University of British Columbia  
Math 300

Midterm 1  

Name (print):  
Student ID Number:  
Signature:

Instructor: Richard Froese

Instructions:

1. No notes, books or calculators are allowed.

2. Read the questions carefully and make sure you provide all the information that is asked for in the question.

3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.

4. Answer the questions in the space provided. Continue on the back of the page if necessary.

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1. (a) (5 points) Write $2 + 2i$ in the form $|z|e^{i\theta}$. What are the all the possible values of $\theta$?

**Solution:** $2 + 2i = \sqrt{2^2 + 2^2}e^{i\pi/4} = 2\sqrt{2}e^{i\pi/4}$. The possible values for $\theta$ are $\pi/4 + 2\pi k$, $k \in \mathbb{Z}$.

(b) (2 points) Write $\frac{1}{2 + 2i}$ in the form $x + iy$ for $x, y$ real.

**Solution:**

$$\frac{1}{2 + 2i} = \frac{2 - 2i}{8} = \frac{1}{4} - \frac{i}{4}$$

(c) (3 points) Find all solutions $z$ to $2z^2 + i = z$ and write them in the form $x + iy$.

**Solution:** The equation can be written $2z^2 + i = z^2 - iz$ or $z^2 + iz + i = 0i$. We can complete the square to obtain $z^2 + 2iz/2 + i = 0i$, $(z - i/2)^2 + 1/4 + i = 0$, $(z - i/2)^2 = -1/4 - i = 0$. Now we write $-1/4 - i = 0$ in polar form. We have $| -1/4 - i | = \sqrt{1/16 + 1} = (17)^{1/2}/4$. Since $-1/4 - i = 0$ is in the left half plane, the argument is $\arctan(4) + \pi$ ($\arctan(4) - \pi$ also works). This results in

$$(z - i/2) = \frac{(17)^{1/4}}{2} e^{i \arctan(4) + \pi/2 + i\pi k}, \quad k \in \mathbb{Z},$$

or

$$z = \frac{i}{2} \pm \frac{(17)^{1/4}}{2} e^{i \arctan(4) + \pi/2}.$$ 

If we write $\alpha = \frac{\arctan(4) + \pi}{2}$ this is

$$z = \pm \frac{(17)^{1/4}}{2} \cos(\alpha) + i \left( \pm \frac{(17)^{1/4}}{2} \sin(\alpha) + 1/2 \right).$$

You could also use the quadratic formula instead of completing the square. (This question actually had a typo in it ... it wasn’t supposed to be this hard)
2. Define the set

\[ S = \{ z : |z| < 1 \text{ and } 0 < \text{Arg}(z) < \pi/4 \} \]

and the complex function

\[ f(z) = e^{i\pi/4}z^3 + 1. \]

(a) (5 points) Is \( S \) open? closed? connected? simply connected? bounded?

**Solution:** \( S \) is open, not closed, connected, simply connected and bounded.

(b) (5 points) Make a sketch of \( S \) and its image \( f(S) \) under \( f \).

**Solution:** Decomposing \( f \) as \( z \mapsto z^3 = z_1, \ z_1 \mapsto e^{i\pi/4}z_1 = z_2, \ z_2 \mapsto z_2 + 1 = f(z) \) results in...
3. (a) (5 points) Is the complex function \( f(z) = \begin{cases} \frac{|z|^2 - z}{z-1} & z \neq 0 \\ 0 & z = 0 \end{cases} \) continuous at 0? Is it complex differentiable at 0?

**Solution:** Since \( |z|^2 - z = zz - z = (\bar{z} - 1)z \) we find that \( f(z) = z \) for all \( z \). Thus \( f \) is both continuous and complex differentiable for any \( z \).

(b) (5 points) Let \( f \) be the complex function given by

\[
f(x + iy) = -2y^2 + 2x + 1 + i(2xy + 2y).
\]

For which \( z \in \mathbb{C} \) is \( f \) complex differentiable? For which \( z \) is \( f \) analytic?

**Solution:** We have \( u = -2y^2 + 2x + 1, \ u_x = 2, \ u_y = -4y, \ v = 2xy + 2y, \ v_x = 2y \) and \( v_y = 2x + 2 \). The partials exist and are continuous everywhere and the Cauchy-Riemann equations hold whenever \( 2 = 2x + 2 \) and \( -4y = -2y \). The only point where this is true is \( x = y = 0 \), i.e., \( z = 0 \). So \( f \) is complex differentiable at \( z = 0 \) and nowhere analytic.
4. If \( u(x, y) \) and \( v(x, y) \) are harmonic functions, which of these statements is always true? Give a reason (either a proof if true or a counterexample if false).

(a) (2 points) \( u(x, y) + iv(x, y) \) is analytic.

**Solution:** False: For example, if \( u(x, y) = v(x, y) = x \) then \( u \) and \( v \) are harmonic but \( f = x + ix \) doesn’t satisfy the Cauchy Riemann equation \( u_x = 1 = v_y = 0 \).

(b) (3 points) \( u(x, y)v(x, y) \) is harmonic.

**Solution:** False: let \( u = v = xy \). Then \( u_{xx} = u_{yy} = v_{xx} = v_{yy} = 0 \) so \( u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0 \). But \( uv = x^2y^2 \) so \( (uv)_{xx} + (uv)_{yy} = 2(y^2 + x^2) \neq 0 \).

(c) (2 points) \( u(x, y) + v(x, y) \) is harmonic.

**Solution:** True: \((u(x, y) + v(x, y))_{xx} + (u(x, y) + v(x, y))_{yy} = u_{xx} + u_{yy} + v_{xx} + v_{yy} = 0 \)

(d) (3 points) If \( v(x, y) \) is a harmonic conjugate of \( u(x, y) \) then \( u(x, y) \) is a harmonic conjugate of \( v(x, y) \).

**Solution:** False: \( y \) is a harmonic conjugate of \( x \) since \( x + iy = z \) is analytic. However \( y + ix = i(x - iy) = iz \) is not analytic.