ON A PROPERTY OF REAL PLANE CURVES OF EVEN DEGREE

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Abstract. F. Cukierman asked whether or not for every smooth real plane curve $X \subset \mathbb{P}^2$ of even degree $d \geq 2$ there exists a real line $L \subset \mathbb{P}^2$ such that $X \cap L$ has no real points. We show that the answer is “yes” if $d = 2$ or 4 and “no” if $n \geq 6$.

1. Introduction

F. Cukierman asked whether or not for every smooth real plane curve $X \subset \mathbb{P}^2$ there exists a real line $L \subset \mathbb{P}^2$ such that the intersections $X \cap L$ has no real points. In other words, can we “see” all real points of $X$ in some affine space of the form $\mathbb{A}^2 = \mathbb{P}^2 \setminus L$?

Note that if $d$ is odd, then the answer is “no” for trivial reasons: $X \cap L$ is cut out by an odd degree polynomial on $L$ and hence, always has a real point. On the other hand, in the case where $d = 2$, the answer is readily seen to be “yes”. Indeed, given a real conic $X$ in $\mathbb{P}^2$, choose a complex point $z \in X(\mathbb{C}) \setminus X(\mathbb{R})$ which is not real, and let $L$ be the (real) line passing through $z$ and its complex conjugate $\overline{z}$. If $X$ is smooth, then $L$ is not contained in $X$. Hence, the intersection $(X \cap L)(\mathbb{C}) = \{z, \overline{z}\}$ contains no real points.

The main result of this note, Theorem 1 below, asserts that the answer to Cukierman’s question is “yes” if $d = 2$ or 4 and “no” if $n \geq 6$.

Theorem 1. (a) Suppose $d = 2$ or 4. Then for every smooth plane curve $X \subset \mathbb{P}^2$ of degree $d$ defined over the reals, there exists a real line $L \subset \mathbb{P}^2$ such that $(X \cap L)(\mathbb{R}) = \emptyset$.

(b) Suppose $d \geq 6$ is an even integer. Then there exists a smooth plane curve $X \subset \mathbb{P}^2$ of degree $d$ defined over the reals, such that $(X \cap L)(\mathbb{R}) \neq \emptyset$ for every real line $L \subset \mathbb{P}^2$.

The proof of Theorem 1 presented in in Sections 3 and 4 uses deformation arguments. These arguments, in turn, rely on the preliminary material in Section 2.

2. Continuity of minimizer and maximizer functions

Lemma 2. Let $V, W$ and $F$ be topological manifolds. Assume that $F$ be compact, $\pi : V \to W$ is an $F$-fibration, and $f : V \to \mathbb{R}$ is a continuous function. Then the minimizer $\mu(w) := \min\{f(v) | \pi(v) = w\}$ and the maximizer $\mathcal{M}(w) := \max\{f(v) | \pi(v) = w\}$ are continuous functions $W \to \mathbb{R}$.

Proof. Since $F$ is compact, $f$ assumes its minimal and maximal values on every fiber $\pi^{-1}(w)$. Hence, the functions $\mu$ and $\mathcal{M}$ are well defined. Note also that if we replace $f$ by $-f$, we will change $\mu(w)$ to $-\mathcal{M}(w)$. Thus it suffices to show that $\mu$ is continuous. Finally, to show that $\mu$ is continuous at $w \in W$, we may replace $W$ by a small neighborhood of

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Corollary 3. Let $d \geq 2$ be an even integer, $\text{Pol}_d$ be the affine space of homogeneous polynomials of even degree $d$ in 3 variables, and $Y = \mathbb{P}^2$ be the dual projective plane $\mathbb{P}^2$, parametrizing the lines in $\mathbb{P}^3$.

Corollary 3. Let $d \geq 2$ be an even integer, $\text{Pol}_d$ be the affine space of homogeneous polynomials of even degree $d$ in 3 variables, and $Y = \mathbb{P}^2$ be the dual projective plane $\mathbb{P}^2$, parametrizing the lines in $\mathbb{P}^3$. Then the functions

$$m_p(L) \text{ and } M_p(L) : \text{Pol}_d(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R}) \to \mathbb{R}$$

given by $m_p(L) = \min \{ p(x) \mid x \in L(\mathbb{R}) \}$ and $M_p(L) = \max \{ p(x) \mid x \in L(\mathbb{R}) \}$ are well-defined and continuous.

Note that a polynomial $p(x, y, z)$ of even degree $d$ gives rise to a continuous function $\mathbb{P}^2(\mathbb{R}) \to \mathbb{R}$ given by

$$(x : y : z) \mapsto \frac{p(x, y, z)}{(x^2 + y^2 + z^2)^{d/2}}.$$  

By a slight abuse of notation, we will continue to denote this function by $p$.

Proof of Corollary 3. We will apply Lemma 2 in the following setting. Let $W := \text{Pol}_d \times \mathbb{P}^2$ and

$$V := \{(p, L, a) \mid a \in L\} \subset \text{Pol}_d \times \mathbb{P}^2 \times \mathbb{P}^2.$$  

In other words, $V = \text{Pol}_d \times \text{Flag}(1, 2)$, where $\text{Flag}(1, 2)$ is the flag variety of $(1, 2)$-flags in a 3-dimensional vector space. Clearly $V$ and $W$ are smooth algebraic varieties defined over $\mathbb{R}$. Their sets of real points, $V(\mathbb{R})$ and $W(\mathbb{R})$, are topological manifolds and the projection $\pi : V(\mathbb{R}) \to W(\mathbb{R})$ to the first two components is a topological fibration with compact fiber $F = \mathbb{P}^1(\mathbb{R})$.

Applying Lemma 2 to the continuous function $f : V(\mathbb{R}) \to \mathbb{R}$ given by $f(p, L, a) := p(a)$, where $p(a)$ is evaluated as in (2.1), we deduce the continuity of the real-valued functions $m_p(L) = \mu(p, L)$ and $M_p(L) = M(p, L)$ on $\text{Pol}_d(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R})$.  

Proposition 4. Let $p \in \mathbb{R}[x, y, z]$ be a homogeneous polynomial of even degree and $X \subset \mathbb{P}^2$ be the zero locus of $p$. Set $m(p) := \max_{L \in \mathbb{P}^2} m_p(L)$ and $M(p) := \min_{L \in \mathbb{P}^2} M_p(L)$, where $L$ ranges over the real lines in $\mathbb{P}^2$. Then

(a) $m(p)$ and $M(p)$ are well-defined continuous functions $\text{Pol}_d(\mathbb{R}) \to \mathbb{R}$.

(b) $m(p) \leq M(p)$.

(c) $(X \cap L)(\mathbb{R}) \neq \emptyset$ for every real line $L \subset \mathbb{P}^2$ if and only if $m(p) \leq 0 \leq M(p)$.

(d) $p$ assumes both positive and negative values on each real line $L \subset \mathbb{P}^2$ if and only if $m(p) < 0 < M(p)$.

(e) If $m(p) = M(p) = 0$, then $X$ is not a smooth curve.

Proof. By Corollary 3, $M_p(L) \text{ and } m_p(L)$ are continuous functions $\text{Pol}_d(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R}) \to \mathbb{R}$. Since $\mathbb{P}^2(\mathbb{R})$ is compact, Lemma 2 tells us that the functions $m(p)$ and $M(p) : \text{Pol}_d(\mathbb{R}) \to \mathbb{R}$ are well-defined and continuous. This proves (a).

(c) and (d) are immediate consequences of the definition of $m(p)$ and $M(p)$.  

w and thus assume that $V = W \times F$ and $\pi : V \to W$ is projection to the first factor. In this special case, the continuity of $\mu$ is well known; see, e.g., [Wo] (cf. also [Da]).  

□
To prove (b) and (e), choose lines \( L_1, L_2 \subset \mathbb{P}^2 \) such that \( m_p(L) \) attains its maximal value \( m(p) \) at \( L = L_1 \) and \( M_p(L) \) attains its minimal value \( M(p) \) at \( L = L_2 \). If \( L_1 \) and \( L_2 \) intersect at a point \( a \in \mathbb{P}^2(R) \), then
\[
m(p) = m_p(L_1) \leq p(a) \leq M_p(L_2) = M(p).
\]
This proves (b).

In part (e), where we further assume that \( m(p) = M(p) = 0 \), the inequalities (2.2) tell us that \( p(a) = 0 \) is the maximal value of \( p \) on \( L_1(R) \) and the minimal value of \( p \) on \( L_2(R) \). Hence, \( a \) lies on \( X \), and both \( L_1 \) and \( L_2 \) are tangent to \( X \) at \( a \). We want to show that \( X \) cannot be a smooth curve. Assume the contrary. Then \( X \) has a unique tangent line at \( a \). Thus \( L_1 = L_2 \), and \( 0 = m_p(L_1) = M_p(L_2) = M_p(L_1) \). We conclude that \( p \) is identically zero on \( L_1(R) = L_2(R) \). Consequently, \( L_1 = L_2 \subset X \), contradicting our assumption that \( X \) is a smooth curve. \( \square \)

3. Proof of Theorem 1(a)

The case where \( d = 2 \) was handled in the Introduction; we will thus assume that \( d = 4 \).

**Lemma 5.** Let \( p \in \mathbb{R}[x, y, z] \) be a homogeneous polynomial of degree 4 cutting out a smooth quartic curve \( X \) in \( \mathbb{P}^2 \). Then either \( m(p) \geq 0 \) or \( M(p) \leq 0 \).

**Proof.** By a theorem of H. G. Zeuthen [Zeu], \( X \) has a real bitangent line \( L \subset \mathbb{P}^2 \). (For a modern proof of Zeuthen’s theorem, we refer the reader to [Ru, Corollary 4.11]; cf. also [PSV].) The restriction of \( p(x, y, z) \) to \( L \) is a real quartic polynomial with two double roots, i.e., a polynomial of the form \( \pm q(s, t)^2 \), where \( s \) and \( t \) are linear coordinates on \( L \), and \( q \in \mathbb{R}[s, t] \) is a binary form of degree 2. In particular, \( p \) does not change sign on \( L \), i.e., either (i) \( p(a) \geq 0 \) for every \( a \in L(R) \) or (ii) \( p(a) \leq 0 \) for every \( a \in L(R) \). In case (i), \( m(p) \geq m_p(L) \geq 0 \) and in case (ii), \( M(p) \leq M_p(L) \leq 0 \). \( \square \)

We are now ready to finish the proof of Theorem 1(a) for \( d = 4 \). Assume the contrary: there exists a smooth real quartic curve \( X \subset \mathbb{P}^2 \) such that \((X \cap L)(R) \neq \emptyset \) for every real line \( L \subset \mathbb{P}^2 \). Let \( p \in \mathbb{R}[x, y, z] \) be a defining polynomial for \( X \). By Proposition 4(c), \( m(p) \leq 0 \leq M(p) \). In view of Lemma 5, after possibly replacing \( p \) by \(-p\), we may assume that \( m(p) = 0 \). Proposition 4(e) now tell us that \( m(p) = 0 < M(p) \). Let \( p_t(x, y, z) = p(x, y, z) - t(x^2 + y^2 + z^2)^2 \), where \( t \) is a real parameter, and let \( X_t \subset \mathbb{P}^2 \) be the quartic curve cut out by \( p_t \). Note that \( X_t \) can be singular for only finitely many values of \( t \in \mathbb{R} \). Thus we can choose \( t \in (0, M(p)) \) so that \( X_t \) is smooth. Since \( x^2 + y^2 + z^2 \) is identically 1 on \( \mathbb{P}^2(R) \) (cf. (2.1)), we have
\[
m(p_t) = m(p) - t < 0 < M(p) - t = M(p_t).
\]
This contradicts Lemma 5, which asserts that \( m(p_t) \geq 0 \) or \( M(p_t) \leq 0 \). \( \square \)

4. Proof of Theorem 1(b)

Given an even integer \( d \geq 6 \), set \( p(x, y, z) := (x^3 + y^3 + z^3)^2(x^2 + y^2 + z^2)^{(d-6)/2} \) and
\[
p_t(x, y, z) = p(x, y, z) - t(x^d + y^d + z^d),
\]
where \( t \) is a real parameter. In view of Proposition 4(c), it suffices to show that if \( t > 0 \) is sufficiently small, then (i) \( X_t \) is smooth and (ii) \( m(p_t) < 0 < M(p_t) \).
Since the Fermat curve, \( x^d + y^d + z^d = 0 \), is smooth, \( X_t \) is singular for only finitely many values of \( t \), and (i) follows.

To prove (ii), note that \( p \) is non-negative but is not identically 0 on any real line \( L \subset \mathbb{P}^2 \). Thus \( M_p(L) > 0 \) and consequently, \( M(p) > 0 \). By Proposition 4(a), \( M(p_t) > 0 \) for small \( t \). On the other hand, for every real line \( L \subset \mathbb{P}^2 \), the cubic polynomial \( x^3 + y^3 + z^3 \) vanishes at some real point \( a \) of \( L \). Hence for every \( t > 0 \), we have \( p_t(a) < 0 \) and thus \( m_{p_t}(L) < 0 \). We conclude that \( m(p_t) < 0 \), as desired. \( \square \)

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**References**


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