ESSENTIAL DIMENSION, SPINOR GROUPS,
AND QUADRATIC FORMS

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Abstract. We prove that the essential dimension of the spinor group
Spin$_n$ grows exponentially with $n$ and use this result to show that qua-
dratic forms with trivial discriminant and Hasse-Witt invariant are more
complex, in high dimensions, than previously expected.

1. Introduction

Let $K$ be a field of characteristic different from 2 containing a square
root of $-1$, $W(K)$ be the Witt ring of $K$ and $I(K)$ be the ideal of classes
of even-dimensional forms in $W(K)$; cf. [Lam73]. By abuse of notation, we
will write $q \in I^a(K)$ if the Witt class on the non-degenerate quadratic form
$q$ defined over $K$ lies in $I^a(K)$. It is well known that every $q \in I^a(K)$ can be
expressed as a sum of the Witt classes of $a$-fold Pfister forms defined over
$K$; see, e.g., [Lam73, Proposition II.1.2]. If dim($q$) = $n$, it is natural to ask
how many Pfister forms are needed. When $a = 1$ or 2, it is easy to see that
$n$ Pfister forms always suffice; see Proposition 4.1. In this paper we will
prove the following result, which shows that the situation is quite different
when $a = 3$.

Theorem 1.1. Let $k$ be a field of characteristic different from 2 and $n \geq 2$ be
an even integer. Then there is a field extension $K/k$ and an $n$-dimensional
quadratic form $q \in I^3(K)$ with the following property: for any finite field
extension $L/K$ of odd degree $q_L$ is not Witt equivalent to the sum of fewer
than
\[
\frac{2(n+4)/4 - n - 2}{7}
\]
3-fold Pfister forms over $L$.

Our proof of Theorem 1.1 is based on new results on the essential di-

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interest. In particular, Theorem 3.3 gives new lower bounds on the essential
dimension of $\text{Spin}_n$ and, in many cases, computes the exact value.

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$\text{Pf}_k(a,n)$ to our attention and for contributing Proposition 4.2. We also
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## 2. Essential dimension

Let $k$ be a field. We will write $\text{Fields}_k$ for the category of field extensions
$K/k$. Let $F: \text{Fields}_k \to \text{Sets}$ be a covariant functor.

Let $L/k$ be a field extension. We will say that $a \in F(L)$ descends to
an intermediate field $k \subseteq K \subseteq L$ if $a$ is in the image of the induced map
$F(K) \to F(L)$.

The essential dimension $\text{ed}(a)$ of $a \in F(L)$ is the minimum of the transcen-
dence degrees $\text{tr deg}_k K$ taken over all fields $k \subseteq K \subseteq L$ such that $a$
descends to $K$.

The essential dimension $\text{ed}(a; p)$ of $a$ at a prime integer $p$ is the minimum
of $\text{ed}(a_{L'})$ taken over all finite field extensions $L'/L$ such that the degree
$[L': L]$ is prime to $p$.

The essential dimension $\text{ed} F$ of the functor $F$ (respectively, the essential
dimension $\text{ed}(F; p)$ of $F$ at a prime $p$) is the supremum of $\text{ed}(a)$ (respectively,
of $\text{ed}(a; p)$) taken over all $a \in F(L)$ with $L$ in $\text{Fields}_k$.

Of particular interest to us will be the Galois cohomology functors, $F_G$
given by $K \to \text{H}^1(K, G)$, where $G$ is an algebraic group over $k$. Here, as
usual, $\text{H}^1(K, G)$ denotes the set of isomorphism classes of $G$-torsors over
$\text{Spec}(K)$, in the fppf topology. The essential dimension of this functor is a
numerical invariant of $G$, which, roughly speaking, measures the complexity
of $G$-torsors over fields. We write $\text{ed} G$ for $\text{ed} F_G$ and $\text{ed}(G; p)$ for $\text{ed}(F_G; p)$.

Essential dimension was originally introduced in this context; see [BR97,
Rei00, RY00]. The above definition of essential dimension for a general
functor $F$ is due to A. Merkurjev; see [BF03].

Recall that an action of an algebraic group $G$ on an algebraic variety $k$-
variety $X$ is called “generically free” if $X$ has a dense open subset $U$
such that $\text{Stab}_G(x) = \{1\}$ for every $x \in U(k)$.

**Lemma 2.1.** If an algebraic group $G$ defined over $k$ has a generically free
linear $k$-representation $V$ then $\text{ed}(G) \leq \dim(V) - \dim(G)$.

*Proof.* See [Rei00, Theorem 3.4] or [BF03, Lemma 4.11].

**Lemma 2.2.** If $G$ is an algebraic group and $H$ is a closed subgroup of
codimension $e$ then

(a) $\text{ed}(G) \geq \text{ed}(H) - e$, and

(b) $\text{ed}(G; p) \geq \text{ed}(H; p) - e$ for any prime integer $p$.
Proof. Part (a) is [BF03, Theorem 6.19]. Both (a) and (b) follow directly from [Bro07, Principle 2.10].

If $G$ is a finite abstract group, we will write $\text{ed}_k G$ (respectively, $\text{ed}_k (G; p)$) for the essential dimension (respectively, for the essential dimension at $p$) of the constant group scheme $G_k$ over the field $k$. Let $C(G)$ denote the center of $G$.

**Theorem 2.3.** Let $G$ be a finite $p$-group whose commutator $[G, G]$ is central and cyclic. Then $\text{ed}_k (G; p) = \text{ed}_k G = \sqrt{|G/C(G)|} + \text{rank } C(G) - 1$ for any base field $k$ of characteristic $\neq p$ containing a primitive root of unity of degree equal to the exponent of $G$.

Note that with the above hypotheses, $|G/C(G)|$ is a complete square. Theorem 2.3 was originally proved in [BRV07] as a consequence of our study of essential dimension of gerbes banded by $\mu_{p^n}$. Karpenko and Merkurjev [KM07] have subsequently refined our arguments to show that the essential dimension of any finite $p$-group over any field $k$ containing a primitive $p$th root of unity is the minimal dimension of a faithful linear $k$-representation of $G$. Using [KM07, Remark 4.7] Theorem 2.3 is easily seen to be a special case of their formula. For this reason we omit the proof here.

3. Essential dimension of Spin groups

As usual, we will write $\langle a_1, \ldots, a_n \rangle$ for the quadratic form $q$ of rank $n$ given by $q(x_1, \ldots, x_n) = \sum_{i=1}^n a_i x_i^2$. Let

\begin{equation}
q^{\text{split}}_n = \begin{cases} 
    h \oplus n/2, & \text{if } n \text{ is even,} \\
    h \oplus (n-1/2) \oplus \langle 1 \rangle, & \text{if } n \text{ is odd.}
\end{cases}
\end{equation}

Let $\text{Spin}_n \overset{\text{def}}{=} \text{Spin}(q^{\text{split}}_n)$ be the split form of the spin group. We will also denote the split forms of the orthogonal and special orthogonal groups by $\text{O}_n \overset{\text{def}}{=} \text{O}(q^{\text{split}}_n)$ and $\text{SO}_n \overset{\text{def}}{=} \text{SO}(q^{\text{split}}_n)$ respectively.

M. Rost [Ros99] computed the following values of $\text{ed}(\text{Spin}_n)$ for $n \leq 14$:

- $\text{ed} \text{Spin}_3 = 0$
- $\text{ed} \text{Spin}_4 = 0$
- $\text{ed} \text{Spin}_5 = 0$
- $\text{ed} \text{Spin}_6 = 0$
- $\text{ed} \text{Spin}_7 = 4$
- $\text{ed} \text{Spin}_8 = 5$
- $\text{ed} \text{Spin}_9 = 5$
- $\text{ed} \text{Spin}_{10} = 4$
- $\text{ed} \text{Spin}_{11} = 5$
- $\text{ed} \text{Spin}_{12} = 6$
- $\text{ed} \text{Spin}_{13} = 6$
- $\text{ed} \text{Spin}_{14} = 7$,

for a detailed exposition of these results; see [Gar08]. V. Chernousov and J.–P. Serre [CS06] recently proved the following lower bounds:

\begin{equation}
\text{ed}(\text{Spin}_n; 2) \geq \begin{cases} 
    \lfloor n/2 \rfloor + 1 & \text{if } n \geq 7 \text{ and } n \equiv 1, 0 \text{ or } -1 \pmod 8 \\
    \lfloor n/2 \rfloor & \text{for all other } n \geq 11.
\end{cases}
\end{equation}
Recall that the Clifford algebra $A_e$.

The structure of the finite 2-group $G$ for all $i = ± 1$. Moreover, if $e$ is central, and the commutator $e$ is odd, and the commutator $e = 0 \mod 4$.}

Theorem 3.3. (a) Let $k$ be a field of characteristic $\neq 2$ and $n \geq 15$ be an integer.

$\text{ed}(\text{Spin}_n; 2) \geq \begin{cases} 2^{(n-1)/2} - \frac{n(n-1)}{2}, & \text{if } n \text{ is odd}, \\ 2^{(n-2)/2} - \frac{n(n-1)}{2}, & \text{if } n \equiv 2 \pmod{4}, \\ 2^{(n-2)/2} - \frac{n(n-1)}{2} + 1, & \text{if } n \equiv 0 \pmod{4}. \end{cases}$

(b) Moreover, if $\text{char}(k) = 0$ then

$\text{ed}(\text{Spin}_n) = 2^{n-1/2} - \frac{n(n-1)}{2}, \text{if } n \text{ is odd},$

$\text{ed}(\text{Spin}_n) = 2^{n-2/2} - \frac{n(n-1)}{2}, \text{if } n \equiv 2 \pmod{4}, \text{ and }$

$\text{ed}(\text{Spin}_n; 2) \leq \text{ed}(\text{Spin}_n) \leq 2^{n-2/2} - \frac{n(n-1)}{2} + n, \text{if } n \equiv 0 \pmod{4}.$

Note that while the proof of part (a) below goes through for any $n \geq 3$, our lower bounds become negative (and thus vacuous) for $n \leq 14$.

Proof. (a) Since replacing $k$ by a larger field $k'$ can only decrease the value of $\text{ed}(\text{Spin}_n; 2)$, we may assume without loss of generality that $\sqrt{-1} \in k$.

The $n$-dimensional split quadratic form $q^\text{split}_n$ is then $k$-isomorphic to

$$q(x_1, \ldots, x_n) = -(x_1^2 + \cdots + x_n^2).$$

over $k$ and hence, we can write $\text{Spin}_n$ as $\text{Spin}(q)$, $\text{O}_n(q)$ as $\text{O}_n(q)$ and $\text{SO}_n(q)$ as $\text{SO}_n(q)$.

Let $\Gamma_n \subseteq \text{SO}_n$ be the subgroup consisting of diagonal matrices. This subgroup is isomorphic to $\mu_2^{n-1}$. Let $G_n$ be the inverse image of $\Gamma_n$ in $\text{Spin}_n$; this is a constant group scheme over $k$. By Lemma 2.2(b)

$$\text{ed}(\text{Spin}_n; 2) \geq \text{ed}(G_n; 2) - \frac{n(n-1)}{2}.$$

Thus in order to prove the lower bounds of part (a), it suffices to show that

$$\text{ed}(G_n; 2) = \text{ed}(G_n) = \begin{cases} 2^{(n-1)/2}, & \text{if } n \text{ is odd}, \\ 2^{(n-2)/2}, & \text{if } n \equiv 2 \pmod{4}, \\ 2^{(n-2)/2} + 1, & \text{if } n \text{ is divisible by 4}. \end{cases}$$

The structure of the finite 2-group $G_n$ is well understood; see, e.g., [Woo89]. Recall that the Clifford algebra $A_n$ of the quadratic form $q$, as in (3.4) is the algebra given by generators $e_1, \ldots, e_n$, and relations $e_i^2 = -1, e_i e_j + e_j e_i = 0$ for all $i \neq j$. For any $I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_r$, set $e_I \equiv e_{i_1} \cdots e_{i_r}$. Here $e_0 = 1$. The group $G_n$ consists of the elements of $A_n$ of the form $\pm e_I$, where the cardinality $r = |I|$ of $I$ is even. The element $-1$ is central, and the commutator $[e_I, e_J]$ is given by $[e_I, e_J] = (-1)^{|I \cap J|}$. It is
clear from this description that $G_n$ is a 2-group of order $2^n$, the commutator subgroup $[G_n, G_n] = \{ \pm 1 \}$ is cyclic, and the center $C(G)$ is as follows:

$$C(G_n) = \begin{cases} 
\{ \pm 1 \} \simeq \mathbb{Z}/2\mathbb{Z}, & \text{if } n \text{ is odd}, \\
\{ \pm 1, \pm e_{\{1,\ldots,n\}} \} \simeq \mathbb{Z}/4\mathbb{Z}, & \text{if } n \equiv 2 \pmod{4}, \\
\{ \pm 1, \pm e_{\{1,\ldots,n\}} \} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, & \text{if } n \text{ is divisible by 4}.
\end{cases}$$

Formula (3.5) now follows from Theorem 2.3.

(b) Clearly $\text{ed}(\text{Spin}_n; 2) \leq \text{ed}(\text{Spin}_n)$. Hence, we only need to show that for $n \geq 15$

$$\text{ed}(\text{Spin}_n) \leq \begin{cases} 
2^{(n-1)/2} - \frac{n(n-1)}{2}, & \text{if } n \text{ is odd}, \\
2^{(n-2)/2} - \frac{n(n-1)}{2}, & \text{if } n \equiv 2 \pmod{4}, \\
2^{(n-2)/2} - \frac{n(n-1)}{2} + n, & \text{if } n \equiv 0 \pmod{4}.
\end{cases}$$

In view of Lemma 2.1 it suffices to show that $\text{Spin}_n$ has a generically free linear representation $V$ of dimension

$$\dim(V) = \begin{cases} 
2^{(n-1)/2}, & \text{if } n \text{ is odd}, \\
2^{(n-2)/2}, & \text{if } n \equiv 2 \pmod{4}, \\
2^{(n-2)/2} + n & \text{if } n \equiv 0 \pmod{4}.
\end{cases}$$

In the case where $n$ is not divisible by 4 such a representation is given by the following lemma.

**Lemma 3.7.** (cf. [PV94, Theorem 7.11]) *If $n \geq 15$ then, over a field of characteristic 0, the following representations of $\text{Spin}_n$ of characteristic 0 are generically free:*

(i) the spin representation, of dimension $2^{(n-1)/2}$, if $n$ is odd,

(ii) either of the two half-spin representation, of dimension $2^{(n-2)/2}$, if $n \equiv 2 \pmod{4}$.

**Proof.** For $n \geq 27$ this follows directly from [AP71, Theorem 1]. For $n$ between 15 and 25 this is proved in [Po85]. ♠

In the case where $n \geq 16$ is divisible by 4, we define $V$ as the sum of the half-spin representation $W$ of $\text{Spin}_n$ and the natural representation $k^n$ of $\text{SO}_n$, which we will view as a $\text{Spin}_n$-representation via the projection $\text{Spin}_n \to \text{SO}_n$. It remains to check that $V = W \times k^n$ is a generically free representation of $\text{Spin}_n$. Indeed, for $a \in k^n$ in general position, $\text{Stab}(a)$ is conjugate to $\text{Spin}_{n-1}$ (embedded in $\text{Spin}_n$ in the standard way). Thus it suffices to show that the restriction of $W$ to $\text{Spin}_{n-1}$ is generically free. Since $W$ restricted to $\text{Spin}_{n-1}$ is the spin representation of $\text{Spin}_{n-1}$ (see, e.g., [Ada96, Proposition 4.4]), and $n \geq 16$, this follows from Lemma 3.7(i). This completes the proof of Theorem 3.3. ♠

**Remark 3.8.** The characteristic 0 assumption in part (b) is used only in the proof of Lemma 3.7. It seems likely that Lemma 3.7 (and thus Theorem 3.3(b)) remain true if $\text{char}(k) = p > 2$ but we have not checked this.
If \( \text{char}(k) \neq 2 \) and \( \sqrt{-1} \in k \), we have the weaker (but asymptotically equivalent) upper bound \( \text{ed}(\text{Spin}_n) \leq \text{ed}(G_n) \), where \( \text{ed}(G_n) \) is given by (3.5). This is a consequence of the fact that every \( \text{Spin}_n \)-torsor admits reduction of structure to \( G_n \), i.e., the natural map \( H^1(K, G_n) \to H^1(K, \text{Spin}_n) \) is surjective for every field \( K/k \); cf. [BF03, Lemma 1.9].

**Remark 3.9.** A. S. Merkurjev (unpublished) recently strengthened our lower bound on \( \text{ed}(\text{Spin}_n; 2) \), in the case where \( n \equiv 0 \pmod{4} \) as follows:

\[
\text{ed}(\text{Spin}_n; 2) \geq 2^{(n-2)/2} - \frac{n(n-1)}{2} + 2^m, 
\]

where \( 2^m \) is the highest power of 2 dividing \( n \). If \( n \geq 16 \) is a power of 2 and \( \text{char}(k) = 0 \) this, in combination with the upper bound of Theorem 3.3(b), yields

\[
\text{ed}(\text{Spin}_n; 2) = \text{ed}(\text{Spin}_n) = 2^{(n-2)/2} - \frac{n(n-1)}{2} + n.
\]

In particular, \( \text{ed}(\text{Spin}_{16}) = 24 \). The first value of \( n \) for which \( \text{ed}(\text{Spin}_n) \) is not known is \( n = 20 \), where \( 326 \leq \text{ed}(\text{Spin}_{20}) \leq 342 \).

**Remark 3.10.** The same argument can be applied to the half-spin groups yielding

\[
\text{ed}(\text{HSpin}_n; 2) = \text{ed}(\text{HSpin}_n) = 2^{(n-2)/2} - \frac{n(n-1)}{2}
\]

for any integer \( n \geq 20 \) divisible by 4 over any field of characteristic 0. Here, as in Theorem 3.3, the lower bound

\[
\text{ed}(\text{HSpin}_n; 2) \geq 2^{(n-2)/2} - \frac{n(n-1)}{2}
\]

is valid for over any base field \( k \) of characteristic \( \neq 2 \). The assumptions that \( \text{char}(k) = 0 \) and \( n \geq 20 \) ensure that the half-spin representation of \( \text{HSpin}_n \) is generically free; see [PV94, Theorem 7.11].

**Remark 3.11.** Theorem 3.3 implies that for large \( n \), \( \text{Spin}_n \) is an example of a split, semisimple, connected linear algebraic group whose essential dimension exceeds its dimension. Previously no examples of this kind were known, even for \( k = \mathbb{C} \).

Note that no complex connected semisimple adjoint group \( G \) can have this property. Indeed, let \( \mathfrak{g} \) be the adjoint representation of \( G \) on its Lie algebra. If \( G \) is an adjoint group then \( V = \mathfrak{g} \times \mathfrak{g} \) is generically free; see, e.g., [Rich88, Lemma 3.3(b)]. Thus \( \text{ed}G \leq \dim(G) \) by Lemma 2.1.

**Remark 3.12.** Since \( \text{ed} \text{SO}_n = n - 1 \) for every \( n \geq 3 \) (cf. [Rei00, Theorem 10.4]), it follows that, for large \( n \), \( \text{Spin}_n \) is also an example of a split, semisimple, connected linear algebraic group \( G \) with a central subgroup \( Z \) such that \( \text{ed}G > \text{ed}G/Z \). To the best of our knowledge, this example is new as well.
4. Pfister numbers

Let $K$ be a field of characteristic not equal to 2 and $a \geq 1$ be an integer. We will continue to denote the Witt ring of $K$ by $W(K)$ and its fundamental ideal by $I(K)$. If non-singular quadratic forms $q$ and $q'$ over $K$ are Witt equivalent, we will write $q \sim q'$.

As we mentioned in the introduction, the $a$-fold Pfister forms generate $I^a(K)$ as an abelian group. In other words, every $q \in I^a(K)$ is Witt equivalent to $\sum_{i=1}^r \pm p_i$, where each $p_i$ is an $a$-fold Pfister form over $K$. We now define the $a$-Pfister number of $q$ to be the smallest possible number $r$ of Pfister forms appearing in any such sum. The $(a,n)$-Pfister number $Pf_k(a,n)$ is the supremum of the $a$-Pfister number of $q$, taken over all field extensions $K/k$ and all $n$-dimensional forms $q \in I^n(K)$.

**Proposition 4.1.** Let $k$ be a field of characteristic $\neq 2$ and let $n$ be a positive even integer. Then (a) $Pf_k(1,n) \leq n$ and (b) $Pf_k(2,n) \leq n - 2$.

**Proof.** (a) Immediate from the identity

$$\langle a_1, a_2 \rangle \sim \langle 1, a_1 \rangle - \langle 1, -a_2 \rangle = \ll -a_1 \gg - \ll a_2 \gg$$

in the Witt ring.

(b) Let $q = \langle a_1, \ldots, a_n \rangle$ be an $n$-dimensional quadratic form over $K$. Recall that $q \in I^2(K)$ iff $n$ is even and $d_\pm(q) = 1$, modulo $(K^*)^2$ [Lam73, Corollary II.2.2]. Here $d_\pm(q)$ is the signed discriminant given by $(-1)^{n(n-1)/2}d(q)$ where $d(q) = \prod_{a_i} a_n$ is the discriminant of $q$; cf. [Lam73, p. 38].

To explain how to write $q$ in terms of $n - 2$ Pfister forms, we will temporarily assume that $\sqrt{-1} \in K$. In this case, without loss of generality, $a_1 \ldots a_n = 1$. Since $\langle a, a \rangle$ is hyperbolic for every $a \in K^*$, we see that $q = \langle a_1, \ldots, a_n \rangle$ is Witt equivalent to

$$\ll a_2, a_1 \gg \oplus \ll a_3, a_1a_2 \gg \oplus \cdots \oplus \ll a_{n-1}, a_1 \ldots a_{n-2} \gg .$$

By inserting appropriate powers of $-1$, we can modify this formula so that it remains valid even if we do not assume that $\sqrt{-1} \in K$, as follows:

$$q = \langle a_1, \ldots, a_n \rangle \sim \sum_{i=2}^n (-1)^i \ll (-1)^{i+1}a_i, (-1)^{i-1/2+1}a_1 \ldots a_{i-1} \gg \star$$

We do not have an explicit upper bound on $Pf_k(3,n)$; however, we do know that $Pf_k(3,n)$ is finite for any $k$ and any $n$. To explain this, let us recall that $I^3(K)$ is the set of all classes $q \in W(K)$ such that $q$ has even dimension, trivial signed discriminant and trivial Hasse-Witt invariant [KMRT98]. The following result was suggested to us by Merkurjev and Totaro.

**Proposition 4.2.** Let $k$ be a field of characteristic different from 2. Then $Pf_k(3,n)$ is finite.

**Sketch of proof.** Let $E$ be a versal torsor for $\text{Spin}_n$ over a field extension $L/k$; cf. [GMS03, Section I.V]. Let $q_L$ be the quadratic form over $L$ corresponding to $E$ under the map $H^1(L, \text{Spin}_n) \to H^1(L, \text{O}_n)$. The 3-Pfister
number of $q_L$ is then an upper bound for the 3-Pfister number of any $n$-dimensional form in $I^3$ over any field extension $K/k$.

Remark 4.3. For $a > 3$ the finiteness of $\text{Pf}_k(a, n)$ is an open problem.

5. Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1 stated in the introduction, which says, in particular, that

$$\text{Pf}_k(3, n) \geq \frac{2^{(n+4)/4} - n - 2}{7}$$

for any field $k$ of characteristic different from 2 and any positive even integer $n$. Clearly, replacing $k$ by a larger field $k'$ strengthens the assertion of Theorem 1.1. Thus, we may assume without loss of generality that $\sqrt{-1} \in k$.

This assumption will be in force for the remainder of this section.

For each extension $K$ of $k$, denote by $T_n(K)$ the image of $H^1(K, \text{Spin}_n)$ in $H^1(K, \text{SO}_n)$. We will view $T_n$ as a functor $\text{Fields}_k \to \text{Sets}$. Note that $T_n(K)$ is the set of isomorphism classes of $n$-dimensional quadratic forms $q \in I_3(K)$.

Lemma 5.1. We have the following inequalities:

(a) $\text{ed} \text{Spin}_n - 1 \leq \text{ed} T_n \leq \text{ed} \text{Spin}_n$,
(b) $\text{ed}(\text{Spin}_n; 2) - 1 \leq \text{ed}(T_n; 2) \leq \text{ed}(\text{Spin}_n; 2)$.

Proof. In the language of [BF03, Definition 1.12], we have a fibration of functors

$$H^1(\ast, \mu_2) \sim H^1(\ast, \text{Spin}_n) \longrightarrow T_n(\ast).$$

The first inequality in part (a) follows from [BF03, Proposition 1.13] and the second from Proposition [BF03, Lemma 1.9]. The same argument proves part (b).

Let $K/k$ be a field extension. Let $h_K = \langle 1, -1 \rangle$ be the 2-dimensional hyperbolic form over $K$. (Note in §3 we wrote $h$ in place of $h_k$; see (3.1).) For each $n$-dimensional quadratic form $q \in I^3(K)$, let $\text{ed}_n(q)$ denote the essential dimension of the class of $q$ in $T_n(K)$.

Lemma 5.2. Let $q$ be an $n$-dimensional quadratic form in $I^3(K)$. Then

$$\text{ed}_{n+2s}(h_K^{\otimes s} \oplus q) \geq \text{ed}_n(q) - \frac{s(s + 2n - 1)}{2}$$

for any integer $s \geq 0$.

Proof. Set $m \overset{\text{def}}{=} \text{ed}_{n+2s}(h_K^{\otimes s} \oplus q)$. By definition, $h_K^{\otimes s} \oplus q$ descends to an intermediate subfield $k \subset F \subset K$ such that $\text{tr} \deg_k(F) = m$. In other words, there is an $(n + 2s)$-dimensional quadratic form $\bar{q} \in I^3(F)$ such that $\bar{q}_k$ is $K$-isomorphic to $h_K^{\otimes s} \oplus q$. Let $X$ be the Grassmannian of $s$-dimensional subspaces of $F^{n+2s}$ which are totally isotropic with respect to $\bar{q}$. The dimension of $X$ over $F$ is $s(s + 2n - 1)/2$. 
The variety \(X\) has a rational point over \(K\); hence there exists an intermediate extension \(F \subseteq E \subseteq K\) such that \(\text{tr deg}_F E \leq s(s + 2n - 1)/2\), with the property that \(q_E\) has a totally isotropic subspace of dimension \(s\). Then \(q_E\) splits as \(h_E^s \oplus q'\), where \(q' \in I^3(E)\). By Witt’s Cancellation Theorem, \(q_K\) is \(K\)-isomorphic to \(q\); hence

\[
ed_n(q) \leq \text{tr deg}_K E = \text{tr deg}_k F + \text{tr deg}_F E = m + s(s + 2n - 1)/2,
\]
as claimed.

\[\blacksquare\]

We now proceed with the proof of Theorem 1.1. For \(n \leq 10\) the statement of the theorem is vacuous, because \(2^{(n+4)/4} - n - 2 \leq 0\). Thus we will assume from now on that \(n \geq 12\).

Lemma 5.1 implies, in particular, that \(\text{ed}(T_n; 2)\) is finite. Hence, there exist a field \(K/k\) and an \(n\)-dimensional form \(q \in I^3(K)\) such that \(\text{ed}_n(q) = \text{ed}(T_n; 2)\). We will show that this form has the properties asserted by Theorem 1.1. In fact, it suffices to prove that if \(q\) is Witt equivalent to

\[
\sum_{i=1}^r \ll a_i, b_i, c_i \gg.
\]

over \(K\) then \(r \geq \frac{2(n+4)/4 - n - 2}{7}\). Indeed, by our choice of \(q\), \(\text{ed}_n(q_L) = \text{ed}(T_n; 2)\) for any finite odd degree extension \(L/K\). Thus if we can prove the above inequality for \(q\), it will also be valid for \(q_L\).

Let us write a 3-fold Pfister form \(\ll a, b, c \gg\) as \(\langle 1 \rangle \ll a, b, c \gg_0\), where

\[
\ll a, b, c \gg_0 \overset{\text{def}}{=} \langle a_i, b_i, c_i, a_i b_i, a_i c_i, b_i c_i, a_i b_i c_i \rangle.
\]

Set

\[
\phi \overset{\text{def}}{=} \begin{cases} 
\sum_{i=1}^r \ll a_i, b_i, c_i \gg_0, & \text{if } r \text{ is even}, \\
\langle 1 \rangle \oplus \sum_{i=1}^r \ll a_i, b_i, c_i \gg_0, & \text{if } r \text{ is odd}.
\end{cases}
\]

Then \(q\) is Witt equivalent to \(\phi\) over \(K\); in particular, \(\phi \in I^3(K)\). The dimension of \(\phi\) is \(7r\) or \(7r + 1\), depending on the parity of \(r\).

We claim that \(n < 7r\). Indeed, assume the contrary. Then \(\text{dim}(q) \leq \text{dim}(\phi)\), so that \(q\) is isomorphic to a form of type \(h_K^s \oplus \phi\) over \(K\). Thus

\[
\frac{3n}{7} \geq 3r \geq \text{ed}_n(q) = \text{ed}(T_n; 2) \geq \text{ed}(\text{Spin}_n; 2) - 1.
\]

The resulting inequality fails for every even \(n \geq 12\) because for such \(n\)

\[
\text{ed}(\text{Spin}_n; 2) \geq n/2;
\]

see (3.2).

So, we may assume that \(7r > n\), i.e., \(\phi\) is isomorphic to \(h_K^s \oplus q\) over \(K\), for some \(s \geq 1\). By comparing dimensions we get the equality \(7r = n + 2s\) when \(r\) is even, and \(7r + 1 = n + 2s\) when \(r\) is odd. The essential dimension of the form \(\phi\), as an element of \(T_{7r}(K)\) or \(T_{7r+1}(K)\) is at most \(3r\), while Lemma 5.2 tells us that this essential dimension is at least \(\text{ed}_n(q) - s(s + 2n - 1)/2\).
From this, Lemma 5.1 and Theorem 3.3(a) we obtain the following chain of inequalities

\[ 3r \geq \text{ed}_n(q) - \frac{s(s + 2n - 1)}{2} = \text{ed}(T_n; 2) - \frac{s(s + 2n - 1)}{2} \]

\[ \geq \text{ed}(	ext{Spin}_n; 2) - 1 - \frac{s(s + 2n - 1)}{2} \geq 2^{(n-2)/2} - \frac{n(n - 1)}{2} - 1 - \frac{s(s + 2n - 1)}{2}. \] (5.3)

Now suppose \( r \) is even. Substituting \( s = (7r - n)/2 \) into inequality (5.3), we obtain

\[ \frac{49r^2 + (14n + 10)r - 2^{(n+4)/2} - n^2 + 2n - 8}{8} \geq 0. \]

We interpret the left hand side as a quadratic polynomial in \( r \). The constant term of this polynomial is negative for all \( n \geq 8 \); hence this polynomial has one positive real root and one negative real root. Denote the positive root by \( r_+ \). The above inequality is then equivalent to \( r \geq r_+ \). By the quadratic formula

\[ r_+ = \frac{\sqrt{49 \cdot 2^{(n+4)/2} + 168n - 367} - (7n + 5)}{49} \geq \frac{2^{(n+4)/4} - n - 2}{7}. \]

This completes the proof of Theorem 1.1 when \( r \) is even. If \( r \) is odd then substituting \( s = (7r + 1 - n)/2 \) into (5.3), we obtain an analogous quadratic inequality whose positive root is

\[ r_+ = \frac{\sqrt{49 \cdot 2^{(n+4)/2} + 168n - 199} - (7n + 12)}{49} \geq \frac{2^{(n+4)/4} - n - 2}{7}, \]

and Theorem 1.1 follows.

References


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