VERSALITY OF ALGEBRAIC GROUP ACTIONS AND RATIONAL POINTS ON TWISTED VARIETIES

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Abstract. We formalize and study several competing notions of versality for an action of a linear algebraic group on an algebraic variety X. Our main result is that these notions of versality are equivalent to various statements concerning rational points on twisted forms of X (existence of rational points, existence of a dense set of rational points, etc.) We give applications of this equivalence in both directions, to study versality of group actions and rational points on algebraic varieties. We obtain similar results on p-versality for a prime integer p. An appendix, by J.-P. Serre, puts the notion of versality in a historical perspective.

1. Introduction

Let k be a base field and G be a linear algebraic k-group. We say that a G-action on an irreducible k-variety X is

- *weakly versal*, if for every field K/k, with K infinite, and every G-torsor $T \to \mathrm{Spec}(K)$ there is a G-equivariant k-morphism $f : T \to X$, and
- *versal*, if every G-invariant dense open subset of X is weakly versal.

The advantage of the second notion over the first is that it only depends on X up to (a G-equivariant) birational isomorphism.

Of particular interest to us will be the case where the G-action on X is generically free. This means that some dense open subset $U \subset X$ is the total space of a G-torsor $U \to B$. Passing to the generic point $\mathrm{Spec}(K) \to B$ of B we obtain a G-torsor $\pi : U_K \to \mathrm{Spec}(K)$, where $K = k(B) = k(X)^G$. If, in addition, X is generically smooth, our definition of versality for X coincides with the usual definition of versality for the G-torsor π or equivalently, of the cohomology class of π in $H^1(K, G)$ [GMS03, Definition 5.1]. Our definition of weak versality is related to the one in [GMS03, Remark 5.8].

Versal objects play an important role in the theory of central simple algebras, under the name of "universal" or "generic" division algebras [Pro67,

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Ami72]. (Here G is the projective linear group PGL_n.) Closely related concepts of "generic field extension" and "generic polynomial" have been extensively studied in Galois theory; for details and further references, see [JLY02]. (Here G is a finite group.) We also note that that a related notion of versality plays a key role in deformation theory (see, e.g., [Art69]) and that classifying versal actions of finite groups on algebraic surfaces is an open problem [Dun09, Tok06] (cf. also [DI09]). For a historical perspective on the notion of versality we refer the reader to the appendix.

Versal G-varieties exist for every linear algebraic group G but they are not unique. In fact, the word "versal" is meant to be interpreted as "similar to universal but without the uniqueness property". However, versal actions with special properties are often of particular interest. For example, the existence of a versal G-variety X such that the field $k(X)^G$ is *rational* over k is related to the Noether problem for the group G; see [DM03]. The minimal value of $\dim(X)-\dim(G)$, over all generically free versal G-varieties X, is the *essential dimension* of G, an important numerical invariant of G; see [Rei00] or [BF03].

Our main focus in this paper will be on the "recognition problem" of determining whether or not a given group action is versal. Our main result, Theorem 1.1 below, relates versality of X to the existence of K-points on certain K-forms of X, for field extensions K/k. To state it, we need the following additional definitions, which will be used throughout the paper.

We will say that a G-action on X is

- *very versal*, if there exists a linear representation $G \to \mathrm{GL}(V)$ and a G-equivariant dominant rational map $V \dashrightarrow X$,
- *birationally linear*, if there exists a linear representation $G \to \mathrm{GL}(V)$ and a G-equivariant birational isomorphism between V and X,
- *stably birationally linear*, if there exists a linear representation $G \to \mathrm{GL}(W)$ such that $X \times W$ is birationally linear.

Note that if the G-action on X is generically free then, by the no-name lemma, $X \times W$ is G-equivariantly birationally equivalent to $X \times \mathbb{A}^n$, where $n = \dim(W)$ and G acts trivially on $\mathbb{A}^n$. In other words, if the G-action on X is generically free, then X is stably birationally linear if and only if $X \times \mathbb{A}^n$ is birationally linear for some $n \geq 0$. We also remark that for generically free actions of $G = \mathrm{PGL}_n$, the notions of versal and very versal actions are closely related to the "dense" and "rational" specialization properties of central simple algebras, introduced by D. J. Saltman in [Sal99, Section11] (cf. also [RV95], where it is shown that universal division algebras have the rational specialization property).

If K/k is a field extension, with K infinite, and $\pi\colon T \to \mathrm{Spec}(K)$ is a G-torsor, we will refer to (T, K) as a *twisting pair* (see Definition 4.1) and write ${}^TX$ for the twist of X_K by T (see Section 3).

Theorem 1.1. *Let G be a linear algebraic group defined over k. A G-action on an irreducible quasiprojective k-variety X is*

(a) *weakly versal if and only if, for every twisting pair* (T, K), ${}^{T}X(K) \neq \emptyset$,
(b) *versal if and only if, for every twisting pair* (T, K), *K-points are dense in* ${}^{T}X$,
(c) *very versal if and only if, for every twisting pair* (T, K), ${}^{T}X$ *is K-unirational,*
(d) *stably birationally linear if and only if, for every twisting pair* (T, K), ${}^{T}X$ *is stably K-rational.*

The rest of this paper is structured as follows. Section 2 is devoted to notation and preliminaries and Section 3 to a detailed discussion of the twisting operation. A proof of Theorem 1.1 is given in Section 4, with part (b) requiring the most delicate arguments.

We then give numerous applications of Theorem 1.1 to study versality of group actions and rational points on algebraic varieties. In particular, in Section 5, we use Theorem 1.1 to show that any K-form of the moduli space $\overline{M_{0,n}}$ of stable curves of genus 0 with $n \geq 5$ marked points is unirational over K. For $n = 5$, we recover a theorem of Enriques and Swinnerton-Dyer about the existence of rational points on del Pezzo surfaces of degree 5. In Section 6, we use Theorem 1.1 in the other direction to give a versality criterion for the action of a closed subgroup $G \subset \Gamma$ on a homogeneous space Γ/H; see Theorem 6.4.

In Section 7, we define and study the related notions of p-versality, where p is a prime integer. This notion turns out to be related to 0-cycles on twisted varieties, rather than K-points. We show that for smooth varieties weak p-versality turns out to be equivalent to p-versality; see Theorem 7.4. This makes p-versality quite accessible.

In Section 8, we relate versality to universal torsors, introduced by J.-L. Colliot-Thélène and J.-J. Sansuc [CTS87]. These results are used in Section 9 to study versality of group actions on toric varieties. For actions on toric varieties we show that various definitions of versality coincide; see Theorem 9.1. In Section 10 we discuss versality criteria for group actions on quadric and cubic hypersurfaces. As an application, we show that a recent conjecture of I. Dolgachev on the Cremona dimension is incompatible with a long-standing conjecture of J. W. S. Cassels, P. Swinnerton-Dyer about rational points on cubic hypersurfaces.

2. Notation and preliminaries

Let k be a field; we will denote the algebraic closure of k by $\bar{k}$.

A *k-variety* X is a reduced, quasiprojective scheme of finite type over k (not necessarily irreducible). A *morphism of k-varieties* is a morphism of schemes respecting the structure morphism to k.

An *algebraic group G over k* is a smooth affine group scheme of finite type over k. An *action of G on X* will always be a morphic action, i.e., a morphism $\sigma : G \times X \to X$ satisfying the standard conditions [MFK94]. We will sometimes refer to X with an action of G as a G-variety.

Given a k-variety X and a field extension K/k, the symbol X_K denotes the K-variety $X \otimes_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$. A *$k$-form of X* is a k-variety X' such that $X_{\bar{k}} \simeq X'_{\bar{k}}$.

A *right* (resp. *left*) *G-torsor over Y* is a morphism $\psi : X \to Y$ of k-schemes such that G acts on X on the right (resp. left), $\psi : X \to Y$ is flat, and the map $G \times_Y X \to X \times_Y X$ defined via $(g, x) \mapsto (x, x \cdot g)$ (resp. $(g, x) \mapsto (x, g \cdot x)$) is an isomorphism. The set of G-torsors over a field K is in bijection with the Galois cohomology set $H^1(K, G)$.

A k-variety is *rational* if it is k-birationally equivalent to $\mathbb{A}^n$, for some positive integer n. A k-variety X is *unirational* if there exists a dominant rational k-map $\mathbb{A}^n \dashrightarrow X$.

A G-action on X is *generically free* if there exists a dense G-invariant open subvariety $X_0 \subset X$ such that the scheme-theoretic stabilizer of every point $x \in X_0$ is trivial. This is equivalent to the existence of a dense G-invariant open subvariety U of X which is the total space of a G-torsor $\pi : U \to B$; see [BF03, Theorem 4.7]. If G transitively permutes the components of X (i.e. B is irreducible), we say that X is a *primitive* G-variety. Thus, the set of G-equivariant birational equivalence classes of generically free primitive G-varieties X are in bijective correspondence with G-torsors over $\mathrm{Spec}(k(X)^G)$. We record the following remark, which is an immediate consequence of this correspondence.

Remark 2.1. A G-action on an irreducible k-variety X is weakly versal if and only if, for every generically free primitive G-variety Y, defined over k, there exists a G-equivariant k-rational map $Y \dashrightarrow X$.

Proposition 2.2. *Let X be an irreducible G-variety defined over k. If G has a fixed k-point $x \in X(k)$ then X is weakly versal.*

Proof. The constant map $T \to X$, sending all of T to x is G-equivariant, for every field K/k and every G-torsor $T \to \mathrm{Spec}(K)$. □

Proposition 2.3. *Let G be an algebraic k-group. Then there exists a generically free linear representation $G \to \mathrm{GL}(V)$. Moreover, we can choose V so that $k(V)^G$ is an infinite field.*

Proof. Embed G into GL_n as a closed subgroup. The G-representation on the space $V = \mathrm{M}_n$ of $n \times n$ matrices via left multiplication is easily seen to be generically free.

If $F := k(V)^G$ is infinite, we are done (in particular, this will be automatic if k is infinite). Otherwise we simply add a copy of the trivial representation to V. This will have the effect of replacing F by $F(t)$, where t is a variable, and the latter field is always infinite. □

Remark 2.4. Let X be a projective irreducible weakly versal G-variety defined over k. Suppose $H \subset G$ is a closed k-subgroup such that every finite-dimensional k-representation of H has a 1-dimensional invariant subspace. Then X has an H-fixed k-point.

To prove this, let V be a generically free G-representation. By Remark 2.1, there is a G-equivariant (and hence, H-equivariant) rational map $V \dashrightarrow X$. Since V has a smooth H-fixed point (the origin), the "Going Down Theorem" implies that X has an H-fixed point; see [RY00, Proposition A.2 and Remark A.7]. □

We will not use Remark 2.4 in the sequel. We note however, that this versality criterion is a convenient way (and often the only known way) to show that a given action is not versal. It is particularly effective if $H \subset G$ is chosen to be a finite abelian subgroup of exponent e and k contains a primitive eth root of unity; see, e.g., [Dun09], [Dun10], [Bea11] and [Tok06].

Proposition 2.5. *If X is a versal irreducible G-variety then X is geometrically irreducible.*

Proof. Let $X_1, \dots, X_n$ denote the irreducible components of X_{k^s}, where k^s denotes the separable closure of k. We want to show that $n = 1$. We will argue by contradiction. Assume $n \geq 2$. Since X is irreducible over k, the absolute Galois group $\mathrm{Gal}(k)$ permutes $X_1, \dots, X_n$ transitively, and $X_1 \cap \dots \cap X_n \neq X$ is a closed subset of X defined over k. After removing this closed subset we may assume that that $X_1 \cap \dots \cap X_n = \emptyset$.

Let V be a generically free linear G-representation; see Proposition 2.3. By Remark 2.1 there exists a G-equivariant rational k-map $f \colon V \dashrightarrow X$. Since V is geometrically irreducible, the image of f is geometrically irreducible and, hence, is contained in one of the components X_i. Since $\mathrm{Gal}(k)$ transitively permutes the components, it is also contained in every irreducible component $X_1, \dots, X_n$ and thus in $X_1 \cap \dots \cap X_n = \emptyset$, a contradiction. □

3. Twisting

Let G/k be an algebraic group, X/k be a G-variety, and $T \to \mathrm{Spec}(k)$ be a right G-torsor. The diagonal action of G on $T \times X$ makes $T \times X$ into the total space of a G-torsor $T \times X \to B$. The base space B of this torsor is unique (it is the geometric quotient of $T \times X$ by G); it is usually called *the twist* of X by T and denoted by ${}^T X$. For details of this construction (which relies on our assumption that X is quasiprojective), see, e.g., [Flo06, Section 2] or [CTKPR11, Section 2].

Note that there is no natural G-action on ${}^T X$; we lose the G-action in the course of constructing ${}^T X$. However, ${}^T X$ carries a natural action of the twisted group ${}^T G$; see Propositions 3.7 and 3.8 below.

If T is split over k, it is easy to see that ${}^T X$ is k-isomorphic to X. Hence, ${}^T X$ is a k-form of X, i.e., X and ${}^T X$ become isomorphic over any splitting field L/k of T.

The following lemma tells us that twisting is performed entirely within the category of quasiprojective varieties.

Lemma 3.1. *Let X be a G-variety and $T \to \operatorname{Spec}(k)$ be a G-torsor, as above. If X is quasiprojective then so is ${}^T X$.*

Proof. Let L/k be a finite Galois splitting field for T, with Galois group W. Then ${}^T X$ is the quotient of $X \times \operatorname{Spec}(L)$ by the split finite group W. Note that $\operatorname{Spec}(L)$ is an affine k-variety, and hence, $X \times \operatorname{Spec}(L)$ is quasiprojective. Thus, by [Knu77, Proposition 4.1.5], the geometric quotient ${}^T X = (X \times \operatorname{Spec}(L))/W$ is also quasiprojective. □

Twisting is functorial in that a G-equivariant morphism $f \colon X \to Y$ (respectively, a rational map $f \colon X \dashrightarrow Y$) of G-varieties gives rise to a K-morphism ${}^T f \colon {}^T X \to {}^T Y$ (respectively, a K-rational map ${}^T f \colon {}^T X \dashrightarrow {}^T Y$). For details, see [Flo08, Lemma 2.2] (where only rational maps are considered, but the construction of ${}^T f$ is even more straightforward if f is regular).

Theorem 3.2. *Let k be a field, G be a k-group and $T \to \operatorname{Spec}(k)$ be a G-torsor. Denote by* **Var** *the category of k-varieties and by G-***Var** *the category of k-varieties with a G-action. Morphisms in the latter category are G-equivariant k-maps.*

Let $\mathcal{L}_T : \mathbf{Var} \to G\text{-}\mathbf{Var}$ be the functor $T \times -$ which takes a k-variety Y to $T \times Y$ (viewed as a G-variety, with G acting trivially on Y). Let $\mathcal{R}_T : G\text{-}\mathbf{Var} \to \mathbf{Var}$ be the twisting functor (${}^T -$) described above.

Then the functors $(\mathcal{L}_T, \mathcal{R}_T)$ form an adjoint pair. In other words, for any $Y \in \mathbf{Var}$ and $X \in G\text{-}\mathbf{Var}$, we have an isomorphism

$$\operatorname{Hom}_{G\text{-}\mathbf{Var}}(T \times Y, X) \simeq \operatorname{Hom}_{\mathbf{Var}}(Y, {}^T X) \tag{3.1}$$

which is functorial in both X and Y.

Proof. For clarity, we define all maps and actions on $\overline{k}$-points. The careful reader will find that all our constructions can be expressed as compositions of projections, action maps, etc. In particular, the resulting maps and actions are defined over k.

Given $X \in G\text{-}\mathbf{Var}$ and $Y \in \mathbf{Var}$, we construct a map

$$\phi_{Y,X} \colon \operatorname{Hom}_{G\text{-}\mathbf{Var}}(T \times Y, X) \to \operatorname{Hom}_{\mathbf{Var}}(Y, {}^T X)$$

as follows. Let $\pi : T \times X \to {}^T X$ be the canonical quotient map. Given a G-equivariant map $\alpha : T \times Y \to X$, we construct the map $F \colon T \times Y \to T \times X$, via $F(p, y) = (p, \alpha(p, y))$ for $(p, y) \in T \times Y$. The map F is G-equivariant; after taking its quotient, we have the desired map $\phi_{Y,X}(\alpha)$.

We use [ML98, Theorem IV.1.2(iii)] to construct the inverse map and prove the adjunction. To do this, we need to construct a natural transformation $\varepsilon \colon \mathcal{L}_T \mathcal{R}_T \to \mathrm{Id}_{G\text{-}\mathbf{Var}}$ and show that, for every $X \in G\text{-}\mathbf{Var}$, there is a universal arrow $\varepsilon_X : T \times {}^T X \to X$ from $\mathcal{L}_T$ to X.

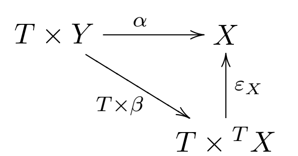
Recall that there is a G-equivariant map $\omega : T\times T \to G$, where $g \in G$ acts via $g \cdot (p, q) = (pg^{-1}, q)$ on $T \times T$, and via left translations on G. Informally, this map is $\omega(p, q) = p^{-1}q$.

Consider $T \times T \times X$ as a $(G \times G)$-variety, where (g, h) acts by $(p, q, x) \mapsto (pg^{-1}, qh^{-1}, hx)$. Consider X as a $(G \times G)$-variety where (g, h) acts via $x \mapsto gx$. The map $E_X : T \times T \times X \to X$, defined by $E_X(p, q, x) = \omega(p, q) \cdot x$, is $(G \times G)$-equivariant. The map

$$\mathcal{L}_T(\pi) = (T \times \pi)\colon T \times T \times X \to T \times {}^T X$$

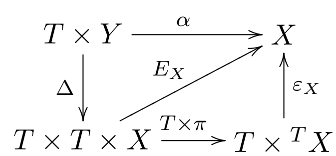
is the quotient of $T\times T\times X$ by the second factor of $(G\times G)$. By the universal property of quotients, we obtain a G-equivariant map $\varepsilon_X : T \times {}^T X \to X$. Furthermore, this construction is functorial in X. Thus, we obtain the desired natural transformation ε.

Now, we verify that each ε_X is a universal arrow. In other words, given a variety Y and any G-equivariant morphism $\alpha : T \times Y \to X$ we require a unique morphism $\beta : Y \to {}^T X$ such that the diagram

(3.2)


commutes. We shall see that $\beta = \phi_{Y,X}(\alpha)$.

One checks that the diagram

(3.3)


commutes, where Δ takes $(p, y) \mapsto (p, p, \alpha(p, y))$. Furthermore, if we consider the G-action on $T \times T \times X$ where each $g \in G$ acts via $(p, q, x) \mapsto (pg^{-1}, qg^{-1}, gx)$, then the diagram is G-equivariant.

Let $\delta = (T \times \pi) \circ \Delta : T \times Y \to T \times {}^T X$ be the composition with the quotient map. We need to show that $\delta = T \times \beta$.

Choose any $y \in Y$, and any $p, q \in T$. The exists a unique $g \in G$ such that $p = qg$. We see that

$$\begin{aligned}\pi(p, \alpha(p, y)) &= \pi(qg, \alpha(p, y))\\ &= \pi(q, g\alpha(p, y))\\ &= \pi(q, \alpha(pg^{-1}, y)\\ &= \pi(q, \alpha(q, y)) \ .\end{aligned}$$

Thus, $(\mathrm{pr}_2 \circ\delta)(p, y) = (\mathrm{pr}_2 \circ\delta)(q, y)$; so the map δ is a product of the maps id_T and $\beta = \phi_{Y,X}(\alpha)$, as desired.

It remains only to show β is the unique arrow fitting into diagram (3.2). Let $\beta' : Y \to {}^T X$ be another such arrow. Fix a point $p \in T$. There exists a unique set map $s : Y(\bar{k}) \to X(\bar{k})$ such that $(p, s(y)) \in T \times X$ is in the preimage of $\beta'(y) \in {}^T X$. By the commutativity of the diagram, $\alpha(p, y) = s(y)$ for every $y \in Y$. Thus $\beta' = \beta$. □

Remark 3.3. Theorem 3.2 strengthens [CTKPR11, Lemma 3.4], whose proof associates morphisms $\alpha : T \times X \to Y$ to morphisms $\beta : Y \to {}^T X$, and vice versa. However, there, the association from $\beta : Y \to {}^T X$ to $\alpha : T \times X \to Y$ is not canonical. Theorem 3.2 asserts that, given β, there is, in fact, a canonical choice of α (namely, the composition $\alpha = \varepsilon_X \circ (T \times \beta)$).

Corollary 3.4. *Let X be a G-variety, and $T \to \mathrm{Spec}(k)$ be a G-torsor. Let L/k be a splitting field of T, let s be a point in $T(L)$, and let $t_s : ({}^T X)_L \to X_L$ be the L-isomorphism such that $s \times t_s$ is a section of $\pi_L : T_L \times X_L \to ({}^T X)_L$. If $\alpha \colon T \to X$ and $\beta \colon \mathrm{Spec}(k) \to {}^T X$ are corresponding maps under the adjunction of Theorem 3.2 (with $Y = \mathrm{Spec}(k)$) then $\alpha_L(s) = t_s(\beta_L)$ in $X(L)$.*

Proof. By definition, α and β fit into the following commutative diagram of G-equivariant k-morphisms:

$$\begin{array}{ccc} T & \xrightarrow{\mathrm{id} \times \alpha} & T \times X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{\beta} & {}^T X \,. \end{array}$$

The vertical maps are G-torsors; we split them by base-changing from k to L. By the definition of t_s the resulting diagram

$$\begin{array}{ccc} T_L & \xrightarrow{\mathrm{id} \times \alpha_L} & T_L \times X_L \\ s \uparrow \; \downarrow & & \downarrow \; \uparrow s \times t_s \\ \mathrm{Spec}(L) & \xrightarrow{\beta_L} & ({}^T X)_L \,. \end{array}$$

is commutative. Tracing from the lower left corner to the upper right, we see that $s \times \alpha_L(s) = s \times t_s(\beta_L)$ as morphisms $\mathrm{Spec}(L) \to T_L \times X_L$. Composing these morphisms with the projection $T_L \times X_L \to X_L$, we see that $\alpha_L(s) = t_s(\beta_L)$ as maps $\mathrm{Spec}(L) \to X$. □

Corollary 3.5. *Let X and Y be G-varieties defined over k, and let $T \to \mathrm{Spec}(k)$ be a G-torsor. Then*

(a) *${}^T(X \times Y)$ is canonically isomorphic to ${}^T X \times {}^T Y$.*
(b) *Let $f \colon X \to Y$ be a G-equivariant closed (resp. open) immersion. Then ${}^T f \colon {}^T X \to {}^T Y$ is also a closed (resp. open) immersion.*

(c) *If $f \colon X \dashrightarrow Y$ is a G-equivariant dominant rational map then the induced rational map ${}^T f \colon {}^T X \dashrightarrow {}^T Y$ is also dominant.*

Proof. (a) The twisting functor is a right adjoint and, thus, is left exact. See also [CTKPR11, Lemma 3.5].

Parts (b) and (c) follow from [EGA IV, Proposition 2.7.1]. For the sake of completeness we will supply a short proof here. The properties of being a closed or open immersion (in part (b)) and of being dominant (in part (c)) are geometric. In other words, for the purpose of checking that ${}^T f$ has these properties, we may pass to any field extension L/k. In particular, we may replace k by a splitting field of T and thus assume without loss of generality that $T \to \operatorname{Spec}(k)$ is split. In this case ${}^T X$, ${}^T Y$ and ${}^T f$ are all isomorphic to X, Y, and f, and the assertions of parts (b) and (c) become obvious. □

Proposition 3.6. *A map $f\colon Y \to {}^T X$ is dominant (respectively, an isomorphism; respectively, a birational isomorphism) if and only if so is the pullback $F : T \times Y \to T \times X$.*

Proof. We have the following commutative diagrams where the vertical maps are G-torsors:

$$\begin{array}{ccc} T \times_k Y & \xrightarrow{F} & T \times_k X \\ \downarrow & & \downarrow \pi \\ Y & \xrightarrow{f} & {}^T X \,. \end{array}$$

By a well known property of torsors, f is dominant (respectively, an isomorphism; respectively, a birational isomorphism) if and only if so is F. □

Proposition 3.7. (*cf.* [CTKPR11, Lemma 3.5]) *Suppose Γ and G are algebraic k-groups, and G acts on Γ by group automorphisms. Let $T \to \operatorname{Spec}(k)$ be a G-torsor. Then ${}^T\Gamma$ is a k-form of the algebraic group Γ. In particular, ${}^T\Gamma$ is an affine algebraic k-group.*

Proof. The commutative diagrams defining the group scheme structure on Γ are all G-equivariant. Applying the twisting functor to these diagrams, and using Corollary 3.5(a), we see that ${}^T\Gamma$ is an algebraic group. If L/k is a splitting field of T then clearly $({}^T\Gamma)_L$ and Γ_L are isomorphic.

As pointed out in [CTKPR11, p. 444], the assertion that Γ is affine follows by descent; see [EGA IV, Proposition 2.7.1(xiii)]. □

Proposition 3.8. *Let $T \to \operatorname{Spec}(k)$ be a G-torsor and let ${}^T G$ denote the twist by T of the conjugation action of G on itself. For every G-variety X, the G-action on X induces a ${}^T G$-action on ${}^T X$. Moreover, for every G-equivariant morphism f the morphism ${}^T f$ is ${}^T G$ equivariant. In other words, the twisting functor factors through the category of ${}^T G$-varieties.*

Proof. The action map $G \times X \to X$ and associated commutative diagrams are all G-equivariant. As in the proof of Proposition 3.7, we obtain an action map ${}^T G \times {}^T X \to {}^T X$ and commutative diagrams which show that ${}^T f$ is ${}^T G$ equivariant. □

Proposition 3.9. *Let V be a linear representation $G \to \mathrm{GL}(V)$ viewed as a G-variety and $T \to \mathrm{Spec}(k)$ be a G-torsor. Then ${}^TV$ is k-isomorphic to V. In particular, ${}^TV(k)$ is dense in ${}^TV$.*

Proof. Since the G-action commutes with the operations $+\colon V \times V \to V$ and $\cdot\colon \mathbb{A}^1 \times V \to V$ of addition and scalar multiplication on V, twisting by T gives rise to operations ${}^T+\colon {}^TV \times {}^TV \to {}^TV$ and ${}^T\cdot\colon \mathbb{A}^1 \times {}^TV \to {}^TV$. The fact that V is a vector space can be expressed by saying that $+$ and $\cdot$ fit into suitable commutative diagrams. Twisting these diagrams by T (as in the proof of Proposition 3.7), we see that ${}^TV$ is a k-vector space with respect to the operations ${}^T+$ and ${}^T\cdot$.

The k-forms of V (as a vector space) are classified by $H^1(k, \mathrm{GL}(V))$, which is trivial by Hilbert's Theorem 90; see, e.g., [Ser79, Proposition X.1.3]. Thus, every k-form of V is k-isomorphic to V_k. In particular, ${}^TV$ is k-isomorphic to V_k. □

4. Proof of Theorem 1.1

We will use repeatedly the fact that twisting commutes with base field extension. Given a k-variety X, a field extension K/k, and a G-torsor $T \to \mathrm{Spec}(K)$, we will use the shorthand notation ${}^TX$ to denote ${}^T(X_K)$.

For brevity, we use the following terminology throughout the paper.

Definition 4.1. Let k be a field and G be an algebraic k-group. By a *G-twisting pair* (T, K) we shall mean a choice of a field extension K/k, with K infinite, and a G-torsor $T \to \mathrm{Spec}(K)$. In situations where the choice of G is clear from the context and there is no risk of ambiguity, we will simply refer to (T, K) as a *twisting pair*.

Proof of Theorem 1.1(a). Let (T, K) be any twisting pair. Setting $Y = \mathrm{Spec}(K)$ in Theorem 3.2, we see that the K-points of ${}^TX$ are in a natural $1-1$ correspondence with G-equivariant maps $T \to X$. □

The proof of Theorem 1.1(b), is considerably more delicate. Before we proceed with the details, we would like to explain a new obstacle our argument will have to overcome.

Given a G-action on an irreducible X, and a G-twisting pair (T, K), let us say that X is (T, K)-weakly versal if there exists a morphism $T \to X$ defined over k. The G-action on X is, by definition, weakly versal if it is (T, K)-weakly versal for every twisting pair (T, K). Note that our proof of Theorem 1.1(a) establishes the following stronger statement:

Choose a G-twisting pair (T, K). Then an irreducible G-variety X is (T, K)-weakly versal if and only if ${}^TX$ has a K-point.

Similarly, given a G-twisting pair (T, K), we will say that an irreducible G-variety X is (T, K)-versal if every dense G-invariant open subvariety of X is (T, K)-weakly versal. One is thus naturally led to try to prove Theorem 1.1(b) by showing that for any given G-twisting pair (T, K), X is

(T, K)-versal if and only if K-points are dense in ${}^T X$. The following example shows that this stronger version of Theorem 1.1(b) fails.

Example 4.2. Let $k = \mathbb{C}$ and let X be a smooth irreducible projective complex curve of genus $g \geq 2$, whose automorphism group $G := \mathrm{Aut}(X)$ is non-trivial. By Hurwitz's theorem, G is finite.

Let $\pi \colon X \to X/G$ be the quotient map, let $K := k(X)^G = k(X/G)$, and let $T \to \mathrm{Spec}(K)$ be the G-torsor obtained by pulling back π via the generic point $\mathrm{Spec}(K) \to X/G$. Then the G-action on X is (T, K)-versal, since the identity map $X \to X$ restricts to a G-equivariant morphism $T \to U$ for any G-invariant open subset $U \subset X$.

On the other hand, we claim that the K-curve ${}^T X$ has only finitely many K-points, and hence, K-points cannot be dense in ${}^T X$. Indeed, arguing as in the proof of Theorem 1.1(a), we see that K-points of ${}^T X$ are in a natural bijective correspondence with G-equivariant k-morphisms $T \to X$ or equivalently, with G-equivariant rational maps $X \dashrightarrow X$, or equivalently (since X is a smooth complete curve) with G-equivariant morphisms $X \to X$. The latter can be of two types: (i) dominant and (ii) constant (i.e., the image is a single point of X). It thus suffices to show that there are only finitely many morphisms $X \to X$ of each type.

(i) Since $g \geq 2$, the Hurwitz formula tells us that any dominant morphism $X \to X$ is, in fact, an automorphism of X. As we mentioned above, X has only finitely many automorphisms.

(ii) If the image of f is a point of X, this point has to be fixed by G, and X has only finitely many G-fixed points. This completes the proof of the claim. □

The above example demonstrates that, given a twisting pair (T, K), we cannot hope to deduce the density of K-points in ${}^T X$ from the fact that X is (T, K)-versal. We will deduce the density of K-points in ${}^T X$, for every twisting pair (T, K), from the fact that X is (S, F)-versal, where S and F are as follows.

Definition 4.3. For the rest of this section

- V will denote a generically free linear representation of G,
- F will denote the field $k(V)^G$. We will choose V so that F is infinite; this is possible by Proposition 2.3.
- $S \to \mathrm{Spec}(F)$ will denote the torsor associated to to the generically free linear representation V of G. That is, S is G-equivariantly birationally k-isomorphic to V.

We now proceed with the proof of Theorem 1.1(b).

Lemma 4.4. *Let X/k be a geometrically irreducible G-variety, and suppose X is (T, K)-versal for some twisting pair (T, K). Then, for any field extension L/k and for any proper closed subset $Y \subset X_L$ (not necessarily*

G-invariant), there exists a G-invariant k-morphism $\psi : T \to X$ such that the image of ψ_L is not contained in Y.

Proof. First assume $L = k$. If Y is G-invariant, the lemma reduces to the definition of (T, K)-versality. The issue is that Y may not be G-invariant. Let Z be the closure of the union $\bigcup \mathrm{im}(\psi)$ where $\psi : T \to X$ varies over all G-equivariant k-morphisms whose image is contained in Y. Since each ψ is G-equivariant, each $\mathrm{im}(\psi)$ is G-invariant, as is the closure of their union. In other words, Z is G-invariant. Note that $Z \subseteq Y \subsetneq X$. Since X is (T, K)-versal, there is a map $\psi : T \to X$ whose image is in the complement of Z. By the construction of Z, the image of any such map is not contained in Y. The completes the proof of the lemma in the case where $k = L$.

Now assume L/k is arbitrary. Let $X = U_1 \cup \cdots \cup U_m$ be an open affine cover of X defined over k. (We do not assume that the U_i are G-invariant.) The defining equations of Y in each U_i involve only a finite number of elements of L. Let R be the k-subalgebra of L generated by all these elements. Then Y is, in fact, defined over $\mathrm{Spec}(R)$. In other words, there exists a closed k-subvariety $Y_0 \subset X_R = X \times_k R$ such that $Y = Y_0 \times_R L$.

Since $Y \neq X_L$, clearly $Y_0 \neq X_R$. Let $X_R = X \times \mathrm{Spec}(R) \to X$ be the natural projection and C be the closed subset of X consisting of all points of X whose fibres are completely contained in Y_0. Since $Y_0 \neq X_R$, $C \neq X$. Since C is defined over k, we have shown that there is a G-equivariant k-morphism $\psi : T \to X$ whose image is not contained in C. Then the image of ψ_R is not contained in Y_0; and thus, the image of ψ_L is not contained in Y. □

Corollary 4.5. *Let X be a geometrically irreducible k-variety, and let L/k be a field extension. Note that there is a natural inclusion of sets $X(k) \hookrightarrow X_L(L)$ by pulling back $\mathrm{Spec}(k) \to X$ by $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$. Then $X(k)$ is dense in X if and only if $X(k)$ is dense in X_L.*

Proof. If L is finite then the result is immediate: in this case $X(k)$ is dense in X if and only if $X(k)$ is dense in $X(L)$ if and only if X is a point. Thus we may assume that L is infinite. The morphism $X_L \to X$ is dominant, so one direction is obvious. The other implication is a special case of Lemma 4.4 with $K = L$, $G = \{1\}$ and $T = \mathrm{Spec}(K)$. □

Lemma 4.6. *Let X/k be a geometrically irreducible G-variety, let (T, K) be a twisting pair, and let L/K be a field extension which splits T. Fix an L-point $s \in T(L)$. Then the following are equivalent:*

(a) *$({}^T X)(K)$ is dense in ${}^T X$,*

(b) *the set of points $f(s)$, where f varies over all G-equivariant k-morphism $f : T \to X$, is dense in X_L.*

Note that condition (b) is considerably stronger than the condition that the union of $f(T)$ is dense in X_L, which came up in Lemma 4.4. This discrepancy is precisely the source of the difficulty which is exemplified by Example 4.2.

Proof. Since X is geometrically irreducible, so is ${}^TX$. By Corollary 4.5, condition (a) is equivalent to

(c) $({}^TX)(K)$ is dense in $({}^TX)_L$.

Let t_s be an L-isomorphism between $({}^TX)_L$ and X_L, chosen so that

$$(s, t_s)\colon\ ({}^TX)_L \to T_L \times X_L$$

is a section (defined over L) of the G-torsor $T \times X \to {}^TX$, as in the statement of Corollary 3.4. Then (c) is equivalent to

(d) the set of L-points of the form $t_s(q)$, where q varies over $({}^TX)(K)$, is dense in X_L.

By Corollary 3.4, (d) is equivalent to (b). □

Proof of Theorem 1.1(b). $\Longleftarrow$ (cf. [FF08, Proposition 1.12]): Assume K-points are dense in ${}^TX$ for every twisting pair (T, K). We want to show that every dense G-invariant open subset $U \subset X$ is weakly versal. By Theorem 1.1(a) it suffices to show that ${}^TU$ contains a K-point for every twisting pair (T, K), as above. This follows from the fact that ${}^TU$ is a dense open subset of ${}^TX$; see Corollary 3.5(b).

$\Longrightarrow$: Assume X is versal. Recall that, by Proposition 2.5, this means X is geometrically irreducible. Fix a twisting pair (T, K). We want to show K-points are dense in ${}^TX$. Let L be a splitting field for T, and let s be a point in $T(L)$. By Lemma 4.6, it suffices to show that for every closed subset $Y \subsetneq X_L$ defined over L, there exists a G-equivariant k-morphism $f\colon T \to X$ such that $f(s) \notin Y$.

As explained above, we cannot construct f directly, based simply on the fact that X is (T, K)-versal. We will instead construct f in two steps, as a composition of a G-equivariant k-morphism $f_1\colon T \to V$ and a G-equivariant rational map $f_2\colon V \dashrightarrow X$. Here V, S and F are as in Definition 4.3.

Let us begin by constructing f_2. By Lemma 4.4, there exists a G-equivariant k-morphism $\psi\colon S \to X$ such that the image of ψ_L is not contained in Y. Equivalently, there exists a G-equivariant rational k-map $f_2\colon V \dashrightarrow X$ such that the image of $(f_2)_L$ is not contained in Y. (Note that our construction of f_2 makes use of the fact that X is (S, F)-versal, not just (T, K)-versal.)

Now let U be an open subset of V_L such that $(f_2)_L$ is regular on U and $(f_2)_L(U) \cap Y = \emptyset$. Replacing X by V simplifies matters considerably, because we know that ${}^TV(K)$ is dense in ${}^TV \simeq V_K$; see Proposition 3.9. Thus, by Lemma 4.6, there exists a G-equivariant k-morphism $f_1\colon T \to V$ such that $f_1(s) \in U$. In particular, there exists a component of T whose image under f_1 intersects the domain of definition of f_2. Since T is a torsor over a field K, and f_2 is G-equivariant, the composition $f_2 f_1$ will be a regular G-equivariant morphism $T \to X$. Moreover, since $f_1(s) \in U$, we have $f_2 f_1(s) \notin Y$, as desired. This completes the proof of Theorem 1.1(b). □

The following lemma will be instrumental in the proof of Theorem 1.1(c) and (d). Let V, F, and S be as in Definition 4.3.

Lemma 4.7. *Let X, Y be geometrically irreducible k-varieties and suppose X has a G-action. If there exists a dominant rational (resp. birational) F-map $Y_F \dashrightarrow {}^S X$ then there exists a G-equivariant dominant rational (resp. birational) k-map $V \times Y \dashrightarrow V \times X$.*

Proof. By Proposition 3.6, the G-equivariant map $S \times_F Y_F \dashrightarrow S \times_F X_F$ associated to $Y_F \dashrightarrow {}^S X$ is dominant (resp. birational).

Since V and S are birational, their generic points are both isomorphic to $\mathrm{Spec}(k(V))$. Consider the fibre product Z of the generic points of S and X_F in $S \times_F X_F$. We see that Z is an affine scheme with coordinate ring

$$k[Z] = k(V) \otimes_F F(X) \simeq k(V) \otimes_F (F \otimes_k k(X)) \simeq k(X)(t_1, \dots, t_n)$$

where $k[V] \simeq k[t_1, \dots t_n]$. Thus, $k[Z]$ is a field and, so, Z is the generic point of $S \times X_F$. In addition, Z is isomorphic to the generic point of $V \times X$. We see $S \times X_F$ is equivariantly k-birational to $V \times X$.

Similarly, $S \times Y_F$ is equivariantly k-birational to $V \times Y$. Thus, the rational map $S \times Y_F \dashrightarrow S \times X_F$ is equivalent to an inclusion of fields

$$k(X)(t_1, \dots, t_n) \hookrightarrow k(Y)(t_1, \dots, t_n)$$

which is clearly defined over k. The lemma follows. □

Proof of Theorem 1.1(c). $\Longrightarrow$: Suppose X is very versal, i.e., there exists a dominant G-equivariant map $f \colon W \dashrightarrow X$. Then for any twisting pair (T, K), the K-rational map ${}^T f \colon {}^T W \dashrightarrow {}^T X$ is dominant; see Corollary 3.5. By Proposition 3.9, ${}^T W \simeq_K W_K$. Thus ${}^T X$ is K-unirational.

$\Longleftarrow$: By assumption, there exists a dominant rational map $\mathbb{A}^n_F \dashrightarrow {}^S X$ for some integer n. By Lemma 4.7, we obtain a dominant rational G-equivariant k-map $V \times \mathbb{A}^n_k \dashrightarrow V \times X$, where G acts trivially on $\mathbb{A}^n_k$. Composing this map with the projection $V \times X \to X$, we see that X is very versal. □

Proof of Theorem 1.1(d). $\Longrightarrow$: Suppose the G-action on X is stably birationally linear, i.e., there exists a G-equivariant birational isomorphism $\phi \colon X \times W_1 \overset{\sim}{\dashrightarrow} W_0$ for some linear representations $G \to \mathrm{GL}(W_1)$ and $G \to \mathrm{GL}(W_0)$ defined over k. Twisting ϕ by a twisting pair (T, K), we obtain a birational isomorphism

$${}^T\phi \colon ({}^T X) \times_K ({}^T W_1) \overset{\sim}{\dashrightarrow} {}^T W_0 \,.$$

Since ${}^T W_1$ and ${}^T W_0$ are affine spaces over K (cf. Proposition 3.9), this tells us that ${}^T X$ is stably rational over K.

$\Longleftarrow$: By assumption, there is a birational isomorphism $\mathbb{A}^n_F \overset{\sim}{\dashrightarrow} {}^S X \times \mathbb{A}^m_F$ defined over F. Now note that ${}^S X \times \mathbb{A}^m_F \simeq {}^S(X \times \mathbb{A}^m_k)$, where G acts trivially on $\mathbb{A}^m_k$; cf. Corollary 3.5(a). By Lemma 4.7, we obtain a G-equivariant birational isomorphism $V \times \mathbb{A}^n_k \overset{\sim}{\dashrightarrow} V \times X \times \mathbb{A}^m_k$, defined over k. Here G

acts trivially on both $\mathbb{A}^n_k$ and $\mathbb{A}^m_k$. This shows that the G-action on X is stably birationally linear. □

Remark 4.8. Let G be an algebraic group and X be a G-variety. Then X is stably birationally linear $\Longrightarrow$ X is very versal $\Longrightarrow$ X is versal $\Longrightarrow$ X is weakly versal. This is immediate from Theorem 1.1.

Note also that if G is trivial, Theorem 1.1 tells us that X is weakly versal iff X has a k-point, versal iff k-points are dense in X (cf. Corollary 4.5), very versal iff X is k-unirational, and stably birationally linear iff X is stably rational over k. Since there are unirational k-varieties which are not stably rational, k-varieties with a dense set of k-points which are not unirational, and k-varieties where rational points exist but are not dense, we conclude that none of the above implications can be reversed in general, even if $G = \{1\}$.

Remark 4.9. Let Y be a geometrically unirational variety defined over a field K. This means that Y is defined over K and becomes unirational over the algebraic closure $\overline{K}$ of K. Consider the following properties of Y: (i) Y is unirational over K, (ii) K-points are dense in Y, and (iii) Y has a K-point. Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) but it is not known whether or not converse implications hold. For a discussion of this question and further references, see [Kol02, Question 1.3].

It is thus conceivable (even though this seems unlikely), that if X is geometrically unirational and G is an algebraic k-group then

$$\text{Weakly versal} \Longleftrightarrow \text{Versal} \Longleftrightarrow \text{Very versal}$$

for every G-action on X.

Remark 4.10. Suppose X is of minimal dimension among generically free versal G-varieties. Then X must be very versal. To see this, let V be a generically free linear representation of G, as in Proposition 2.3. Let $U \subset X$ be a G-invariant open subset which is the total space of a G-torsor $U \to B$. Then U is weakly versal and, by Remark 2.1, there exists a G-equivariant rational map $f : V \dashrightarrow U$. The closure Z of the image of f is very versal and, since U is a G-torsor, the action on Z is generically free. By minimality of $\dim(X)$, we conclude that $\dim(Z) = \dim(X)$ and thus f is dominant.

Additional examples where the various notions of versality coincide will be discussed in Sections 6, 9 and 10.

5. Forms of $\overline{M_{0,n}}$

In this section, we assume that $\mathrm{Char}(k) = 0$.

A theorem of F. Enriques [Enr97] asserts that a del Pezzo surface of degree 5 over any field has a rational point. The first modern proof of this fact is due to H. P. F. Swinnerton-Dyer [SD72]; subsequent shorter proofs have been published by N. I. Shepherd-Barron [SB92] and A. N. Skorobogatov [Sko93].

In this section we will prove the following theorem.

Theorem 5.1. *Let $X = \overline{M_{0,n}}$ be the moduli space of stable curves of genus 0 with $n \geq 5$ marked points and let K/k be a field extension. Then every K-form of X is K-unirational.*

It is well known that $\overline{M_{0,5}}$ is a Del Pezzo surface of degree 5 and that every Del Pezzo surface of degree 5 over a field K/k is a K-form of $\overline{M_{0,5}}$. Thus, for $n = 5$, Theorem 5.1 reduces to the above-mentioned theorem of Enriques and Swinnerton-Dyer.

Proof. The natural action of S_n on $X = M_{0,n}$ permuting the n points on $\mathbb{P}^1$ extends to $\overline{M_{0,n}}$. Our proof relies on a recent theorem of A. Bruno and M. Mella [BM10] which says that S_n is, in fact, the full automorphism group of $\overline{M_{0,n}}$. (In [BM10] the base field is assumed to be the field of complex numbers. However, using the Lefschetz principle one easily deduces that the automorphism group of $\overline{M_{0,n}}$ is S_n over any base field k of characteristic 0.) Consequently, every K-form of $\overline{M_{0,n}}$, over a field extension K/k is isomorphic to ${}^T X$ for some S_n-torsor $T \to \mathrm{Spec}(K)$.

By Theorem 1.1(c) it suffices to show that the S_n-action on $\overline{M_{0,n}}$ is very versal. To do this, consider dominant S_n-equivariant maps

$$(\mathbb{A}^2)^n \dashrightarrow (\mathbb{P}^1)^n \dashrightarrow \overline{M_{0,n}}\,.$$

Here the first map is the nth power of the natural projection $\mathbb{A}^2 \setminus \{(0,0)\} \to \mathbb{P}^1$, and the second map takes an n-tuple of distinct points on $\mathbb{P}^1$ to its class in $M_{0,n}$. The symmetric group S_n acts on the $2n$-dimensional affine space $(\mathbb{A}^2)^n$ linearly, by permuting the n factors of $\mathbb{A}^2$. □

Remark 5.2. It was observed by Yu. I. Manin [Man66] that if a del Pezzo surface X of degree 5 over a field K, has a rational point then it is, in fact, rational. Thus, by Theorem 1.1, the S_5-action on $\overline{M_{0,5}}$ is stably birationally linear.

6. Homogeneous Spaces

Our main tool in this section is the following theorem, due to C. Chevalley in characteristic 0, and M. Rosenlicht and A. Grothendieck in prime characteristic.

Theorem 6.1. *Let K be a field and Γ be a connected algebraic K-group. If Γ is reductive or K is perfect, then the underlying variety of Γ is K-unirational.*

Proof. See [Bor91, Theorem 18.2(ii)]. □

Remark 6.2. Note that, in applying Theorem 1.1, we have to consider all possible extensions K/k. In prime characteristic, even if k is perfect, K may no longer be perfect. For this reason we can only use Theorem 6.1 in this context if we assume that Γ is reductive. More generally, the fact that an extension of a perfect field need not be perfect is one of the underlying reasons for restrictions on the characteristic of k in the remainder of the paper.

The following proposition amplifies [CTKPR11, Proposition 3.3].

Proposition 6.3. *Let Γ be a connected algebraic group. If* $\mathrm{Char}(k) > 0$*, assume that* Γ *is reductive. Then every action of an algebraic group G on Γ by group automorphisms is very versal.*

Proof. Let (T, K) be a twisting pair. By Proposition 3.7, the twisted variety ${}^T\Gamma$ is an algebraic group over K. Hence, by Theorem 6.1, ${}^T\Gamma$ is unirational over K. The desired conclusion now follows from Theorem 1.1(c). □

Theorem 6.4. *Let Γ be a (not necessarily connected) algebraic group. If* $\mathrm{Char}(k) > 0$*, assume that Γ is reductive. Suppose G and H are closed subgroups of Γ, and $X := \Gamma/H$ is geometrically irreducible. Consider X as a G-variety. The following are equivalent:*

(a) *X is very versal,*
(b) *X is versal,*
(c) *X is weakly versal,*
(d) *the image of the natural map $H^1(K, G) \to H^1(K, \Gamma)$ is contained in the image of the natural map $H^1(K, H) \to H^1(K, \Gamma)$ for every field extension K/k where K is infinite.*

Proof. Let (T, K) be a twisting pair. Note that in view of Proposition 3.8, ${}^TX$ is a homogeneous space for the twisted group ${}^TG$.

By Theorem 1.1 it suffices to show that the following conditions are equivalent:

(i) ${}^TX$ has a K-point,
(ii) There exists a dominant Γ-equivariant map $f\colon {}^T\Gamma \to {}^TX$ defined over K,
(iii) ${}^TX$ is K-unirational,
(iv) K-points are dense in ${}^TX$,
(v) The class of T lies in the image of the natural map $H^1(k, H) \to H^1(k, \Gamma)$.

Proof of (i) $\Longrightarrow$ (ii): If $p \in {}^TX(K)$, set $f(g) := g \cdot p$.

The implication (ii) $\Longrightarrow$ (iii) is immediate from Theorem 6.1.

The implications (iii) $\Longrightarrow$ (iv) and (iv) $\Longrightarrow$ (i) are obvious. This shows that (i), (ii), (iii) and (iv) are equivalent. The equivalence of (ii) and (v) is proved in [Spr66, Proposition 1.11]. □

In the following examples, we assume Γ is reductive if $\mathrm{Char}(k) > 0$.

Example 6.5. Setting $H = \{1\}$ we see that the translation action of a subgroup G on Γ is versal if and only if the natural map $H^1(K, G) \to H^1(K, \Gamma)$ is trivial for every field extension K/k, where K is infinite.

The same is true for the G-action on Γ/H whenever H is a special group (this means that $H^1(K, H)$ is trivial for every K/k).

Example 6.6. Setting $H = G$ yields the following: For any closed subgroup $G \subset \Gamma$, the translation action of G on Γ/G is versal.

Example 6.7. Setting $H = N :=$ normalizer of a maximal torus in Γ, shows that the translation action of G on Γ/N is versal for any $G \subset \Gamma$. This is because, by a theorem of T. A. Springer, the natural map $H^1(K, N) \to H^1(K, \Gamma)$ is surjective for every field extension K/k; see [Ser02, III.4.3, Lemma 6].

Remark 6.8. Let Λ be an algebraic group over a field K and let Y be a Λ-variety which contains a Λ-homogeneous space Y_0 as a dense open subvariety defined over K. (Sometimes one refers to such Y as a *quasi-homogenerous space* for Λ.) A theorem of M. Florence [Flo06] asserts that if Y has a smooth K-point then Y_0 has a K-point.

Applying this result to the ${}^T\Gamma$-action on ${}^TX$, it is easy to modify our proof to show that Theorem 6.4 remains valid if $X = \Gamma/H$ is replaced by any smooth Γ-variety, defined over k, and containing Γ/H as a dense open subset.

7. p-VERSALITY

Throughout this section, p is a prime number.

Definition 7.1. Let G/k be an algebraic group and let X/k be an irreducible G-variety. We say that X is

- *weakly p-versal* if for every twisting pair (T, K), there exists a field extension L/K of degree prime to p and a G-equivariant k-morphism $T_L \to X$.
- *p-versal* if every G-invariant dense open subset $U \subset X$ is weakly p-versal (cf. [Mer09, Section 2.2]).

As we saw in the previous sections, versality has to do with K-points on the twisted forms of X. The following lemma shows that p-versality is related to 0-cycles of degree prime to p on twisted forms of X.

Recall that a field L is called *p-closed* if the degree of every finite field extension of L is a power of p. For every field K, there exists an algebraic extension $K^{(p)}/K$, such that $K^{(p)}$ is p-closed and, for every finite subextension $K \subset K' \subset L$, the degree $[K' : K]$ is prime to p. The field $K^{(p)}$ satisfying these conditions is unique up to K-isomorphism; it usually called the *p-closure* of K and is denoted by $K^{(p)}$. For details, see [EKM08, Proposition 101.16].

Lemma 7.2. *Let X be an irreducible G-variety. Then the following conditions are equivalent:*

(a) *X is weakly p-versal,*
(b) *for every twisting pair (T, K), ${}^TX$ has a point whose degree over K is prime to p,*

(c) *for every twisting pair* (T, K), ${}^T X$ *has a* 0*-cycle whose degree is prime to* p*,*
(d) *for every twisting pair* (T, K)*, the variety* $({}^T X)_{K^{(p)}}$ *has a* 0*-cycle of degree* 1*,*
(e) *for every twisting pair* (T, K)*, the variety* ${}^T X$ *has a* $K^{(p)}$*-point.*

Proof. (a) $\iff$ (b): By Theorem 3.2, L-points of ${}^T X$ are in bijective correspondence with G-equivariant k-morphisms $\mathrm{Spec}(L) \to {}^T X$.

(b) $\implies$ (c) is obvious.

(c) $\implies$ (d): Suppose $Z \subset {}^T X$ is a zero cycle of degree d, where d is prime to p. Since the degree of every point of $({}^T X)_{K^{(p)}}$ is a power of p, there exists a 0-cycle $Z' \subset ({}^T X)_{K^{(p)}}$ whose degree is a power of p. A desired 0-cycle of degree 1 on $({}^T X)_{K^{(p)}}$ can then be constructed as a linear combination of Z and Z'.

(d) $\implies$ (e): (cf. [Ful85, Example 13.1]) This is immediate from the fact that the degree of every closed point on $({}^T X)_{K^{(p)}}$ is a power of p.

(e) $\implies$ (b): Every $K^{(p)}$-point of ${}^T X$ descends to a finitely generated subextension $K \subset L \subset K^{(p)}$. The field L is then a finite extension of K whose degree is prime to p. □

Finding 0-cycles of algebraic varieties is usually more accessible than finding rational points; for this reason p-versality is often easier to establish than versality. In the sequel we shall need the following simple facts about 0-cycles.

Lemma 7.3. *Let* K *be a field,* Y *be a smooth irreducible* K*-variety.*

(a) *Suppose* U *be a dense open subset of* Y *defined over* K*. If* Y *has a* 0*-cycle then* U *also has a* 0*-cycle of the same degree.*
(b) *Let* p *be a prime and let* $\phi\colon Y \to Y'$ *be a dominant rational map between geometrically integral smooth* G*-varieties of the same dimension. Assume that the degree of* ϕ *(which is defined as* $m := [k(Y)\colon k(Y')]$*) is prime to* p*. Then* Y *has a* 0*-cycle whose degree is prime to* p *if and only if so does* Y'*.*

Proof. We will work with 0-cycles up to linear equivalence, i.e., with elements of the Chow group $A_0(Y)$. Note that linear equivalent 0-cycles have the same degree.

(a) follows from a special case of Chow's Moving Lemma, which asserts that if Z is a 0-cycle then U contains a 0-cycle linearly equivalent to Z. For a modern proof of the Moving Lemma and further references, see [R70].

(b) Choose dense open K-subsets of $U \subset Y$ and $U' \subset Y'$ so that ϕ restricts to a finite flat morphism $U \to U'$. By part (a), we may replace Y by U and Y' by U'. In other words, we may assume that ϕ is a finite and flat morphism.

One direction is obvious, since the 0-cycles $Z \in A_0(Y)$ and $\phi_*(Z) \in A_0(Y')$ have the same degree. Conversely, suppose the degree of $Z' \in A_0(Y)$

is prime to p. Set $Z := \phi^*(Z')$. By [Ful85, Example 1.7.4], $\phi_*(Z) = \deg(\phi)Z'$ in $A_0(Y')$. This shows that the degree of Z is prime to p as well. □

Theorem 7.4. *Let G be an algebraic k-group acting on a smooth geometrically irreducible k-variety X. Then X is p-versal if and only if X is weakly p-versal.*

Proof. We will assume that X is weakly p-versal and prove that X is p-versal; the other direction is obvious.

Let $U \subset X$ be a G-invariant dense open subset. We want to show that U is weakly versal. Let (T, K) be a twisting pair. By Lemma 7.2, it suffices to prove that if ${}^TX$ has a 0-cycle whose degree is prime to p then so does ${}^TU$. Recall that, by Corollary 3.5(b), ${}^TU$ is a dense open subset of ${}^TX$. The theorem now follows from Lemma 7.3(a). □

Corollary 7.5. *Let $f\colon X \dashrightarrow Y$ be a dominant rational map of geometrically irreducible generically smooth G-varieties, of the same dimension. Assume that the degree of f is prime to p. Then X is p-versal if and only if Y is p-versal.*

Proof. After replacing X and Y we may assume that X and Y are smooth. Let (T, K) be a twisting pair. We then have a rational K-map

$$ {}^Tf\colon {}^TX \dashrightarrow {}^TY $$

where ${}^TX$ and ${}^TY$ are again smooth. By Lemma 7.3(b), ${}^TX$ has a 0-cycle of degree prime to p if and only if so does ${}^TY$. Lemma 7.2 now tells us that X is weakly versal if and only Y is weakly versal. By Theorem 7.4, the same is true if "weakly versal" is replaced by "versal". □

Corollary 7.6. *Let X/k be a geometrically irreducible generically smooth G-variety and let H be a closed subgroup of G of finite index. Assume that the index $[G : H]$ is prime to p. A G-action on X is p-versal if and only if the restricted H-action is p-versal.*

Proof. After replacing X by its smooth locus, we may assume that X is smooth.

From the proof of [MR09, Lemma 4.1], for any field K/k, the map $H^1(K, H) \to H^1(K, G)$ is p-surjective. That is, for any $\alpha \in H^1(K, G)$ there exists a finite extension L/K of degree prime to p such that α_L lies in the image of the natural map $H^1(L, H) \to H^1(L, G)$. If K is p-closed, then $[L : K]$ is a power of p, so $L = K$, and the map $H^1(K, H) \to H^1(K, G)$ is surjective. In other words, for any H-torsor $T \to \mathrm{Spec}(K)$, there exists a G-torsor $T' \to \mathrm{Spec}(K)$ such that ${}^TX$ and ${}^{T'}X$ become isomorphic over $K^{(p)}$. In particular, ${}^TX$ has a $K^{(p)}$-point if and only if ${}^{T'}X$ has a $K^{(p)}$-point. Lemma 7.2 now tells us that the G-action on X is weakly p-versal if and only the H-action is weakly p-versal. By Theorem 7.4, the same is true if "weakly versal" is replaced by "versal". □

Corollary 7.7. *Let X/k be a geometrically irreducible G-variety. Suppose there exists a smooth k-point $x \in X(k)$ such that the orbit $G \cdot x$ is finite and* $\deg([G \cdot x])$ *is prime to p. Then the G-action on X is p-versal.*

Proof. After replacing X by its smooth locus, we may assume that X is smooth. We will give two proofs.

First proof: Let C be the G-orbit of x and $i\colon C \to X$ be the natural inclusion. For any twisting pair (T, K), the map ${}^T i\colon {}^T C \to {}^T X$ places a 0-cycle into ${}^T X$ whose degree is the same as $[G \cdot x]$. We conclude that X is weakly p-versal by Lemma 7.2 and consequently, p-versal by Theorem 7.4.

Second proof: Let H be the stabilizer of x in G. Then H is a closed subgroup and the index $[G : H] = \deg([G \cdot x])$ is finite and prime to p. By Proposition 2.2, the H-action on X is weakly versal. By Theorem 7.4, the H-action on X is p-versal. By Corollary 7.6, the G-action on X is p-versal as well. □

We note the following immediate corollary of Lemma 7.2, in the spirit of Theorem 1.1.

Corollary 7.8. *A G-action on a smooth geometrically irreducible variety X is p-versal for every prime p if and only if, for every twisting pair (T, K), ${}^T X$ has a 0-cycle of degree 1.*

Every versal G-variety is clearly p-versal for every prime p. However, the converse is not true in general, even when $G = 1$. Indeed, there exist k-varieties which possess 0-cycles of degree 1 but no k-points; cf. Remark 4.8.

Nevertheless, in view of Corollary 7.8, p-versality for every prime p is a natural (if not perfect) approximation to versality. We will see that these two notions are equivalent if X is a toric variety (Theorem 9.1) or a quadric hypersurface (Theorem 10.2). On the other hand, no counterexample is known for the following weaker statement:

Conjecture 7.9 (cf. [Dun09])**.** *Let G be a finite constant group, X be a G-variety and G_p be a Sylow p-subgroup of G. If X is G_p-versal for prime p, then X is G-versal.*

Note that the key assumption here is that X is versal and not just p-versal as a G_p-variety.

Remark 7.10. Combining the argument in Remark 2.4 with the "Going Up Theorem" ([RY00, Proposition A.4]), we obtain the following fixed point criterion for p-versality.

Assume the base field k has a primitive eth root of unity and $H \subset G$ is a finite abelian p-group of exponent e. Then every p-versal complete G-variety has an H-fixed point.

Remark 7.11. It is natural to define a G-variety X to be "very p-versal" if there exists a linear representation V, and a diagram of dominant rational

G-equivariant maps of the form

$$\begin{array}{ccc} V' & & \\ \downarrow & \searrow & \\ V & & X\,, \end{array}$$

where the degree of $V' \dashrightarrow V$ is prime to p. (Note that V' is not required to be a vector space.) Under mild assumptions on X this notion also turns out to be equivalent to p-versality.

8. Universal Torsors

For simplicity, throughout this section we assume $\mathrm{Char}(k) = 0$.

Let X be a smooth, geometrically integral, projective k-variety such that the Picard group $\mathrm{Pic}(X_L)$ is a finitely generated free abelian group for any algebraically closed field L/k. Examples of such varieties include smooth, proper, geometrically rational varieties [CTS87, Corollaire 2.A.2] and smooth Fano varieties [IP99, Proposition 2.1.2].

We recall the notion of *universal torsors* introduced in [CTS87]. Let R be the k-torus dual to the $\mathrm{Gal}(\bar{k}/k)$-module $\mathrm{Pic}(X_{\bar{k}})$. From [CTS87, 2.0.2], we have the following exact sequence in étale cohomology:

$$1 \longrightarrow H^1(k,R) \longrightarrow H^1(X,R) \xrightarrow{\ m\ } \mathrm{End}_{\mathrm{Gal}(\bar{k}/k)}(\mathrm{Pic}(X_{\bar{k}})) \tag{8.1}$$

An R-torsor $\mathcal{X} \to X$ is *universal* if $m([\mathcal{X}])$ is equal to the identity morphism $\mathrm{Pic}(X_{\bar{k}}) \to \mathrm{Pic}(X_{\bar{k}})$.

The existence of a universal torsor is closely related to the existence of k-points as illustrated by the following:

Lemma 8.1. *The three properties of X below,*

(a) *X has a k-point,*
(b) *X has a 0-cycle of degree 1, and*
(c) *X has a universal torsor,*

are related as follows: (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c).

Proof. (a) $\Longrightarrow$ (b) is immediate, while (b) $\Longrightarrow$ (c) follows from [CTS87, Proposition 2.2.2] and [CTS87, Proposition 2.2.8(iii)]. □

Now, suppose there is an action of an algebraic group G on X. By [Sko09, Lemma 1.1], if X possesses a universal torsor $\mathcal{X}$, then there exists an algebraic group $\mathcal{G}$ with an action on $\mathcal{X}$ fitting into an exact sequence

$$1 \to R \to \mathcal{G} \to G \to 1 \tag{8.2}$$

where R acts on the fibres and $\mathcal{G}$ preserves the fibres of the projection $\mathcal{X} \to X$.

The main theorem of this section generalizes [Dun09, Theorem 3.2].

Theorem 8.2. *If X is G-versal, then X has a universal torsor $\mathcal{X}$. Furthermore, for any such $\mathcal{X}$, the corresponding sequence* (8.2) *splits.*

Lemma 8.3. *For every field extension K/k, the morphism $H^1(K, \mathcal{G}) \to H^1(K, G)$ is surjective if and only if, for every twisting pair (T, K), ${}^TX$ has a universal torsor.*

Proof. By [Sko09, Lemma 1.2], given a G-torsor $T \to \mathrm{Spec}(K)$, every universal torsor $\mathcal{X}'$ over ${}^TX$ is of the form ${}^{\mathcal{T}}\mathcal{X}$, where $\mathcal{T} \to \mathrm{Spec}(K)$ is a $\mathcal{G}$-torsor such that the class of T is in the image of the class of $\mathcal{T}$ under the map $H^1(K, \mathcal{G}) \to H^1(K, G)$. □

Lemma 8.4. *Consider an exact sequence of algebraic groups*

$$1 \to R \to \mathcal{G} \to G \to 1\,.$$

Suppose there exists an $\mathcal{G}$-action on R (as a variety) such that $R \subset \mathcal{G}$ acts by translations. Then the sequence splits.

Proof. Let e_R be the identity element of $R(k)$. We will show that $H = \mathrm{Stab}_{\mathcal{G}}(e_R)$ is isomorphic to G under the restriction of the map $\mathcal{G} \to G$. First, we note that $H \times_{\mathcal{G}} R$ is $e_{\mathcal{G}}$, so the kernel of $H \to G$ is trivial. Next, consider any point $g \in G(\bar{k})$: it has a lift $\tilde{g} \in \mathcal{G}(\bar{k})$. Since R acts freely on itself, there exists an element $s \in R(\bar{k})$ such that $s\tilde{g}e_R = e_R$. Thus, $H(\bar{k}) \to G(\bar{k})$ is surjective. Since G is smooth, by [KMRT98, Proposition 22.3] the morphism $H \to G$ is surjective. Thus $H \simeq G$ and we have a splitting $G \to \mathcal{G}$. □

Proof of Theorem 8.2. Clearly, a versal variety has a k-point. Thus, we have universal torsor by Lemma 8.1. Furthermore, we have a universal torsor for every twist ${}^TX$. Note that it suffices to show (8.2) splits for only one universal torsor. Indeed, by [Sko09, Lemma 1.2], we can obtain any other splitting by twisting the group homomorphisms in $G \to \mathcal{G} \to G$ by an R-torsor.

By Lemma 8.3, $H^1(K, \mathcal{G}) \to H^1(K, G)$ is surjective for all field extensions K/k. Since X is G-versal, there exists a G-equivariant rational map $f : V \dashrightarrow X$, where V is a generically free linear representation of G; see Proposition 2.3. Let $Y \subset V$ be the domain of definition of f, and consider $\mathcal{Y} \to Y$ the pullback of $\mathcal{X}$ via f.

Since X is proper, the complement of Y has codimension ≥ 2 in V. Thus, $(\bar{k}[Y])^\times = (\bar{k}[V])^\times = \bar{k}^\times$. Since $\mathrm{Pic}(Y_{\bar{k}}) = 0$, we have that $H^1(k, R) = H^1(Y, R)$ by the sequence (8.1). Thus, there exists an R-torsor T such that ${}^T\mathcal{Y}$ is a trivial R-torsor over Y. Taking $\mathcal{Y}' = {}^T\mathcal{Y}$, $\mathcal{X}' = {}^T\mathcal{X}$, and $\mathcal{G}' = {}^T\mathcal{G}$, we obtain a $\mathcal{G}'$-equivariant map $\mathcal{Y}' \to \mathcal{X}'$ of R-torsors over the G-equivariant map $Y \to X$.

Since all multiplicatively invertible functions on $Y_{\bar{k}}$ are constant, all maps $Y_{\bar{k}} \to R_{\bar{k}}$ are constant. Thus, any automorphism of the torsor $\mathcal{Y}' \simeq Y \times R$ is an element of $\mathrm{Aut}(Y) \times \mathrm{Aut}(R)$. In particular, we have a group homomorphism $\mathcal{G}' \to \mathrm{Aut}(R)$. Thus, there is an action of $\mathcal{G}'$ on R such that the restricted action of R on itself is translation. We have a splitting $G \to \mathcal{G}'$ by Lemma 8.4.

□

9. Toric Varieties

Throughout this section, we assume $\mathrm{Char}(k) = 0$.

Let U be a (not necessarily split) torus defined over k. Recall that a *toric variety* X (for U) is a normal geometrically irreducible U-variety containing an open dense U-orbit that is isomorphic to U as a U-variety. In particular, any such X has a k-point. We shall restrict our attention to smooth projective toric varieties; see, e.g., [Vos82]. Note that X satisfies the conditions of section 8.

Note that, when X is smooth and projective, $\mathrm{Aut}(X)$ is an algebraic group with a maximal torus equal to U. Indeed, from [Dem70, 4.2 Propositions 10 and 11] or [Cox95, Section 4], we see that $U_{\bar{k}}$ is a maximal torus in the algebraic group $\mathrm{Aut}(X_{\bar{k}})$. Let N be the normalizer of U in $\mathrm{Aut}(X)$ and let W be the quotient N/U. By Lemma 8.4, there is a splitting $N \simeq U \rtimes W$.

Theorem 9.1. *Let X be a smooth projective toric variety with a G-action. The following are equivalent:*

(a) *X is very versal,*
(b) *X is versal,*
(c) *X is weakly versal,*
(d) *the image of the natural map $H^1(K, G) \to H^1(K, \mathrm{Aut}(X))$ lies in the image of the natural map $H^1(K, W) \to H^1(K, \mathrm{Aut}(X))$ for all field extensions K/k,*
(e) *X is p-versal for every prime p.*

Furthermore, there exists a universal torsor $\mathcal{X}$ with a $\mathcal{G}$-action, as in Section 8, such that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f) $\Leftrightarrow$ (g), where

(f) *for every field extension K/k, $H^1(K, \mathcal{G}) \to H^1(K, G)$ is surjective,*
(g) *the sequence* (8.2) *splits.*

Proof. We begin by proving the equivalence of (a) – (d). First, we note that X is a quasi-homogeneous space (it is birational to N/W). We would like to use Remark 6.8, but this only works if $U \subset X$ is G-invariant; the birational equivalence $X \overset{\sim}{\dashrightarrow} N/W$ is not G-equivariant in general.

We claim that there is a surjection $H^1(K, N) \to H^1(K, \mathrm{Aut}(X))$ for any field K/k. Indeed, the cohomology of $\mathrm{Aut}(X)$ is canonically isomorphic to the cohomology of its reductive quotient by [Ser02, III.2.1 Proposition 6]. The claim then follows from [Ser02, III.4.3 Lemma 6]; cf. Example 6.7.

Thus, for any twisting pair (T, K), there exists an N-torsor $T' \to \mathrm{Spec}(K)$ such that ${}^{T}X \simeq {}^{T'}X$. Both T and T' have the same image in $H^1(K, \mathrm{Aut}(X))$. Thus, the equivalence of (a) – (d) follows by essentially the same argument as Theorem 6.4 using T' in place of T.

For the remaining equivalences, we use the results from Section 8. As we noted above, every toric variety has a k-point. Hence, there exists a universal torsor $\mathcal{X} \to X$ by Lemma 8.1.

The implications (c) $\implies$ (e) $\implies$ (f) follow from Lemmas 8.1 and 8.3. The implication (b) $\implies$ (g) is Theorem 8.2. Note that (g) $\implies$ (f) since the splitting yields a composition of morphisms of functors $H^1(-,G) \to H^1(-,\mathcal{G}) \to H^1(-,G)$ which is equal to the identity.

The only remaining implication is (f) $\implies$ (c). As above, for any G-twisting pair (T,K), there exists a N-twisting pair (T',K) such that ${}^TX \simeq {}^{T'}X$. By Lemma 9.2 below, ${}^{T'}X$ contains a torsor under some twisted form of U. By [CTS87, Exemples 2.2.11(c)], this implies that ${}^{T'}X$ has a universal torsor if and only if it has a K-point. Thus (f) implies that ${}^TX \simeq {}^{T'}X$ has a universal torsor (Lemma 8.3), which is equivalent to ${}^TX$ having a K-point for every twisting pair (T,K). The last condition is equivalent to (c) by Theorem 1.1(a). □

Lemma 9.2. *Suppose $N = U \rtimes W$ is a semidirect product of algebraic k-groups. Let X be a N-variety. Then, for every N-torsor $T_N \to \operatorname{Spec}(k)$, there is an isomorphism of k-varieties ${}^{T_N}X \simeq {}^{T_U}({}^{T_W}(X))$, where $T_W \to \operatorname{Spec}(k)$ is a W-torsor and $T_U \to \operatorname{Spec}(k)$ is a ${}^{T_W}U$-torsor.*

Proof. We have an exact sequence $H^1(k,U) \to H^1(k,N) \to H^1(k,W)$ of pointed sets. Let T_W be a torsor in the class corresponding to the image of the class of T_N in the map $H^1(k,N) \to H^1(k,W)$. The fibre of the class of T_W consists of classes of torsors in $H^1(k,{}^{T_W}U)$. The splitting $W \to N$ induces a map from $H^1(k,W) \to H^1(k,N)$ which takes the class of T_W to the class of an N-torsor which differs from T_N by a ${}^{T_W}U$-torsor T_U. □

The following corollary amplifies [Dun09, Corollary 3.3].

Corollary 9.3. *Let G be a finite group with an action on $\mathbb{P}^n_k$. Then the following conditions are equivalent:*

(a) *the G-action on $\mathbb{P}^n$ is weakly versal,*
(b) *the G-action on $\mathbb{P}^n$ is versal,*
(c) *the G-action on $\mathbb{P}^n$ is very versal,*
(d) *the G-action on $\mathbb{P}^n$ is stably birationally linear,*
(e) *the G-action on $\mathbb{P}^n$ is p-versal for every prime p,*
(f) *the G-action on $\mathbb{P}^n$ lifts to $\mathbb{A}^{n+1}$.*

Proof. Note that the standard projection $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is a universal torsor for X and $\mathcal{G}$ is the preimage of $G \subset \mathrm{PGL}_{n+1}$ in in GL_{n+1}. Thus, the equivalence of all conditions but (d) follow from Theorem 9.1.

A twist $P := {}^T(\mathbb{P}^n)$ a Brauer-Severi variety for any G-twisting pair (T,K). Hence, $P(K) \neq \emptyset$ if and only if P is K-rational. In view of Theorem 1.1, this means that (d) is equivalent to (a). □

Remark 9.4. It suffices to check condition (e) of Corollary 9.3 only for primes p dividing $n+1$.

10. Group actions on quadric and cubic hypersurfaces

Lemma 10.1. *Let V be a finite-dimensional k-vector space and $G \to \mathrm{GL}(V)$ be a linear representation. Then*

(a) *for any twisting pair (T, K), ${}^T\mathbb{P}(V)$ is K-isomorphic to $\mathbb{P}(V)_K$.*

(b) *Suppose X be a closed G-invariant subvariety of $\mathbb{P}(V)$. Then the inclusion $\iota\colon X \hookrightarrow \mathbb{P}(V)$ induces a closed embedding ${}^T\iota\colon {}^TX \hookrightarrow \mathbb{P}(V)_K$ with the same Hilbert polynomial as X.*

Proof. (a) By Proposition 3.9, ${}^TV \simeq V_K$. The (T, K)-twist of the natural projection $V \dashrightarrow \mathbb{P}(V)$, is thus a dominant rational map $V_K \dashrightarrow {}^T\mathbb{P}(V)$. Consequently, the Brauer-Severi variety ${}^T\mathbb{P}(V)$ has a K-point, and part (a) follows.

(b) Since the embeddings ${}^T\iota : {}^TX \to \mathbb{P}(V)_K$ and $\iota : X \to \mathbb{P}(V)$ become projectively equivalent over the algebraic closure $\bar{K}$, they have the same Hilbert polynomial. □

Theorem 10.2. *Let G be an algebraic group over k, $G \to \mathrm{GL}(V)$ be a finite-dimensional k-representation, and $X \subset \mathbb{P}(V)$ be an irreducible, quadratic G-invariant hypersurface. The following are equivalent:*

(a) *X is stably birationally linear,*
(b) *X is very versal,*
(c) *X is versal,*
(d) *X is weakly versal.*
(e) *X is 2-versal.*

Assume further that G is finite, and G_2 is a Sylow 2-subgroup of G. Then conditions (a) – (e) are equivalent to

(f) *X is versal for the action of G_2.*

Proof. Let (T, K) be a twisting pair. By Lemma 10.1 $Q := {}^TX$ is an irreducible quadratic hypersurface in $\mathbb{P}^n_K$, defined over K. The equivalence of conditions (a)–(d) now follows from the following well-known property of irreducible quadric hypersurfaces $Q \subset \mathbb{P}(V)_K$:

$$\text{if } Q \text{ has a } K\text{-point then } X \text{ is } K\text{-rational.} \tag{10.1}$$

The equivalence of (a) and (e) is an immediate consequence of a theorem by T. A. Springer: if Q has an L-point for some odd degree extension L/K then Q has a K-point.

Finally, assume that G is a finite group. In this case, (f) $\Longrightarrow$ (e) by Corollary 7.6 and implies that On the other hand, (b) $\Longrightarrow$ X is very versal as a G_2-variety $\Longrightarrow$ (f). □

If we replace a quadric hypersurface by a cubic hypersurface of dimension ≥ 2 then property (10.1) in the above proof remains true, provided that "rational" is replaced by "unirational", and Springer's Theorem becomes an open conjecture. The precise statements are as follows.

Theorem 10.3. *Let $X \subset \mathbb{P}^n_k$ be a smooth cubic hypersurface where $n \geq 3$. If X has a k-point then X is k-unirational.*

For a proof of this theorem, variations for singular varieties, and bibliographical references to earlier partial results, see [Kol02].

Conjecture 10.4 (J. W. S. Cassels, P. Swinnerton-Dyer)**.** *Suppose $X \subset \mathbb{P}^n_k$ is a cubic hypersurface. If X has a 0-cycle of degree prime to 3, then X has a k-point.*

For a detailed discussion of this conjecture, see [Cor76].

The proof of Theorem 10.2 now yields the following analogous statement for cubic forms.

Theorem 10.5. *Let G be an algebraic k-group, $G \to \mathrm{GL}(V)$ be a finite-dimensional k-representation and $X \subset \mathbb{P}(V)$ be a smooth G-invariant cubic hypersurface. Assume $\dim(V) \geq 4$. Then following are equivalent:*

(a) *X is very versal,*
(b) *X is versal,*
(c) *X is weakly versal,*

Assuming G is finite with Sylow 3-subgroup G_3 and Conjecture 10.4 holds, (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e), where

(d) *X is 3-versal, and*
(e) *X is versal for the action of G_3.* □

Corollary 10.6. *Suppose an algebraic group G acts on a smooth cubic hypersurface X as in Theorem 10.5. If G fixes a k-point $x \in X(k)$ then X is G-versal.*

Proof. By Proposition 2.2 the G-action on X is weakly versal. Theorem 10.5 now tells us that this action is versal. □

For the rest of this section we will assume that $k = \mathbb{C}$ is the field of complex numbers.

Example 10.7. The Klein cubic is the smooth cubic threefold $X \subset \mathbb{P}^4$ cut out by

$$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0 = 0\,.$$

The automorphism group of X is $G = \mathrm{PSL}_2(\mathbb{F}_{11})$. The action of this group on X extends to a linear action on $\mathbb{P}^4$ [Adl78]. It is shown in [Bea11] that X has a G_p-fixed point x_p for any p-Sylow subgroup G_p of G. Hence, by Corollary 10.6, the G_p-action on X is versal for every prime p. Consequently, the G-action on X is p-versal for every prime p; see Corollary 7.6. Conjecture 10.4 now implies that the G-action on X is versal; see Theorem 10.5.

If we knew that the G-action on X is versal, then we would be able to conclude that the essential dimension of $G = \mathrm{PSL}_2(\mathbb{F}_{11})$ equals 3, thus completing the classification of finite simple groups of essential dimension 3 (over $\mathbb{C}$). For details on this, see [Bea11].

Recall that the *Cremona dimension*, $\mathrm{Crdim}(G)$, of a finite group G is the minimal integer n such that G embeds into the complex Cremona group $\mathrm{Cr}(n)$. Equivalently, the Cremona dimension is the minimal integer n such that G admits a faithful action on a n-dimensional complex rational variety. The following conjecture is due to I. Dolgachev (unpublished).

Conjecture 10.8. $\mathrm{Crdim}(G) \le \mathrm{ed}(G)$ *for every finite group* G.

From [Pro09, Remark 1.6] we see that there are no rational complex threefolds with a faithful action of $\mathrm{PSL}_2(\mathbb{F}_{11})$ (in particular, the Klein cubic threefold in Example 10.7 is not rational). Thus $\mathrm{Crdim}(\mathrm{PSL}_2(\mathbb{F}_{11})) \ge 4$. On the other hand, as we mentioned in Example 10.7, Conjecture 10.4 implies that $\mathrm{ed}(\mathrm{PSL}_2(\mathbb{F}_{11})) = 3$. We conclude that Conjectures 10.8 and 10.4 are incompatible: they cannot both be true.

We also remark that since every Sylow subgroup of $G = \mathrm{PSL}_2(\mathbb{F}_{11})$ acts versally on on the Klein cubic X (see Example 10.7), Conjecture 7.9 also implies that the G-action on X is versal. Therefore, Conjecture 10.8 is also incompatible with Conjecture 7.9.

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Appendix: Letter from J.-P. Serre to Z. Reichstein

Paris, June 10, 2010

Dear Reichstein,

About “versal” :

There was first the notion of a “universal object” , a notion which appeared in several branches of mathematics around 1930-1950; there is even a section of Bourbaki’s *Théorie des Ensembles* (chap.IV, §2) on the general properties of this notion. An especially interesting case being the universal G-principal homogeneous space (now “G-torsor”); the case of $G = \mathrm{GL}(n)$ was basically due to Chern. Such spaces ($E_G \to B_G$ was the standard notation) were very useful to topologists; see e.g. Borel’s thesis.

In the definition of “universal”, there is a uniqueness property (up to homotopy, sometimes) which is required. There are many interesting cases where it does not hold (e.g. deformations of complex manifolds, à la Kodaira-Spencer); people called them “almost universal” (or quasi , or semi . . .). I do not know exactly when somebody had the amusing idea to call them “versal”, by deleting the “uni” which suggests uniqueness. I seem to remember that it was Douady who did this (he enjoyed playing with words); the date should be close to 1966, but I have not looked into his publications, and I cannot give you a precise reference.[1]

That this idea applied to Galois cohomology was obvious from the beginning, both to people with a topologist background (such as Rost or myself), and to algebraists trying to parametrize equations (they rather used the word "generic", which I find a bit confusing). But I don't think(*) the word "versal" got into print [in this context] before my UCLA lectures of 2001 (do you know an earlier reference?)[2], even though I had used it in some College lectures around 1990 (especially those on "negligible cohomology", which were never written down).

Note that the definition in UCLA has a rather non standard restriction: it asks for a density property which may seem artificial (but it is essential in Duncan's work!).

Best wishes,
J-P.Serre

(*) I have asked Google about "versal torsor", but all the references there seem to be post 2001.

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[1] The earliest reference I have been able to find is [Dou60], page 2-04. Z.R.

[2] The term *versal* is used in [BR97], Section 7. The *versal polynomials* defined there give rise to versal G-torsors, in the sense of [GMS03, Section I.5], where G is a finite group. Z.R.

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