University of British Columbia Math 300 Final Exam
December 15, 2018, 3:30–6:00pm (2.5 hours)

Last Name (print): 

First Name (print): 

Student ID Number: Signature: 

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>19</td>
<td></td>
</tr>
<tr>
<td>Total:</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

Additional Instructions:

- No notes, books or calculators are allowed.
- Read the questions carefully and make sure you provide all the information that is asked for in the question.
- Show all your work. Correct answers without explanation or accompanying work could receive no credit.
- Answer the questions in the space provided. Continue on the back of the page if necessary.
1. (a) (4 points) Find all solutions $z$ to the equation $z^4 = i$.

Taking modulus gives $|z|^4 = 1$ so $|z| = 1$ and $z = e^{i\varphi}$.

Then $z^4 = e^{4i\varphi} = e^{i\frac{4\pi}{2}} = i$. So $4\varphi = \frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$ or
\[
\varphi = \frac{\pi}{8} + \frac{\pi k}{2}, \quad k \in \mathbb{Z}.
\]
This results in 4 distinct values
\[
e^{i\left[\left(\frac{\pi}{8} + \frac{\pi k}{2}\right)\right]}, \quad k \in \{0, 1, 2, 3\}.
\]

(b) (4 points) What are the values of the multivalued function $i^i$?

\[
i^i = e^{i \log(i)} = e^{i \left(0 + i\frac{\pi}{2} + 2\pi k\right)}, \quad k \in \mathbb{Z}
\]
\[
= e^{-\frac{\pi}{2} - 2\pi k}, \quad k \in \mathbb{Z}.
\]

Note: In parts (c) and (d), $(\cdot)^{1/2}$ denotes the principal value of the square root.

(c) (4 points) Write $(-1 + i)^{1/2}$ in the form $x + iy$.

\[
(-1 + i)^{1/2} = e^{\frac{1}{2} \log(-1+i)} = e^{\frac{1}{2} \left(\ln \sqrt{2} + i \frac{3\pi}{4}\right)}
\]
\[
= e^{\frac{1}{2} \left(\frac{1}{2} \ln 2 + i \frac{3\pi}{4}\right)} = 2^{1/4} e^{i \frac{3\pi}{8}}
\]
\[
= 2^{1/4} \cos \left(\frac{3\pi}{8}\right) + i 2^{1/4} \sin \left(\frac{3\pi}{8}\right)
\]

(d) (4 points) Why are there no solutions $z$ to the equation $(z)^{1/2} = -1 + i$?

The principal values of $z^{1/2}$ all have Re part > 0.

To see this write $z^{1/2} = e^{\frac{1}{2} \left(\ln |z| + i \text{Arg}(z)\right)}$.

Since $\text{Arg}(z) \in (-\pi, \pi]$, $\frac{1}{2} \text{Arg}(z) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This
implies $z^{1/2}$ in right half plane.
2. Let \[ f(x + iy) = x^2 + iy^2. \]

(a) (6 points) For which \( z \) is \( f(z) \) complex differentiable?

Let \( u(x,y) = \text{Re}(f(x+iy)) = x^2, \quad v(x,y) = \text{Im}(f(x+iy)) = y^2. \)

Then \( f \) is complex diff. at \( z = x + iy \) if (1) partials of \( u, v \) exist and are cts and (2) the Cauchy Riemann eqns hold. In this case \( u, v \) have cts. partials of all orders. C.R. eqns are

\[
\begin{align*}
ux &= vy & 2x &= 2y \\
vx &= -uy & 0 &= 0
\end{align*}
\]

This \( f \) is complex diff if \( x = y \).

(b) (4 points) For which \( z \) is \( f(z) \) analytic?

\( f \) is analytic at \( z \) if \( f \) is complex diff for every point in some disk centred at \( z \). There are no such pts so \( f \) is not analytic for any \( z \).
3. (a) (5 points) Compute
\[ \oint_{|z|=1} \frac{e^{z^2}}{z^9} \, dz \]
where the contour traversed once counterclockwise.

\[ e^{z^2} = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!} \quad \text{and} \quad z^{-9} e^{z^2} = \sum_{k=0}^{\infty} \frac{z^{2k-9}}{k!} \]

\[ 2k - 9 = -1 \quad \text{when} \quad k = 4 \quad \text{so the residue at} \quad 0 \quad \text{is} \]
\[ \frac{1}{4!} \quad \text{Thus} \quad f = \frac{2 \pi i}{4!} = \frac{\pi i}{12} \]

(b) (5 points) Compute
\[ \oint_{|z|=1} \frac{e^{1/z^2}}{z^9} \, dz \]
where the contour traversed once counterclockwise.

\[ e^{1/z^2} = 1 + \frac{1}{2z^2} + \frac{1}{2!} \frac{1}{z^4} + \ldots \]

\[ z^{-9} e^{1/z^2} = z^{-9} + z^{-11} + \ldots \]

Residue at 0 is 0

\[ f = 0 \]
4. (a) (8 points) Show that if $f(z)$ is entire and $\text{Re}(f(z)) \leq M$ for all $z$, then $f(z)$ is constant.

If $f(z)$ is entire then so is $e^{f(z)}$.
We can estimate $|e^{f(z)}| = e^{\text{Re}(f(z))} \leq e^M$.
Thus $|e^{f(z)}|$ is bounded. By Liouville's theorem $e^{f(z)} = c$ for some constant.

Differentiating yields $e^{f(z)} f'(z) = 0$, and since $e^{f(z)} \neq 0$ for all $z$ this gives $f'(z) = 0$.

This implies $f(z)$ is constant.

[Note: $e^{f(z)} = c$ is not enough to show $f(z)$ is constant; $f(z)$ could be $0$ for some $z$'s and $\pi i$ for the rest.]
5. Note: in this question $D(r)$ denotes the open disk $\{ z \in \mathbb{C} : |z| < r \}$.

Consider the sequence of functions $f_n(z) = z^n$ for $n = 1, 2, 3, \ldots$.

(a) (4 points) Does $f_n(z)$ converge pointwise in $D(1)$? Does $f_n(z)$ converge uniformly in $D(1)$? Justify your answers.

**Pointwise:** yes: If $z \in D(1)$ then $|z| < 1$ so

$$\lim |z|^n = 0$$

which implies $\lim f_n(z) = \lim z^n = 0$.

**Uniformly:** no: $\sup_{z \in D(1)} |z^n - 0| = \sup_{z \in D(1)} |z|^n = 1$

This does not $\to 0$ as $n \to \infty$.

(b) (4 points) Does $f_n(z)$ converge pointwise in $D(1/2)$? Does $f_n(z)$ converge uniformly in $D(1/2)$? Justify your answers.

**Pointwise:** yes: same as (a)

**Uniformly:** yes: $\sup_{z \in D(1/2)} |z^n - 0| = \sup_{z \in D(1/2)} |z|^n = (\frac{1}{2})^n$

This $\to 0$ as $n \to \infty$.

(c) (4 points) Is there a set $G$ of complex numbers such that $f_n(z)$ converges uniformly for $z \in G$ but not pointwise? If yes, give an example. If no, say why.

No, since uniform convergence $\Rightarrow$ pointwise convergence.
6. Let

\[ f(z) = e^z + e^{1/z} \]

(a) (5 points) Write down the Laurent series of the form \( \sum_{n=-\infty}^{\infty} a_n z^n \) for \( f \). For which \( z \) does this series converge?

\[
e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \text{and} \quad e^{1/z} = \sum_{k=0}^{\infty} \frac{z^{-k}}{k!} = \sum_{-\infty}^{0} \frac{z^k}{|k|!}
\]

\[ f(z) = \sum_{k=\infty}^{\infty} \frac{z^k}{k!} + 2 + \sum_{k=1}^{\infty} \frac{z^{-k}}{k!} \]

Converges for \( z \neq 0 \).

(b) (5 points) Write down the first three terms in the Taylor series for \( f \) centered at \( z_0 = 2 \). Where does this series converge?

\[
\text{Convergence for } |z - 2| < 2.
\]

\[
f(2) = e^2 + e^{1/2}
\]

\[
f'(2) = e^2 - \frac{1}{2}e^{1/2}
\]

\[
f''(2) = e^2 + \frac{1}{2}e^{1/2} + \frac{1}{4}e^{1/2} = e^2 + \frac{2e^{1/2}}{2} + \frac{1}{4}e^{1/2}
\]

\[
f'' = (e^2 + e^{1/2}) + (e^2 - \frac{1}{4}e^{1/2})(z-2) + \left( e^2 + \frac{5}{16}e^{1/2} \right) \left( \frac{1}{2} \right)(z-2)^2
\]

(c) (5 points) Write down the first two terms in the Taylor series for \( e^z - \frac{e^{1/z}}{z^2} \) centered at \( z_0 = 2 \). Where does this series converge?

This is the series for \( f'(2) \). Convergence is same as for \( f \). Series can be obtained by differentiating term by term:

\[
f'(2) = (e^2 - \frac{1}{4}e^{1/2}) + (e^2 + \frac{5}{16}e^{1/2})(z-2) + \ldots
\]
7. (10 points) Compute the integral
\[ \int_0^{2\pi} \frac{1}{1 + \sin(\theta)^2} \, d\theta \]
using the residue calculus.

\[ I = \int_0^{2\pi} \frac{1}{1 + \sin^2 \theta} \, d\theta = \oint_{|z|=1} \frac{1}{1 + (\frac{z - \frac{1}{2}}{z i})^2} \frac{dz}{i z} \]

\[ = \frac{1}{i} \oint \frac{4 z}{(4 - (z - \frac{1}{2})^2) z^2} \, dz = 4 i \oint \frac{z}{4 z^2 - (z^2 - 1)^2} \, dz \]

\[ = 4 i \oint \frac{z}{(z^2 - r_1^2) (z^2 - r_2^2)} \, dz \quad \text{where} \quad r_1 = \frac{6 + \sqrt{36 - 4}}{2} = 3 + 2\sqrt{2} \]

\[ r_2 = 3 - 2\sqrt{2} \]

Singularity inside unit circle are \( z = \sqrt{r_2} \) and \( z = -\sqrt{r_2} \) (note \( r_2 > 0 \)).

\[ \text{Res} \left[ \frac{z}{(z^2 - r_1) (z^2 - r_2)} \right] = \frac{\sqrt{r_2}}{2 \sqrt{r_1} (r_2^2 - r_1)} \neq 0 \]

\[ \text{Res} \left[ \frac{z}{(z^2 - r_1^2)} \right] = -\frac{\sqrt{r_1}}{2 \sqrt{r_1} (r_2^2 - r_1^2)} \neq 0 \]

\[ I = 2\pi i \cdot 4 i \cdot \left( -\frac{\sqrt{r_1}}{16} - \frac{\sqrt{r_2}}{16} \right) = -8\pi \cdot \frac{\sqrt{2}}{8} = -\sqrt{2} \pi. \]
8. Let \[ g(z) = \frac{\pi}{z^2 \sin(\pi z)}. \]

(a) (8 points) Find all isolated singularities of \( g(z) \) and determine whether they are removable, essential or poles. Determine the order of each pole. Find the residue of \( g(z) \) at each singularity.

**Singularities at** \( z \in \mathbb{Z} \)

For \( z \in \mathbb{Z} \setminus \{0\} \) singularity is a simple pole with residue \[ \frac{\pi}{k^2 \cos(\pi k)}. \]

\[ \frac{1}{k^2 \cos(\pi k)} = \frac{(-1)^k}{k^2} \]

For \( z = 0 \)

\[ g(z) = \frac{\frac{\pi}{z^2 (\pi z) \left( 1 - \frac{(\pi z)^2}{3!} + O(z^4) \right)}}{z^2 \left( \frac{(\pi z)^3}{3!} + O(z^5) \right)} \]

\[ = \frac{1}{2^3} \left( 1 + \left[ \frac{\pi^2 z^2}{6} + O(z^4) \right] + O(z^4) \right) \]

Poles has order 3 and residue is \( \frac{\pi^2}{6} \)
(b) (5 points) Let $\Gamma_N$ be the square contour, traversed once counterclockwise, that is centred at 0, whose vertical sides cut the real axis at $\pm (N + 1/2)$ and whose horizontal sides cut the imaginary axis at $\pm (N + 1/2)i$. Use the residue calculus to evaluate $\oint_{\Gamma_N} g(z)dz$.

Singularities inside are $k \in \{-N, -N+1, \ldots, N\}$.

$$\oint_{\Gamma_N} g(z)dz = 2\pi i \left\{ \sum_{k=-N}^{-1} \frac{(-1)^k}{k^2} + \frac{\pi^2}{6} + \sum_{k=1}^{N} \frac{(-1)^k}{k^2} \right\}$$

(c) (3 points) Given that $|\sin(\pi z)| \geq 1$ for $z \in \Gamma_N$, show that

$$\lim_{N \to \infty} \oint_{\Gamma_N} g(z)dz = 0$$

$$\left| \oint_{\Gamma_N} g(z)dz \right| \leq \max_{z \in \Gamma_N} \frac{\pi}{1z^2/|\sin(\pi z)|} \leq \max_{z \in \Gamma_N} \frac{\pi}{1z^2}$$

$$\leq \frac{C}{N^2} \cdot N = \frac{C}{N} \to 0 \quad \text{as} \quad N \to \infty$$

(d) (3 points) Use the calculations of the previous parts to evaluate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

$$\lim_{N \to \infty} \frac{\pi^2}{6} + 2 \sum_{k=1}^{N} \frac{(-1)^k}{k^2} = 0 \implies$$

$$\sum_{h=1}^{\infty} \frac{(-1)^h}{h^2} = -\frac{\pi^2}{12}$$