University of British Columbia  
Math 300 Midterm 1  
October 5, 2018 8:00 - 8:50am  

Last Name (print):  
First Name (print):  
Student ID Number:  
Signature:  

Instructor: Richard Froese  

Instructions:  
1. No notes, books or calculators are allowed.  
2. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.  
3. Answer the questions in the space provided. Continue on the back of the page if necessary.  

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1. (8 points) Let \( z = -1 + i \). Find \( \overline{z} \), \( |z| \), \( 1/z \), \( z^{10} \). Find the set \( \log(z) \) and the principal value \( \text{Log}(z) \).

Solution:

\[
\overline{z} = -1 - i \quad [1]
\]

\[
|z| = \sqrt{2} \quad [1]
\]

\[
1/z = (-1 - i)/2 \quad [1]
\]

\[
z^{10} = (\sqrt{2}e^{i3\pi/4})^{10} = 2^5 e^{i30\pi/4} = 32e^{i(7+1/2)\pi} = 32(-i) \quad [2]
\]

\[
\log(-1 + i) = \{\ln(\sqrt{2}) + i(3\pi/4 + 2\pi k) \mid k \in \mathbb{Z}\} \quad [2]
\]

\[
\text{Log}(-1 - i) = \ln(\sqrt{2}) + i3\pi/4 \quad [1]
\]

2. (6 points) Compute \( \left| \frac{\cos^3(1 + 2i)}{\cos^3(1 - 2i)} \right| \). For each step in your computation indicate a property that you are using (for example, something like \( z + w = \overline{z} + \overline{w} \)).

Solution:

\[
\left| \frac{\cos^3(1 + 2i)}{\cos^3(1 - 2i)} \right| = \left| \frac{\cos(1 + 2i)}{\cos(1 - 2i)} \right|^3 \quad \text{since} \quad \frac{|z|}{|w|} = \frac{|z|}{|w|}
\]

\[
= \left| \frac{\cos(1 + 2i)}{\cos(1 - 2i)} \right|^3 \quad \text{since} \quad |z^3| = |z|^3
\]

\[
= \left| \frac{\cos(1 + 2i)}{\cos(1 + 2i)} \right|^3 \quad \text{since} \quad \cos(z) = \cos(\overline{z})
\]

\[
= \left| \frac{\cos(1 + 2i)}{\cos(1 + 2i)} \right|^3 \quad \text{since} \quad |\overline{z}| = |z|
\]

\[
= 1 \quad \text{since} \quad \frac{z}{\overline{z}} = 1.
\]
3. (6 points) Sketch the set \( U = \{ z : 1 < |z - 2i| \leq 2 \} \). Is \( U \) open, closed, connected, bounded? What is the boundary of \( U \)?

Solution: 6

\( U \) is neither open nor closed [2]. \( U \) is connected and bounded [2]. The boundary of \( U \) is the union the circles of radius 1 and 2 about \( z = 2i \), i.e.,

\[
\partial U = \{ z : |z - 2i| = 1 \} \cup \{ z : |z - 2i| = 2 \} [2].
\]
4. (9 points) Find all solutions $z$ to $e^{4z} + 2e^{2z} + 2 = 0$.

Solution: Let $w = e^{2z}$. Then the equation says that $w^2 + 2w + 2 = 0$ so by the quadratic formula we have $w = (-2 \pm \sqrt{4 - 8})/2 = -1 \pm i$. Thus either $e^{2z} = -1 + i$ or $e^{2z} = -1 - i$. So either

$$2z \in \log(-1 + i) = \{ \ln(\sqrt{2}) + i(3\pi/4 + 2\pi k) : k \in \mathbb{Z} \}$$

or

$$2z \in \log(-1 - i) = \{ \ln(\sqrt{2}) - i(3\pi/4 + 2\pi k) : k \in \mathbb{Z} \}$$

Dividing by 2 yields

$$z \in \left\{ \frac{\ln(2)}{4} + i(3\pi/8 + \pi k) : k \in \mathbb{Z} \right\}$$

or

$$z \in \left\{ \frac{\ln(2)}{4} - i(3\pi/8 + \pi k) : k \in \mathbb{Z} \right\}$$
5. (6 points) Where is the function \( f(z) \) defined by \( f(x + iy) = 2xy + i(x^2 + y^2) \) complex differentiable? Where is it analytic? Give a reason.

Solution: We have \( u(x, y) = 2xy \) and \( v(x, y) = x^2 + y^2 \) [1]. These functions have continuous partial derivatives of all orders [1]. The Cauchy Riemann equations read \( 2y = 2y \ (u_x = v_y) \) and \( 2x = -2x \ (u_y = -v_x) \) [1]. They hold if \( x = 0 \) [1]. Thus \( f \) is differentiable on the imaginary axis [1]. This implies it is nowhere analytic, since there are no interior points on the imaginary axis [1].
6. (5 points) Given that \( \lim_{z \to 2i} z^2 + 4 = 0 \) there must be a \( \delta > 0 \) such that \( |z^2 + 4| < \frac{1}{2} \) whenever \( 0 < |z - 2i| < \delta \). Find such a \( \delta \) and show that it does the job.

**Solution:** We can write

\[
|z^2 + 4| = |(z + 2i)(z - 2i)| \\
= |z + 2i| |z - 2i| \\
= |z - 2i + 4i| |z - 2i| \\
\leq (|z - 2i| + |4i|) |z - 2i| \\
= (|z - 2i| + 4) |z - 2i|.
\]

This shows that if \( |z - 2i| < \delta \), then

\[
|z^2 + 4| < (\delta + 4)\delta.
\]

So any \( \delta \) with \( (\delta + 4)\delta < 1/2 \) will work. Try \( \delta = \frac{1}{10} \). Then \( (\delta + 4)\delta = \left( \frac{1}{10} + \frac{40}{10} \right) \frac{1}{10} = \frac{41}{100} < \frac{1}{2} \) so \( \delta = \frac{1}{10} \) works.

There are other ways of estimating this.