Linear Approximations

- A real valued function $f(x)$ of a real variable $x$ is differentiable at $x_0$ if there is a real number $a$ such that
  \[ f(x_0 + h) = f(x_0) + ah + E(x_0, h) \]  
  (1.1)

  where
  \[ \lim_{h \to 0} \frac{|E(x_0, h)|}{|h|} = 0. \]  
  (1.2)

- A vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at $z_0$ if there is a $2 \times 2$ matrix $A$ such that
  \[ F(z_0 + h) = F(z_0) + Ah + E(z_0, h), \]  
  (1.3)

  where
  \[ \lim_{h \to 0} \frac{||E(z_0, h)||}{||h||} = 0. \]  
  (1.4)

- A complex valued function $f(z)$ of a complex variable $z$ is complex differentiable at $z_0$ if there is a complex number $a$ such that
  \[ f(z_0 + h) = f(z_0) + ah + E(z_0, h) \]  
  (1.5)

  where
  \[ \lim_{h \to 0} \frac{|E(z_0, h)|}{|h|} = 0. \]  
  (1.6)

Remark 1. Equations (1.1) and (1.2) say that $f(x_0 + h)$ is approximated by the linear function $f(x_0) + ah$ up to an error term that goes to zero faster than $h$ as $h \to 0$. To see that this is equivalent to the standard definition we can rearrange (1.1) to read
\[ \frac{f(x_0 + h) - f(x_0)}{h} = a + \frac{E(x_0, h)}{h}. \]

Equation (1.2) says that the right side has a limit equal to $a$. So the limit of the left side must also exist and equal $a$. In other words, $f'(x_0)$ exists and equals $a$. The same argument using complex numbers applied (1.5) and (1.6) show that the $a$ in (1.5) is the complex derivative $f'(z_0)$. 


Remark 2. If \( F(z) = F \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} \) is differentiable at \( z_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \) then the partial derivatives of \( u \) and \( v \) exist at \( x_0, y_0 \) and \( A \) is given by the Jacobian matrix

\[
A = \begin{bmatrix}
u_x(x_0, y_0) & u_y(x_0, y_0) \\
v_x(x_0, y_0) & v_y(x_0, y_0)
\end{bmatrix}.
\]

The existence of the partial derivatives of \( u \) and \( v \) at \( (x_0, y_0) \) is not sufficient to guarantee that \( F \) is differentiable. But if the partial derivatives of \( u \) and \( v \) exist for all \( (x, y) \) in a small disk around \( (x_0, y_0) \) and are continuous at \( (x_0, y_0) \) then \( F \) is differentiable at \( z_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \) with \( A \) given by (1.7).

Remark 3. For (1.2) and (1.6) we could equally well write \( \frac{E(x_0, h)}{h} \to 0 \) and \( \frac{E(z_0, h)}{h} \to 0 \).

**Complex functions as vector fields**

We can identify a complex number \( z = i + iy \) with the vector \( z = \begin{bmatrix} x \\ y \end{bmatrix} \). Then \( |z| = ||z|| \).

Also, if \( a = s + it \) is another complex number then the complex product \( az = (s + it)(x + iy) = sx - ty + i(tx + sy) \) is identified with \( \begin{bmatrix} sx - ty \\ tx + sy \end{bmatrix} = \begin{bmatrix} s & -t \\ t & s \end{bmatrix} z \).

Similarly a complex function \( f(z) \) with \( f(x + iy) = u(x, y) + iv(x, y) \) can be regarded as a vector field \( F \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} \).

With these identifications we can rewrite the conditions for complex differentiability as follows.

- A complex function \( f(z) \) represented by \( F \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} \) is complex differentiable at \( z_0 = x_0 + iy_0 \) if there exists a matrix \( A \) representing complex multiplication by \( a \) such that (1.3) and (1.4) hold.

**Conclusion**

Complex differentiability for \( f \) is equivalent to differentiability of the corresponding \( F \) with the additional condition that the Jacobian matrix \( A \) has the form \( \begin{bmatrix} s & -t \\ t & s \end{bmatrix} \) for some \( s \) and \( t \). Since

\[
A = \begin{bmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{bmatrix}
\]

this is the same as saying

\[
u_x(x_0, y_0) = v_y(x_0, y_0)
\]

\[
u_y(x_0, y_0) = -v_x(x_0, y_0)
\]

These are the Cauchy Riemann equations.
Theorem 1.1 (i) If \( f = u + iv \) is complex differentiable at \( x_0 + iy_0 \) then the partial derivatives of \( u \) and \( v \) exist at \( (x_0, y_0) \) and the Cauchy Riemann equations hold.

(ii) If the partial derivatives of \( u \) and \( v \) exist for all \( (x, y) \) in a small disk around \( (x_0, y_0) \) and are continuous at \( (x_0, y_0) \) and, in addition, the Cauchy-Riemann equations hold, then \( f = u + iv \) is complex differentiable at \( x_0 + iy_0 \).

Example: \( f(z) = |z|^2 \). We have \( f(x + iy) = x^2 + y^2 \) so \( u(x, y) = x^2 + y^2 \) and \( v(x, y) = 0 \). The partial derivatives of \( u \) and \( v \) exist and are continuous everywhere. We compute \( u_x = 2x, u_y = 2y, v_x = 0 \) and \( v_y = 0 \). Thus the Cauchy-Riemann equations hold if

\[
2x = 0 \\
2y = 0
\]

i.e., if \( x = y = 0 \). We conclude that \( f(z) \) is complex differentiable at \( z = 0 \) only. Since every disk about 0 contains non-zero points, \( f \) is nowhere analytic.

Example: \( f(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y) \). Here \( u(x, y) = x^3 + 3xy^2 - 3x \) and \( v(x, y) = y^3 + 3x^2y - 3y \). These are polynomials so the partial derivatives of \( u \) and \( v \) exist and are continuous everywhere. We have \( u_x = 3x^2 + 3y^2 - 3, u_y = 6xy, v_x = 6xy \) and \( v_y = 3y^2 + 3x^2 - 3 \). Thus the Cauchy-Riemann equations hold if

\[
3x^2 + 3y^2 - 3 = 3y^2 + 3x^2 - 3 \\
6xy = -6xy
\]

The first of these is always true, but the second only if \( xy = 0 \). This happens if either \( x = 0 \) or \( y = 0 \). So \( f \) is complex differentiable exactly on the real axis and the imaginary axis. Again, \( f \) is nowhere analytic.

Example: \( f(z) = e^z \). Here \( f(x + iy) = e^{x+iy} = e^x e^{iy} = e^x \cos(y) + ie^x \sin(y) \). So \( u(x, y) = e^x \cos(y) \) and \( v(x, y) = e^x \sin(y) \). Again the partial derivatives of \( u \) and \( v \) exist and are continuous everywhere. We have \( u_x = e^x \cos(y), u_y = -e^x \sin(y), v_x = e^x \sin(y) \) and \( v_y = e^x \cos(y) \). The Cauchy-Riemann equations hold if

\[
e^x \cos(y) = e^x \cos(y) \\
e^x \sin(y) = -e^x \sin(y)
\]

These hold for all \( x \) and \( y \) so \( e^z \) is complex differentiable everywhere and therefore analytic everywhere (i.e., entire).