ON A PROPERTY OF REAL PLANE CURVES OF EVEN DEGREE

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Abstract. F. Cukierman asked whether or not for every smooth real plane curve $X \subset \mathbb{P}^2$ of even degree $d \geqslant 2$ there exists a real line $L \subset \mathbb{P}^2$ such $X \cap L$ has no real points. We show that the answer is "yes" if $d = 2$ or 4 and "no" if $n \geqslant 6$.

1. Introduction

F. Cukierman asked whether or not for every smooth real plane curve $X \subset \mathbb{P}^2$ there exists a real line $L \subset \mathbb{P}^2$ such that the intersections $X \cap L$ has no real points. In other words, can we "see" all real points of X in some affine space of the form $\mathbb{A}^2 = \mathbb{P}^2 \setminus L$?

Note that if d is odd, then the answer is "no" for trivial reasons: $X \cap L$ is cut out by an odd degree polynomial on L and hence, always has a real point. On the other hand, in the case where $d = 2$, the answer is readily seen to be "yes". Indeed, given a real conic X in $\mathbb{P}^2$, choose a complex point $z \in X(\mathbb{C}) \setminus X(\mathbb{R})$ which is not real, and let L be the (real) line passing through z and its complex conjugate $\overline{z}$. If X is smooth, then L is not contained in X. Hence, the intersection $(X \cap L)(\mathbb{C}) = \{z, \overline{z}\}$ contains no real points.

The main result of this note, Theorem 1 below, asserts that the answer to Cukierman's question is "yes" if $d = 2$ or 4 and "no" if $n \geqslant 6$.

Theorem 1. *(a) Suppose $d = 2$ or 4. Then for every smooth plane curve $X \subset \mathbb{P}^2$ of degree d defined over the reals, there exists a real line $L \subset \mathbb{P}^2$ such that $(X \cap L)(\mathbb{R}) = \emptyset$.*

(b) Suppose $d \geqslant 6$ is an even integer. Then there exists a smooth plane curve $X \subset \mathbb{P}^2$ of degree d defined over the reals, such that $(X \cap L)(\mathbb{R}) \neq \emptyset$ for every real line $L \subset \mathbb{P}^2$.

The proof of Theorem 1 presented in in Sections 3 and 4 uses deformation arguments. These arguments, in turn, rely on the preliminary material in Section 2.

2. Continuity of minimizer and maximizer functions

Lemma 2. *Let V, W and F be topological manifolds. Assume that F be compact, $\pi \colon V \to W$ is an F-fibration, and $f \colon V \to \mathbb{R}$ is a continuous function. Then the minimizer $\mu(w) := \min\{f(v) \,|\, \pi(v) = w\}$ and the maximizer $\nu(w) := \max\{f(v) \,|\, \pi(v) = w\}$ are continuous functions $W \to \mathbb{R}$.*

Proof. Since F is compact, f assumes its minimal and maximal values on every fiber $\pi^{-1}(w)$. Hence, the functions μ and ν are well defined. Note also that if we replace f by $-f$, we will change $\mu(w)$ to $-\nu(w)$. Thus it suffices to show that μ is continuous. Finally, to show that μ is continuous at $w \in W$, we may replace W by a small neighborhood of

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w and thus assume that $V = W \times F$ and $\pi \colon V \to W$ is projection to the first factor. In this special case, the continuity of μ is well known; see, e.g., [Wo] (cf. also [Da]). $\Box$

Corollary 3. *Let $d \geqslant 2$ be an even integer, Pol_d be the affine space of homogeneous polynomials of even degree d in 3 variables, and $\check{\mathbb{P}}^2$ be the dual projective plane parametrizing the lines in $\mathbb{P}^2$. Then the functions*

$$m_p(L) \text{ and } M_p(L)\colon \mathrm{Pol}_d(\mathbb{R}) \times \check{\mathbb{P}}^2(\mathbb{R}) \to \mathbb{R}$$

given by $m_p(L) = \min\{p(x) \mid x \in L(\mathbb{R})\}$ and $M_p(L) = \max\{p(x) \mid x \in L(\mathbb{R})\}$ are well defined and continuous.

Note that a polynomial $p(x, y, z)$ of even degree d gives rise to a continuous function $\mathbb{P}^2(\mathbb{R}) \to \mathbb{R}$ given by

$$(x : y : z) \to \frac{p(x, y, z)}{(x^2 + y^2 + z^2)^{d/2}}\,. \tag{2.1}$$

By a slight abuse of notation, we will continue to denote this function by p.

Proof of Corollary 3. We will apply Lemma 2 in the following setting. Let $W := \mathrm{Pol}_d \times \check{\mathbb{P}}^2$ and

$$V := \{(p, L, a) \mid \ a \in L\} \subset \mathrm{Pol}_d \times \check{\mathbb{P}}^2 \times \mathbb{P}^2\,.$$

In other words, $V = \mathrm{Pol}_d \times \mathrm{Flag}(1, 2)$, where $\mathrm{Flag}(1, 2)$ is the flag variety of $(1, 2)$-flags in a 3-dimensional vector space. Clearly V and W are smooth algebraic varieties defined over $\mathbb{R}$. Their sets of real points, $V(\mathbb{R})$ and $W(\mathbb{R})$, are topological manifolds and the projection $\pi\colon V(\mathbb{R}) \to W(\mathbb{R})$ to the first two components is a topological fibration with compact fiber $F = \mathbb{P}^1(\mathbb{R})$.

Applying Lemma 2 to the continuous function $f\colon V(\mathbb{R}) \to \mathbb{R}$ given by $f(p, L, a) := p(a)$, where $p(a)$ is evaluated as in (2.1), we deduce the continuity of the real-valued functions $m_p(L) = \mu(p, L)$ and $M_p(L) = \nu(p, L)$ on $\mathrm{Pol}_d(\mathbb{R}) \times \check{\mathbb{P}}^2(\mathbb{R})$. $\Box$

Proposition 4. *Let $p \in \mathbb{R}[x, y, z]$ be a homogeneous polynomial of even degree and $X \subset \mathbb{P}^2$ be the zero locus of p. Set $m(p) := \max_{L \in \check{\mathbb{P}}^2} m_p(L)$ and $M(p) := \min_{L \in \check{\mathbb{P}}^2} M_p(L)$, where L ranges over the real lines in $\mathbb{P}^2$. Then*

(a) $m(p)$ and $M(p)$ are well-defined continuous functions $\mathrm{Pol}_d(\mathbb{R}) \to \mathbb{R}$.

(b) $m(p) \leqslant M(p)$.

(c) $(X \cap L)(\mathbb{R}) \neq \emptyset$ for every real line $L \subset \mathbb{P}^2$ if and only if $m(p) \leqslant 0 \leqslant M(p)$.

(d) p assumes both positive and negative values on each real line $L \subset \mathbb{P}^2$ if and only if $m(p) < 0 < M(p)$.

(e) If $m(p) = M(p) = 0$, then X is not a smooth curve.

Proof. By Corollary 3, $M_p(L)$ and $m_p(L)$ are continuous functions $\mathrm{Pol}_d(\mathbb{R}) \times \check{\mathbb{P}}^2(\mathbb{R}) \to \mathbb{R}$. Since $\check{\mathbb{P}}^2(\mathbb{R})$ is compact, Lemma 2 tells us that the functions $m(p)$ and $M(p)\colon \mathrm{Pol}_d(\mathbb{R}) \to \mathbb{R}$ are well-defined and continuous. This proves (a).

(c) and (d) are immediate consequences of the definition of $m(p)$ and $M(p)$.

To prove (b) and (e), choose lines $L_1, L_2 \subset \mathbb{P}^2$ such that $m_p(L)$ attains its maximal value $m(p)$ at $L = L_1$ and $M_p(L)$ attains its minimal value $M(p)$ at $L = L_2$. If L_1 and L_2 intersect at a point $a \in \mathbb{P}^2(\mathbb{R})$, then

$$m(p) = m_p(L_1) \leqslant p(a) \leqslant M_p(L_2) = M(p)\,. \tag{2.2}$$

This proves (b).

In part (e), where we further assume that $m(p) = M(p) = 0$, the inequalities (2.2) tell us that $p(a) = 0$ is the maximal value of p on $L_1(\mathbb{R})$ and the minimal value of p on $L_2(\mathbb{R})$. Hence, a lies on X, and both L_1 and L_2 are tangent to X at a. We want to show that X cannot be a smooth curve. Assume the contrary. Then X has a unique tangent line at a. Thus $L_1 = L_2$, and $0 = m_p(L_1) = M_p(L_2) = M_p(L_1)$. We conclude that p is identically zero on $L_1(\mathbb{R}) = L_2(\mathbb{R})$. Consequently, $L_1 = L_2 \subset X$, contradicting our assumption that X is a smooth curve. □

3. Proof of Theorem 1(a)

The case where $d = 2$ was handled in the Introduction; we will thus assume that $d = 4$.

Lemma 5. *Let $p \in \mathbb{R}[x, y, z]$ be a homogeneous polynomial of degree 4 cutting out a smooth quartic curve X in $\mathbb{P}^2$. Then either $m(p) \geqslant 0$ or $M(p) \leqslant 0$.*

Proof. By a theorem of H. G. Zeuthen [Zeu], X has a real bitangent line $L \subset \mathbb{P}^2$. (For a modern proof of Zeuthen's theorem, we refer the reader to [Ru, Corollary 4.11]; cf. also [PSV].) The restriction of $p(x, y, z)$ to L is a real quartic polynomial with two double roots, i.e., a polynomial of the form $\pm q(s, t)^2$, where s and t are linear coordinates on L, and $q \in \mathbb{R}[s, t]$ is a binary form of degree 2. In particular, p does not change sign on L, i.e., either (i) $p(a) \geqslant 0$ for every $a \in L(\mathbb{R})$ or (ii) $p(a) \leqslant 0$ for every $a \in L(\mathbb{R})$. In case (i), $m(p) \geqslant m_p(L) \geqslant 0$ and in case (ii), $M(p) \leqslant M_p(L) \leqslant 0$. □

We are now ready to finish the proof of Theorem 1(a) for $d = 4$. Assume the contrary: there exists a smooth real quartic curve $X \subset \mathbb{P}^2$ such that $(X \cap L)(\mathbb{R}) \neq \emptyset$ for every real line $L \subset \mathbb{P}^2$. Let $p \in \mathbb{R}[x, y, z]$ be a defining polynomial for X. By Proposition 4(c), $m(p) \leqslant 0 \leqslant M(p)$. In view of Lemma 5, after possibly replacing p by $-p$, we may assume that $m(p) = 0$. Proposition 4(e) now tell us that $m(p) = 0 < M(p)$. Let $p_t(x, y, z) = p(x, y, z) - t(x^2 + y^2 + z^2)^2$, where t is a real parameter, and let $X_t \subset \mathbb{P}^2$ be the quartic curve cut out by p_t. Note that X_t can be singular for only finitely many values of $t \in \mathbb{R}$. Thus we can choose $t \in (0, M(p))$ so that X_t is smooth. Since $x^2 + y^2 + z^2$ is identically 1 on $\mathbb{P}^2(\mathbb{R})$ (cf. (2.1)), we have

$$m(p_t) = m(p) - t < 0 < M(p) - t = M(p_t)\,.$$

This contradicts Lemma 5, which asserts that $m(p_t) \geqslant 0$ or $M(p_t) \leqslant 0$. □

4. Proof of Theorem 1(b)

Given an even integer $d \geqslant 6$, set $p(x, y, z) := (x^3 + y^3 + z^3)^2(x^2 + y^2 + z^2)^{(d-6)/2}$ and

$$p_t(x, y, z) = p(x, y, z) - t(x^d + y^d + z^d)\,,$$

where t is a real parameter. In view of Proposition 4(c), it suffices to show that if $t > 0$ is sufficiently small, then (i) X_t is smooth and (ii) $m(p_t) < 0 < M(p_t)$.

Since the Fermat curve, $x^d + y^d + z^d = 0$, is smooth, X_t is singular for only finitely many values of t, and (i) follows.

To prove (ii), note that p is non-negative but is not identically 0 on any real line $L \subset \mathbb{P}^2$. Thus $M_p(L) > 0$ and consequently, $M(p) > 0$. By Proposition 4(a), $M(p_t) > 0$ for small t. On the other hand, for every real line $L \subset \mathbb{P}^2$, the cubic polynomial $x^3+y^3+z^3$ vanishes at some real point a of L. Hence for every $t > 0$, we have $p_t(a) < 0$ and thus $m_{p_t}(L) < 0$. We conclude that $m(p_t) < 0$, as desired. □

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