a maximum value $M$ on the closed disk $|z| \leq 10$. We do not know the value $M$, but whatever it is, we have $|g(z)| \leq \max\{M, 1000\}$ for every $z \in \mathbb{C}$. That is, $g(z)$ is bounded. According to Liouville’s theorem, since $g$ is a bounded entire function, it has to be constant, that is, $g(z) = c$ for some constant $c \in \mathbb{C}$. Recalling the definition of $g$, we get $f(z)/e^z = c$, hence $f(z) = ce^z$ for every $z$. 