The operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$

filename: dbar.tex

September 24, 2014

Define $\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$. These operators provide a compact way of writing the Cauchy Riemann equations and further illuminate the meaning of analyticity.

They act on complex functions in the obvious way: if $f(x + iy) = u(x, y) + iv(x, y)$ then

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left( u(x, y) + iv(x, y) \right) = \frac{1}{2} \left( u_x(x, y) + v_y(x, y) \right) + \frac{i}{2} \left( -u_y(x, y) + v_x(x, y) \right)$$

and

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \left( u(x, y) + iv(x, y) \right) = \frac{1}{2} \left( u_x(x, y) - v_y(x, y) \right) + \frac{i}{2} \left( u_y(x, y) + v_x(x, y) \right)$$

(1) Notice that these expressions make sense for any complex function $f = u + iv$ provided the partial derivatives of $u$ and $v$ exist.

(2) $\frac{\partial f}{\partial \bar{z}} = 0$ is equivalent to the Cauchy Riemann equations. So if this equation is satisfied in some domain, then $f$ is analytic there.

(3) If (2) holds, so that the complex derivative $f'(z)$ exists, then $f'(z) = \frac{\partial f}{\partial z}(z)$.

In some sense (2) is saying that analytic functions are functions that don’t depend on $\bar{z}$. One way of making this more precise is to notice that $\frac{\partial}{\partial z} = 1, \frac{\partial}{\partial \bar{z}} = 0, \frac{\partial}{\partial z} = 0, \frac{\partial}{\partial \bar{z}} = 1$.

Furthermore $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ obey the usual sum and product rules of differentiation. These rules can be used to show that

$$\frac{\partial}{\partial z} (z^n) = nz^{n-1}, \frac{\partial}{\partial \bar{z}} (z^n) = n\bar{z}^{n-1}.$$ 

This means that if $p(z, \bar{z})$ is a polynomial in $z$ and $\bar{z}$ we may compute $\frac{\partial p(z, \bar{z})}{\partial z}$ and $\frac{\partial p(z, \bar{z})}{\partial \bar{z}}$ by pretending that $z$ and $\bar{z}$ are independent variables and using the usual rules of calculus.

To illustrate this last point, consider $p(z, \bar{z}) = z^2\bar{z}$. To compute $\frac{\partial p(z, \bar{z})}{\partial z}$ according to our original definition we first write

$$p(z, \bar{z}) = p(x + iy, x - iy) = (x + iy)^2(x - iy) = (x^3 + xy^2) + i(x^2y + y^3)$$

to determine that $u(x, y) = x^3 + xy^2$ and $v(x, y) = x^2y + y^3$. Then we find that

$$\frac{\partial p(z, \bar{z})}{\partial z} = \frac{1}{2} (u_x + v_y) + \frac{i}{2} (-u_y + v_x)$$

$$= \frac{1}{2} (3x^2 + y^2 + x^2 + 3y^2) + \frac{i}{2} (-2xy + 2xy)$$

$$= 2(x^2 + y^2).$$

On the other hand if we pretend that $z$ and $\bar{z}$ are independent variables and use the usual rules of calculus then we can compute the same quantity.

$$\frac{\partial p(z, \bar{z})}{\partial z} = \frac{\partial z^2 \bar{z}}{\partial z} = 2z\bar{z} = 2(x^2 + y^2).$$

Now we can state a result linking analyticity and independence of $\bar{z}$.

**Proposition 1.1** Let $p(z, w)$ be a polynomial in two complex variables $z$ and $w$. Define a complex function $f(z) = p(z, \bar{z})$. Then $f(z)$ is analytic if and only if $p(z, w)$ does not depend on $w$, that is, $\frac{\partial p(z, w)}{\partial w} = 0$.

**Proof:** We know that $f(z)$ is analytic if and only if $\frac{\partial f}{\partial \bar{z}} = 0$. But we have seen that $\frac{\partial f}{\partial \bar{z}}$ is equal to $\frac{\partial p(z, w)}{\partial w}$ evaluated at $w = \bar{z}$. \qed