

Essential dimension

Zinovy Reichstein

Department of Mathematics
University of British Columbia
, Vancouver, Canada

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Introduction

Informally speaking, the essential dimension of an algebraic object is the minimal number of independent parameters one needs to define it. In the past 15 years this numerical invariant has been extensively studied by a variety of algebraic, geometric and cohomological techniques. The goal of this lecture is to survey some of this research.

To motivate the notion of essential dimension, I will start with a simple example, where the object in question is a quadratic form.

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To motivate the notion of essential dimension, I will start with a simple example, where the object in question is a quadratic form.

The essential dimension of a quadratic form

Let k be a field of characteristic $\neq 2$, K/k be a field extension and q be a non-degenerate quadratic form on K^d .

Denote the symmetric bilinear form associated to q by b . We would like to know if q can be *defined over* (or equivalently, *descends to*) some smaller field $k \subset K_0 \subset K$.

This means that there is a K -basis e_1, \dots, e_d of K^d such that

$$b_{ij} := b(e_i, e_j) \in K_0$$

for every $i, j = 1, \dots, d$.

Equivalently, in this basis $q(x_1, \dots, x_n) = \sum_{i,j=1}^n b_{ij}x_i x_j$ has all of its coefficients in K_0 .

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It is natural to ask if there is a minimal field K_0 (with respect to inclusion) to which q descends. The answer is usually “no”.

So, we modify the question: instead of asking for a minimal field of definition K_0 for q , we ask for a field of definition K_0 of minimal transcendence degree.

The smallest possible value of $\text{trdeg}_k(K_0)$ is called the *essential dimension* of q and is denoted by $\text{ed}(q)$ or $\text{ed}_k(q)$.

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Towards a more general definition

Note that the above definition of $\text{ed}(q)$ is in no way particular to quadratic forms.

In a similar manner one can consider fields of definition of any polynomial in $K[x_1, \dots, x_n]$, any finite-dimensional K -algebra, any algebraic variety defined over K , etc.

In each case the minimal transcendence degree of a field of definition is an interesting numerical invariant which gives us some insight into the “complexity” of the object in question.

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The essential dimension of a functor

Let k be a base field, Fields_k be the category of field extensions K/k , Sets be the category of sets, and

$$\mathcal{F}: \text{Fields}_k \rightarrow \text{Sets}$$

be a covariant functor.

For example, $\mathcal{F}(K)$ could be the set of K -isomorphism classes of quadratic forms on K^n , or of n -dimensional K -algebras, for a fixed integer n , or of elliptic curves defined over K .

In general we think of \mathcal{F} as specifying the type of algebraic object we want to work with, and elements of $\mathcal{F}(K)$ as algebraic objects of this type defined over K .

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Given a field extension K/k , we will say that $\alpha \in \mathcal{F}(K)$ *descends* to an intermediate field $k \subseteq K_0 \subseteq K$ if α is in the image of the induced map $\mathcal{F}(K_0) \rightarrow \mathcal{F}(K)$:

$$\begin{array}{ccc} \alpha_0 & \longrightarrow & \alpha \\ | & & | \\ K_0 & \longrightarrow & K. \end{array}$$

The *essential dimension* $\text{ed}(\alpha)$ of $\alpha \in \mathcal{F}(K)$ is the minimum of the transcendence degrees $\text{trdeg}_k(K_0)$ taken over all fields

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such that α descends to K_0 .

The essential dimension $\text{ed}(\mathcal{F})$ of the functor \mathcal{F} is the supremum of $\text{ed}(\alpha)$ taken over all $\alpha \in \mathcal{F}(K)$ with K in Fields_k .

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The essential dimension of a group

An important class of examples are the Galois cohomology functors $\mathcal{F}_G = H^1(*, G)$ sending a field K/k to the set $H^1(K, G_K)$ of isomorphism classes of G -torsors over $\text{Spec}(K)$. Here G is an algebraic group defined over k .

$\text{ed}(\mathcal{F}_G)$ is a numerical invariant of G . Informally speaking, it is a measure of complexity of G -torsors over fields. This number is usually denoted by $\text{ed}(G)$.

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Classical examples

- F. Klein, 1885: $\text{ed}(S_5) = 2$. (“Kronecker’s theorem”?)
- J.-P. Serre, A. Grothendieck, 1958: Classified “special groups” over an algebraically closed field. Recall that k -group G is called *special* if

$$H^1(K, G_K) = \{pt\}$$

for every field K/k . G is special if and only if $\text{ed}(G) = 0$.

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Techniques for proving lower bounds on $\text{ed}(G)$

- (~ 1996) Bounds related to cohomological invariants of G .
- (~ 2000) Bounds related to non-toral abelian subgroups of G .
- (~ 2008) Bounds related to Brauer classes induced by a central extension

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Cohomological invariants

A morphism of functors $\mathcal{F} \rightarrow H^d(*, \mu_n)$ is called a *cohomological invariant* of degree d ; it is said to be nontrivial if $\mathcal{F}(K)$ contains a non-zero element of $H^d(K, \mu_n)$ for some K/k .

Observation (J.-P. Serre) Suppose k is algebraically closed. If there exists a non-trivial cohomological invariant $\mathcal{F} \rightarrow H^d(*, \mu_n)$ then $\text{ed}(\mathcal{F}) \geq d$.

Proof:

$$\begin{array}{ccc} \mathcal{F}(K) & \longrightarrow & H^d(K, \mu_n) \\ \uparrow & & \uparrow \\ \mathcal{F}(K_0) & \longrightarrow & H^d(K_0, \mu_n). \end{array}$$

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Examples of cohomological invariants

- $\text{ed}(\mathcal{O}_n) = n$. Cohomological invariant
 $H^1(K, \mathcal{O}_n) \rightarrow H^n(K, \mu_2)$: n th Stiefel-Whitney class of a quadratic form.
- $\text{ed}(\mu_p^r) = r$. Cohomological invariant
 $H^1(K, \mu_p^r) \rightarrow H^r(K, \mu_p)$: cup product.
- $\text{ed}(S_n) \geq [n/2]$. Cohomological invariant
 $H^1(K, S_n) \rightarrow H^{[n/2]}(K, \mu_2)$: $[n/2]$ th Stiefel-Whitney class of the trace form of an étale algebra. Alternatively, (c) can be deduced from (b).

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Examples continued

- $\text{ed}(\text{PGL}_{p^r}) \geq 2r$. Cohomological invariant:

$H^1(K, \text{PGL}_n) \xrightarrow{\partial} H^2(K, \mu_{p^r}) \xrightarrow{p^r} H^{2r}(K, \mu_{p^r})$, where p^r is the divided r th power map.

- $\text{ed}(F_4) \geq 5$. Cohomological invariant:

$H^1(K, F_4) \rightarrow H^5(K, \mu_2)$, Rost-Serre invariant.

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Non-toral abelian subgroups

Theorem: (R.-Youssin, 2000; R.-Gille, 2007) If G is connected, A is a finite abelian subgroup of G and $\text{char}(k)$ does not divide $|A|$, then

$$\text{ed}_k(G) \geq \text{rank}(A) - \text{rank } C_G^0(A).$$

Remarks:

- May pass to the algebraic closure \bar{k} .
- If A lies in a torus of G then the above inequality is vacuous.
- Most interesting case: $C_G^0(A)$ is finite. This happens iff A is not contained in any proper parabolic subgroup of G .
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Examples

- $\text{ed}(\text{SO}_n) \geq n - 1$ for any $n \geq 3$,
- $\text{ed}(\text{PGL}_{p^s}) \geq 2s$
- $\text{ed}(\text{Spin}_n) \geq [n/2]$ for any $n \geq 11$.
- $\text{ed}(\text{G}_2) \geq 3$
- $\text{ed}(\text{F}_4) \geq 5$
- $\text{ed}(\text{E}_6^{\text{sc}}) \geq 4$
- $\text{ed}(\text{E}_7^{\text{sc}}) \geq 7$
- $\text{ed}(\text{E}_8) \geq 9$

Minor restrictions on $\text{char}(k)$ apply.

Each inequality is proved by exhibiting a non-toral abelian subgroup $A \subset G$ whose centralizer is finite. For example, in part (a) we assume $\text{char}(k) \neq 2$ and take $A \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$ to be the subgroup of diagonal matrices in SO_n .

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Each inequality is proved by exhibiting a non-toral abelian subgroup $A \subset G$ whose centralizer is finite. For example, in part (a) we assume $\text{char}(k) \neq 2$ and take $A \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$ to be the subgroup of diagonal matrices in SO_n .

Examples

- $\text{ed}(\text{SO}_n) \geq n - 1$ for any $n \geq 3$,
- $\text{ed}(\text{PGL}_{p^s}) \geq 2s$
- $\text{ed}(\text{Spin}_n) \geq [n/2]$ for any $n \geq 11$.
- $\text{ed}(\text{G}_2) \geq 3$
- $\text{ed}(\text{F}_4) \geq 5$
- $\text{ed}(\text{E}_6^{\text{sc}}) \geq 4$
- $\text{ed}(\text{E}_7^{\text{sc}}) \geq 7$
- $\text{ed}(\text{E}_8) \geq 9$

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Central extensions

Theorem: (Brosnan–R.–Vistoli) Suppose $1 \rightarrow C \rightarrow G \rightarrow \overline{G} \rightarrow 1$ is a central exact sequence of k -groups, with $C \simeq_k \mu_p$.

Assume that k is a field of characteristic $\neq p$ containing a primitive p th root of unity. Then

$$\mathrm{ed}_k(G) \geq \gcd \{ \dim(\rho) \} - \dim G,$$

where ρ ranges over all k -representations of G whose restriction to C is faithful.

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Applications

Brosnan–R.–Vistoli: $\text{ed}(\text{Spin}_n)$ increases exponentially with n .

An exponential lower bound can be obtained by applying the theorem to the central sequence

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Two types of problems

Suppose we are given a functor

$$\mathcal{F}: \text{Fields}_k \rightarrow \text{Sets}$$

and we would like to show that some (or every) $\alpha \in \mathcal{F}(K)$ has a certain property.

It is often useful to approach this problem in two steps. For the first step we choose a prime p and ask whether or not α_L has the desired property for some prime-to- p extension L/K . This is what I call a *Type 1 problem*.

If the answer is “no” for some p then we are done.

If the answer is “yes” for every prime p , then the remaining problem is to determine whether or not α itself has the desired property. I refer to problems of this type as *Type 2 problems*.

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Essential dimension at p

Let $\mathcal{F}: \text{Fields}_k \rightarrow \text{Sets}$ be a functor and $\alpha \in \mathcal{F}(K)$ for some field K/k .

The essential dimension $\text{ed}(\alpha; p)$ of α at a prime integer p is defined as the minimal value of $\text{ed}(\alpha_L)$, as L ranges over all finite field extensions L/K such that p does not divide $[L : K]$.

The essential dimension $\text{ed}(\mathcal{F}; p)$ is then defined as the maximal value of $\text{ed}(\alpha; p)$, as K ranges over all field extensions of k and α ranges over $\mathcal{F}(K)$.

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In the case where $\mathcal{F}(K) = H^1(K, G)$ for some algebraic group G defined over k , we will write $\text{ed}(G; p)$ in place of $\text{ed}(\mathcal{F}; p)$.

Clearly, $\text{ed}(\alpha; p) \leq \text{ed}(\alpha)$, $\text{ed}(\mathcal{F}; p) \leq \text{ed}(\mathcal{F})$, and $\text{ed}(G; p) \leq \text{ed}(G)$ for every prime p .

In the context of essential dimension:

Type 1 problem. Find $\text{ed}(\alpha; p)$ or $\text{ed}(\mathcal{F}; p)$ or $\text{ed}(G; p)$ for some (or every) prime p .

Type 2 problem. Assuming $\text{ed}(\alpha; p)$, $\text{ed}(\mathcal{F}; p)$, or $\text{ed}(G; p)$ is known for every prime p , find the “absolute” essential dimension $\text{ed}(\alpha)$, $\text{ed}(\mathcal{F})$, or $\text{ed}(G)$.

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$\text{ed}(G)$ versus $\text{ed}(G; p)$

A closer look at the three techniques we discussed of proving lower bounds of the form $\text{ed}(G) \geq d$ reveals that in every case the argument can be modified to show that in fact $\text{ed}(G; p) \geq d$ for some (naturally chosen) prime p . In other words, these techniques are well suited to Type 1 problems only.

This is a special case of the following more general but admittedly vague phenomenon.

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Examples of Type 2 problems

- The cyclicity problem and the cross product problem for central simple algebras
- The torsion index problem (for simply connected or adjoint groups)
- The problem of computing the canonical dimension of a simple group
- Serre's conjecture on the splitting of a torsor
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Another Type 2 problem

In the context of essential dimension, while we know that for some finite groups G ,

$$\mathrm{ed}(G) > \mathrm{ed}(G; p)$$

for every prime p , the only natural examples where we can prove this are in low dimensions, with $\mathrm{ed}(G) \leq 3$ or (with greater effort) 4.

Open problem 1: What is $\text{ed}(S_n)$?

This is a classical question, loosely related to the algebraic form of Hilbert's 13th problem.

In classical language, $\text{ed}(S_n)$ is a measure of how much the general polynomials,

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_n,$$

where a_1, \dots, a_n are independent variables, can be reduced by a Tschirnhaus transformation. That is, $\text{ed}(S_n)$ is the minimal possible number of algebraically independent elements among the coefficients b_1, \dots, b_n of a polynomial

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More on $\text{ed}(S_n)$

The problem of computing $\text{ed}(S_n)$ turns out to be of Type 2. For simplicity, let us assume that $\text{char}(k) = 0$. Then $\text{ed}(S_n; p) = [n/p]$, is known for every prime p . For the “absolute” essential dimension, we only know that

$$[n/2] \leq \text{ed}(S_n) \leq n - 3$$

for every $n \geq 5$.

In particular, $\text{ed}(S_5) = 2$ and $\text{ed}(S_6) = 3$. It is also easy to see that $\text{ed}(S_2) = \text{ed}(S_3) = 1$ and $\text{ed}(S_4) = 2$.

Theorem (A. Duncan, 2010): $\text{ed}(S_7) = 4$.

The proof relies on recent work in Mori theory, due to Yu. Prokhorov.

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This appears to be out of reach for now, except for a few small values of n .

On the other hand, there has been recent progress on computing $\text{ed}(\text{PGL}_n; p)$?

May assume that $n = p^r$. It is easy to see that $\text{ed}(\text{PGL}_{p^r}; p) = 2$.

Theorem: For $r \geq 2$,

$$(r - 1)p^r + 1 \leq \text{ed}(\text{PGL}_{p^r}; p) \leq p^{2r-2} + 1.$$

The lower bound is due to Merkurjev and the upper bound is due to his student A. Ruozi. In particular,

$$\text{ed}(\text{PGL}_{p^2}; p) = p^2 + 1 \text{ and } \text{ed}(\text{PGL}_8; 2) = 17.$$

Of course, in general there is still a wide gap between $(r - 1)p^r + 1$ and $p^{2r-2} + 1$.

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Open problem 3: essential dimension of exceptional groups

Assume for simplicity that k is an algebraically closed field of characteristic 0.

$G = G_2$. Know $\text{ed}(G_2) = \text{ed}(G_2; 2) = 3$.

$G = F_4$. Here the Type 1 problem has been completely solved:

$$\text{ed}(F_4; 2) = 5, \text{ed}(F_4; 3) = 3, \text{ and } \text{ed}(F_4; p) = 0$$

for all $p > 3$.

The (Type 2) problem of computing $\text{ed}(F_4)$ remains open. The situation is similar for the simply connected group E_6^{sc} .

For the other exceptional groups G , very little is known. In particular, all we know about $\text{ed}(E_8; 2)$ is $9 \leq \text{ed}(E_8; 2) \leq 120$.

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$G = G_2$. Know $\text{ed}(G_2) = \text{ed}(G_2; 2) = 3$.

$G = F_4$. Here the Type 1 problem has been completely solved:

$$\text{ed}(F_4; 2) = 5, \text{ed}(F_4; 3) = 3, \text{ and } \text{ed}(F_4; p) = 0$$

for all $p > 3$.

The (Type 2) problem of computing $\text{ed}(F_4)$ remains open. The situation is similar for the simply connected group E_6^{sc} .

For the other exceptional groups G , very little is known. In particular, all we know about $\text{ed}(E_8; 2)$ is $9 \leq \text{ed}(E_8; 2) \leq 120$.

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