Evaluation of some infinite sums

**Proposition 0.1** Let $p$ and $q$ be polynomials with $\deg(q) \geq \deg(p) + 2$ and let $Q$ denote the (finite) set of roots of $q$. Define

$$f(z) = \frac{p(z) \cot(\pi z)}{q(z)} = \frac{p(z) \cos(\pi z)}{q(z) \sin(\pi z)}$$

Then

$$\sum_{n \in \mathbb{Z} \setminus Q} \frac{p(n)}{q(n)} = -\pi \sum_{w \in Q} \operatorname{Res}[f; w]$$

**Proof** The function $f(z)$ has poles when $z \in \mathbb{Z} \cup Q$. The poles at $n \in \mathbb{Z} \setminus Q$ are simple and for these values of $n$

$$\operatorname{Res}[f; n] = \frac{p(n)}{\pi q(n)}.$$

Let $\Gamma_n$ be a square with corners $(n + 1/2)(\pm 1 \pm i)$. For $n$ large enough, $\Gamma_n$ will enclose all the zeros of $q$.

The residue formula gives

$$\int_{\Gamma_n} f(z) \, dz = 2\pi i \left\{ \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus Q, |k| \leq n} \frac{p(k)}{q(k)} + \sum_{w \in Q} \operatorname{Res}[f; w] \right\}$$

The idea is to show that the integral on the left tends to zero as $n \to \infty$. Standard bounds on polynomials give $|p(z)/q(z)| \leq C|z|^{-2}$ for $|z|$ large. This implies that for large $n$

$$\sup_{z \in \Gamma_n} \left| \frac{p(z)}{q(z)} \right| \leq \frac{C_1}{n^2}.$$
Next we need to show that 

$$\sup_{z \in \Gamma_n} |\cot(\pi z)| \leq C_2.$$ 

Let $z = x + iy$ with $y > 0$. Then

$$|\cot(\pi z)| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| \leq \frac{|e^{i\pi z}| + |e^{-i\pi z}|}{|e^{-i\pi z} - e^{i\pi z}|} = \frac{e^{\pi y} + e^{-\pi y}}{e^{\pi y} - e^{-\pi y}} = \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}}.$$ 

So if $y$ is large enough to ensure $e^{-2y} \leq 1/2$ then the right side is bounded by $(1 + 1/2)/(1 - 1/2) = 3$. Similarly, when $y < 0$ we get and

$$|\cot(\pi z)| \leq \frac{1 + e^{2\pi y}}{1 - e^{2\pi y}},$$

which is also bounded for $y$ negative and large. This implies that $|\cot(\pi z)|$ is bounded by a constant independent of $n$ on the top and bottom of $\Gamma_n$. We could also compute a bound for the left and right sides of $\Gamma_n$ explicitly. Alternatively we can argue that $|\cot(\pi z)|$ is a periodic function so the value of $|\cot(\pi(n+1/2)+iy)|$ is independent of $n$. This implies that we have a uniform bound for $|\cot(\pi z)|$ on the left and right sides of $\Gamma_n$. Altogether, we have $\sup_{z \in \Gamma_n} |\cot(\pi z)| \leq C_2$ as required. Finally

$$\text{length}(\Gamma_n) \leq C_3 n.$$ 

Thus

$$\left| \int_{\Gamma_n} f(z) \, dz \right| = \left| \int_{\Gamma_n} \frac{p(z) \cot(\pi z)}{q(z)} \, dz \right| \leq \frac{C_1 C_2 C_3}{n} \to 0$$

as $n \to \infty$. This implies that

$$\lim_{n \to \infty} \left\{ \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus \{Q \}} \frac{p(k)}{q(k)} + \sum_{w \in Q} \text{Res}[f; w] = 0 \right\}$$

which gives the formula we wish to prove.

**Example**

$$\sum_{n=-\infty}^{\infty} \frac{1}{1 + n^2} = -\pi \text{Res} \left[ \frac{\cot(\pi z)}{1 + z^2}; z = i \right] - \pi \text{Res} \left[ \frac{\cot(\pi z)}{1 + z^2}; z = -i \right]$$

$$= -2\pi \cot(-\pi i) = \pi \coth(\pi) \sim 3.153348095$$

**Example**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^2} = -\frac{\pi}{2} \text{Res} \left[ \frac{\cot(\pi z)}{z^2}; z = 0 \right] = \frac{\pi^2}{6}$$

**Remark** This method doesn't work for $\sum_{n=1}^{\infty} n^{-3}$ because when we add the negative $n$ terms to make the sum go over $\mathbb{Z}$ everything cancels. We can get a formula for $\sum_{n=1}^{\infty} \frac{(-1)^n p(n)}{q(n)}$ using $f(z) = \frac{p(z)}{q(z) \sin(\pi z)}$. 

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