ON A PROPERTY OF REAL PLANE CURVES OF EVEN DEGREE
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Abstract. F. Cukierman asked whether or not for every smooth real plane curve
\( X \subset \mathbb{P}^2 \) of even degree \( d \geq 2 \) there exists a real line \( L \subset \mathbb{P}^2 \) such
\( X \cap L \) has no real points. We show that the answer is “yes” if \( d = 2 \) or 4 and “no” if \( n \geq 6 \).

1. Introduction

F. Cukierman asked whether or not for every smooth real plane curve \( X \subset \mathbb{P}^2 \) there
exists a real line \( L \subset \mathbb{P}^2 \) such that the intersections \( X \cap L \) has no real points. In other
words, can we “see” all real points of \( X \) in some affine space of the form \( \mathbb{A}^2 = \mathbb{P}^2 \setminus L \)?

Note that if \( d \) is odd, then the answer is “no” for trivial reasons: \( X \cap L \) is cut out by
an odd degree polynomial on \( L \) and hence, always has a real point. On the other hand,
in the case where \( d = 2 \), the answer is readily seen to be “yes”. Indeed, given a real conic
\( X \) in \( \mathbb{P}^2 \), choose a complex point \( z \in X(\mathbb{C}) \setminus X(\mathbb{R}) \) which is not real, and let \( L \) be the
(real) line passing through \( z \) and its complex conjugate \( \overline{z} \). If \( X \) is smooth, then \( L \) is not
contained in \( X \). Hence, the intersection \( (X \cap L)(\mathbb{C}) = \{z, \overline{z}\} \) contains no real points.

The main result of this note, Theorem 1 below, asserts that the answer to Cukierman’s
question is “yes” if \( d = 2 \) or 4 and “no” if \( n \geq 6 \).

Theorem 1. (a) Suppose \( d = 2 \) or 4. Then for every smooth plane curve \( X \subset \mathbb{P}^2 \) of
degree \( d \) defined over the reals, there exists a real line \( L \subset \mathbb{P}^2 \) such that \( (X \cap L)(\mathbb{R}) = \emptyset \).

(b) Suppose \( d \geq 6 \) is an even integer. Then there exists a smooth plane curve \( X \subset \mathbb{P}^2 \)
of degree \( d \) defined over the reals, such that \( (X \cap L)(\mathbb{R}) \neq \emptyset \) for every real line \( L \subset \mathbb{P}^2 \).

The proof of Theorem 1 presented in in Sections 3 and 4 uses deformation arguments.
These arguments, in turn, rely on the preliminary material in Section 2.

2. Continuity of minimizer and maximizer functions

Lemma 2. Let \( V, W \) and \( F \) be topological manifolds. Assume that \( F \) be compact, \( \pi: V \to W \)
is an \( F \)-fibration, and \( f: V \to \mathbb{R} \) is a continuous function. Then the minimizer
\( \mu(w) := \min\{f(v) | \pi(v) = w\} \) and the maximizer \( \nu(w) := \max\{f(v) | \pi(v) = w\} \) are
continuous functions \( W \to \mathbb{R} \).

Proof. Since \( F \) is compact, \( f \) assumes its minimal and maximal values on every fiber
\( \pi^{-1}(w) \). Hence, the functions \( \mu \) and \( \nu \) are well defined. Note also that if we replace \( f \) by
\( -f \), we will change \( \mu(w) \) to \( -\nu(w) \). Thus it suffices to show that \( \mu \) is continuous. Finally,
to show that \( \mu \) is continuous at \( w \in W \), we may replace \( W \) by a small neighborhood of
Corollary 3. Let \(d \geq 2\) be an even integer, \(\text{Pol}_d\) be the affine space of homogeneous polynomials of even degree \(d\) in 3 variables, and \(\mathbb{P}^2\) be the dual projective plane parametrizing the lines in \(\mathbb{P}^2\). Then the functions

\[
m_p(L) \text{ and } M_p(L) : \text{Pol}_d(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R}) \to \mathbb{R}
\]

given by \(m_p(L) = \min\{p(x) \mid x \in L(\mathbb{R})\}\) and \(M_p(L) = \max\{p(x) \mid x \in L(\mathbb{R})\}\) are well-defined and continuous.

Proof of Corollary 3. By a slight abuse of notation, we will continue to denote this function by \(p\) and thus assume that \(V = W \times F\) and \(\pi : V \to W\) is projection to the first factor. In this special case, the continuity of \(w\) is well known; see, e.g., [Wo] (cf. also [Da]). □

Proposition 4. Let \(p \in \mathbb{R}[x, y, z]\) be a homogeneous polynomial of even degree and \(X \subset \mathbb{P}^2\) be the zero locus of \(p\). Set \(m(p) := \max_{L \in \mathbb{P}^2} m_p(L)\) and \(M(p) := \min_{L \in \mathbb{P}^2} M_p(L)\), where \(L\) ranges over the real lines in \(\mathbb{P}^2\). Then

(a) \(m(p)\) and \(M(p)\) are well-defined continuous functions \(\text{Pol}_d(\mathbb{R}) \to \mathbb{R}\).

(b) \(m(p) \leq M(p)\).

(c) \((X \cap L)(\mathbb{R}) \neq \emptyset\) for every real line \(L \subset \mathbb{P}^2\) if and only if \(m(p) \leq 0 \leq M(p)\).

(d) \(p\) assumes both positive and negative values on each real line \(L \subset \mathbb{P}^2\) if and only if \(m(p) < 0 < M(p)\).

(e) If \(m(p) = M(p) = 0\), then \(X\) is not a smooth curve.

Proof. By Corollary 3, \(M_p(L)\) and \(m_p(L)\) are continuous functions \(\text{Pol}_d(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R}) \to \mathbb{R}\). Since \(\mathbb{P}^2(\mathbb{R})\) is compact, Lemma 2 tells us that the functions \(m(p)\) and \(M(p) : \text{Pol}_d(\mathbb{R}) \to \mathbb{R}\) are well-defined and continuous. This proves (a).

(c) and (d) are immediate consequences of the definition of \(m(p)\) and \(M(p)\).
To prove (b) and (e), choose lines $L_1, L_2 \subset \mathbb{P}^2$ such that $m_p(L)$ attains its maximal value $m(p)$ at $L = L_1$ and $M_p(L)$ attains its minimal value $M(p)$ at $L = L_2$. If $L_1$ and $L_2$ intersect at a point $a \in \mathbb{P}^2(\mathbb{R})$, then

$$m(p) = m_p(L_1) \leq p(a) \leq M_p(L_2) = M(p).$$

This proves (b).

In part (e), where we further assume that $m(p) = M(p) = 0$, the inequalities (2.2) tell us that $p(a) = 0$ is the maximal value of $p$ on $L_1(\mathbb{R})$ and the minimal value of $p$ on $L_2(\mathbb{R})$. Hence, $p$ lies on $X$, and both $L_1$ and $L_2$ are tangent to $X$ at $a$. We want to show that $X$ cannot be a smooth curve. Assume the contrary. Then $X$ has a unique tangent line at $a$. Thus $L_1 = L_2$, and $0 = m_p(L_1) = M_p(L_2) = M_p(L_1)$. We conclude that $p$ is identically zero on $L_1(\mathbb{R}) = L_2(\mathbb{R})$. Consequently, $L_1 = L_2 \subset X$, contradicting our assumption that $X$ is a smooth curve. \hfill \Box

3. Proof of Theorem 1(a)

The case where $d = 2$ was handled in the Introduction; we will thus assume that $d = 4$.

**Lemma 5.** Let $p \in \mathbb{R}[x, y, z]$ be a homogeneous polynomial of degree 4 cutting out a smooth quartic curve $X$ in $\mathbb{P}^2$. Then either $m(p) \geq 0$ or $M(p) \leq 0$.

**Proof.** By a theorem of H. G. Zeuthen [Zeu], $X$ has a real bitangent line $L \subset \mathbb{P}^2$. (For a modern proof of Zeuthen’s theorem, we refer the reader to [Ru, Corollary 4.11]; cf. also [PSV].) The restriction of $p(x, y, z)$ to $L$ is a real quartic polynomial with two double roots, i.e., a polynomial of the form $\pm q(s, t)^2$, where $s$ and $t$ are linear coordinates on $L$, and $q \in \mathbb{R}[s, t]$ is a binary form of degree 2. In particular, $p$ does not change sign on $L$, i.e., either (i) $p(a) \geq 0$ for every $a \in L(\mathbb{R})$ or (ii) $p(a) \leq 0$ for every $a \in L(\mathbb{R})$. In case (i), $m(p) \geq m_p(L) \geq 0$ and in case (ii), $M(p) \leq M_p(L) \leq 0$. \hfill \Box

We are now ready to finish the proof of Theorem 1(a) for $d = 4$. Assume the contrary: there exists a smooth real quartic curve $X \subset \mathbb{P}^2$ such that $(X \cap L)(\mathbb{R}) \neq \emptyset$ for every real line $L \subset \mathbb{P}^2$. Let $p \in \mathbb{R}[x, y, z]$ be a defining polynomial for $X$. By Proposition 4(c), $m(p) \leq 0 \leq M(p)$. In view of Lemma 5, after possibly replacing $p$ by $-p$, we may assume that $m(p) = 0$. Proposition 4(e) now tell us that $m(p) = 0 < M(p)$. Let $p_t(x, y, z) = p(x, y, z) - t(x^2 + y^2 + z^2)^2$, where $t$ is a real parameter, and let $X_t \subset \mathbb{P}^2$ be the quartic curve cut out by $p_t$. Note that $X_t$ can be singular for only finitely many values of $t \in \mathbb{R}$. Thus we can choose $t \in (0, M(p))$ so that $X_t$ is smooth. Since $x^2 + y^2 + z^2$ is identically 1 on $\mathbb{P}^2(\mathbb{R})$ (cf. (2.1)), we have

$$m(p_t) = m(p) - t < 0 < M(p) - t = M(p_t).$$

This contradicts Lemma 5, which asserts that $m(p_t) \geq 0$ or $M(p_t) \leq 0$. \hfill \Box

4. Proof of Theorem 1(b)

Given an even integer $d \geq 6$, set $p(x, y, z) := (x^3 + y^3 + z^3)^2(x^2 + y^2 + z^2)^{(d-6)/2}$ and

$$p_t(x, y, z) = p(x, y, z) - t(x^d + y^d + z^d),$$

where $t$ is a real parameter. In view of Proposition 4(c), it suffices to show that if $t > 0$ is sufficiently small, then (i) $X_t$ is smooth and (ii) $m(p_t) < 0 < M(p_t)$. 
Since the Fermat curve, \( x^d + y^d + z^d = 0 \), is smooth, \( X_t \) is singular for only finitely many values of \( t \), and (i) follows.

To prove (ii), note that \( p \) is non-negative but is not identically 0 on any real line \( L \subset \mathbb{P}^2 \). Thus \( M_p(L) > 0 \) and consequently, \( M(p) > 0 \). By Proposition 4(a), \( M(p_t) > 0 \) for small \( t \). On the other hand, for every real line \( L \subset \mathbb{P}^2 \), the cubic polynomial \( x^3 + y^3 + z^3 \) vanishes at some real point \( a \) of \( L \). Hence for every \( t > 0 \), we have \( p_t(a) < 0 \) and thus \( m_{p_t}(L) < 0 \). We conclude that \( m(p_t) < 0 \), as desired.

\[ \square \]

Acknowledgments. The author is grateful to Fernando Cukierman, Corrado de Concini, Kee Yuen Lam, Grigory Mikhalkin and Benedict Williams for stimulating discussions.

References


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