A NON-SPLIT TORSOR WITH TRIVIAL FIXED POINT OBSTRUCTION

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Abstract. Let $G$ be a linear algebraic group and $X$ be an irreducible algebraic variety with a generically free $G$-action, all defined over an algebraically closed base field of characteristic zero. It is well known that $X$ can be viewed as a $G$-torsor, representing a class $[X]$ in $H^1(K,G)$, where $K$ is the field of $G$-invariant rational functions on $X$. We have previously shown that if $X$ has a smooth $H$-fixed point for some non-toral diagonalizable subgroup of $G$ then $[X] \neq 1$. It is natural to ask if the converse is true, assuming $G$ is connected and $X$ is projective and smooth. In this note we show that the answer is “no”.

1. Introduction

Let $G$ be a linear algebraic group defined over an algebraically closed base field $k$ of characteristic zero. By a $G$-variety we shall mean an algebraic variety $X$ with a regular action of $G$ (defined over $k$). We shall say that $X$ is generically free if $G$ acts freely on a dense open subset of $X$. Birational isomorphism classes of $G$-varieties $X$ with $k(X)^G = K$ are in 1-1 correspondence with $H^1(K,G)$; see [6, 1.3]. We will call $X$ split if one (and thus all) of the following equivalent conditions hold.

• $X$ represents the trivial class in $H^1(K,G)$.
• $X$ is birationally isomorphic to $Y \times G$ as a $G$-variety. Here $Y$ is an algebraic variety with trivial $G$-action, and $G$ acts on $Y \times G$ by left translations on the second factor.
• The (rational) quotient map $X \twoheadrightarrow X/G$ has a rational section;

cf. [6, 1.4]. We shall say that a subgroup of $G$ is toral if it lies in a subtorus of $G$ and non-toral otherwise. The starting point for this note is the following:

Proposition 1. ([10, Lemma 4.3]) Let $X$ be a generically free $G$-variety. If $X$ has a smooth $H$-fixed point for some non-toral diagonalizable subgroup of $H$ of $G$, then $X$ is not split.

In other words, the presence of a smooth $H$-fixed point on $X$ is an obstruction to $X$ being split; we shall refer to it as the fixed point obstruction.

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In the case where $H$ is a non-toral finite abelian subgroup of $G$, we have described this obstruction in a more quantitative way by giving lower bounds on the essential dimension [9, Theorem 1.2], splitting degree [10, Theorem 1.1], and the size of a splitting group of $X$ [10, Theorem 1.2] in terms of $H$. (Recall that a split variety has essential dimension 0, splitting degree 1 and splitting group $\{1\}$.)

The question that remained unanswered in [9] and [10] is whether or not the converse to Proposition 1 is also true. Of course, in stating the converse, we need to assume that the $G$-variety $X$ is smooth and complete; otherwise the fixed point obstruction may not be “visible” because it may “hide” in the “boundary” or in the singular locus of $X$. Fortunately, every class in $H^1(K, G)$ can be represented by a smooth complete (and even projective) $G$-variety; see [10, Proposition 2.2]. Moreover, the fixed point obstruction is detectable on any such model in the following sense. Suppose $X$ is a generically free $G$-variety and $Y$ is a smooth complete $G$-variety birationally isomorphic to $X$. If $X$ has a smooth $H$-fixed point for some non-toral diagonalizable subgroup $H \subset G$ then so does $Y$; see [9, Proposition A2].

We also remark that if $H$ is toral then $X^H \neq \emptyset$ by the Borel Fixed Point Theorem [1, Theorem 10.4]; thus only non-toral subgroups $H$ are of interest here. To sum up, we will address the following:

**Question 2.** Is the fixed point obstruction the only obstruction to splitting? In other words, if $X$ is a smooth projective generically free $G$-variety such that $X^H = \emptyset$ for every diagonalizable non-toral subgroup $H \subset G$, is $X$ necessarily split?

**Example 3.** If $G$ is a finite group then the answer is “no”, because $G$ can be made to act freely on an irreducible smooth projective curve $X$. Over $\mathbb{C}$ such a curve can be constructed as follows. Suppose $G$ is generated by $n$ elements, $g_1, \ldots, g_n$. Let $Y$ be a curve of genus $n$. Then the fundamental group $\pi_1(Y)$ is given by $2n$ generators $a_1, \ldots, a_n, b_1, \ldots, b_n$ and one relation

$$\prod_{i=1}^n a_i b_i a_i^{-1} b_i^{-1} = 1.$$ 

The surjective homomorphism $\pi_1(Y) \rightarrow G$, sending $a_i$ to $g_i$ and $b_i$ to 1, gives rise to an unramified $G$-cover $X \rightarrow Y$ of Riemann surfaces. By the Riemann Embedding Theorem, $X$ is a smooth projective algebraic curve with a free $G$-action. The same argument goes through over any algebraically closed base field $k$ of characteristic zero, provided that $\pi_1(Y)$ is interpreted as Grothendieck’s algebraic fundamental group of $Y$; see [4, Expose XIII, Corollaire 2.12].

Question 2 becomes more delicate if we $G$ is assumed to be connected. The purpose of this note is to show that under this assumption the answer is still “no”. Our main result is the following:

**Theorem 4.** Let $p$ be an odd prime. Then there exists a smooth projective generically free $\text{PGL}_p$-variety $X$ with the following properties:
(a) $X$ is not split,
(b) $X^H = \emptyset$ for every diagonalizable non-toral subgroup $H$ of $\text{PGL}_p$,
(c) $k(X)^{\text{PGL}_p}$ is a purely transcendental extension of $k$.

The rest of this paper is devoted to proving Theorem 4. In Sections 2 and 3 we reduce the proof to the question of existence of a certain division algebra of degree $p$; see Proposition 7. Our construction of this algebra in Section 4 relies on a criterion of Fein, Saltman and Schacher [2].

2. Nontoral subgroups of $\text{PGL}_p$

Consider the $p \times p$-matrices
\[
\sigma = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & \zeta & \ldots & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \ldots & \zeta^{p-1}
\end{pmatrix}
\quad \text{and} \quad
\tau = \begin{pmatrix}
0 & 0 & \ldots & 1 \\
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
0 & 0 & \ldots & 1
\end{pmatrix}
\]
where $\zeta$ is a primitive $p$th root of unity in $k$. Note that
\[
\sigma \tau = \zeta \tau \sigma .
\]
Thus the elements $\sigma, \tau \in \text{PGL}_p$, represented, respectively, by $\sigma$ and $\tau$, generate an abelian subgroup; we shall denote this subgroup by $A$. Clearly $A \cong (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$. It is well known that, up to conjugacy, $A$ is the unique non-toral elementary abelian subgroup of $\text{PGL}_p$; cf., e.g., [3, Theorem 3.1]. In the sequel we will need to know that $A$ is in fact the unique diagonalizable subgroup with this property. For lack of a suitable reference, we give a direct elementary proof of this fact below.

Lemma 5. Let $H$ be a non-toral diagonalizable subgroup of $\text{PGL}_p$, where $p$ is a prime. Then $H$ is conjugate to $A$.

In the sequel we will only need this lemma for odd $p$; however, for the sake of completeness, we will treat the case $p = 2$ as well.

Proof. Let $\tilde{H}$ be the preimage of $H$ in $\text{SL}_p$. Then for every $x, y \in \tilde{H}$, $x y x^{-1} y^{-1}$ is a scalar matrix in $\text{SL}_p$, i.e., a matrix of the form $f(x, y)I$, where $I$ is the $p \times p$ identity matrix and $f(x, y)$ is a $p$th root of unity. If $f(x, y) = 1$ for every $x, y \in \tilde{H}$ then $\tilde{H}$ is a commutative subgroup of $\text{SL}_p$ consisting of semisimple elements. This implies that $\tilde{H}$ is toral in $\text{SL}_p$ (see, e.g., [1, Proposition 8.4]) and thus $H$ is toral in $\text{PGL}_p$, contradicting our assumption. Therefore, $f(x, y)$ is a primitive $p$th root of unity for some $x, y \in \tilde{H}$. Replacing $x$ by $y^i$ for an appropriate $i$, we may assume $f(x, y) = \zeta$, i.e.,
\[
xy = \zeta yx .
\]
Suppose $v$ is an eigenvector of $x$ with associated eigenvalue $\lambda \neq 0$. Then (2) shows that $v_i = y^i(v)$ is an eigenvector of $x$ with eigenvalue $\lambda \zeta^i$. These
eigenvalues are distinct for \( i = 0, 1, \ldots, p - 1 \), and hence, the eigenvectors \( v = v_0, v_1, \ldots, v_{p-1} \) form a basis of \( k^p \). Moreover, since \( y^p(v) \) is an eigenvector for \( x \) with eigenvalue \( \lambda \) and the \( \lambda \)-eigenspace of \( x \) is 1-dimensional, \( y^p(v) = cv_0 \) for some \( c \in k \). Writing \( x \) and \( y \) in the basis \( v_0, \ldots, v_{p-1} \), we see that

\[
x = \lambda \sigma \quad \text{and} \quad y = \text{diag}(c, 1, \ldots, 1) \tau ,
\]

where \( \sigma \) and \( \tau \) are as in (1). Since \( \det(y) = 1 \), we see that \( c = (-1)^{p+1} \). We now consider two cases:

(i) \( p \) is odd. Then \( c = 1 \) and \( x, y \in \text{SL}_p \) represent, respectively, \( \sigma \) and \( \tau \) in \( \text{PGL}_p \).

(ii) \( p = 2 \). Here \( c = -1 \), and in the basis \( v_0, v_1 \),

\[
x = \lambda \sigma = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

Let \( g = \text{diag}(1, i) \), where \( i \) is a primitive 4th root of unity. Then \( gxg^{-1} \) and \( gyyg^{-1} \) represent, respectively, \( \sigma \) and \( \tau \) in \( \text{PGL}_p \).

Thus, after conjugation, we may assume that \( A \subset H \). Since \( A \) is self-centralizing in \( \text{PGL}_p \) (cf. [9, Lemma 8.12(b)]), we conclude that \( H = A \). \( \square \)

3. Division algebras

Let \( F \) be a finitely generated field extension of \( k \). Recall that elements of \( H^1(F, \text{PGL}_n) \) may be interpreted in two ways:

- as central simple algebras of degree \( n \) with center \( F \); see [11, Section 10.5] or [5, p. 396], and
- as birational isomorphism classes of irreducible generically free \( \text{PGL}_n \)-varieties \( X \) such that \( k(X)^{\text{PGL}_n} = F \); see [6, Section 1.3] (cf. also [12, Section I.5.2]).

Thus to every central simple algebra \( D \) of degree \( n \) over \( F \) we can associate a generically free \( \text{PGL}_n \)-variety \( X_D \) with \( k(X_D)^{\text{PGL}_n} = F \). Moreover, \( X_D \) is uniquely defined up to birational isomorphism of \( \text{PGL}_n \)-varieties, and \( D \) can be recovered from \( X_D \) as the algebra of \( \text{PGL}_n \)-equivariant rational maps \( X_D \to M_n \); see [7, Proposition 8.6 and Lemma 9.1]. We shall write \( D = \text{RMaps}_{\text{PGL}_n}(X_D, M_n) \). Note that \( D \cong M_n(F) \) if and only if the \( \text{PGL}_n \)-variety \( X_D \) is split.

**Proposition 6.** Let \( D \) be a division algebra of degree \( p \) with center \( K \) and \( X_D \) be an algebraic variety representing the class of \( D \) in \( H^1(K, \text{PGL}_n) \). Let \( A \) be the subgroup of \( \text{PGL}_p \) defined in Section 2. If \( D \) has an element of (reduced) trace 0 and norm 1 then \( X_D \) does not have a smooth \( A \)-fixed point.

**Proof.** The proposition is proved in [8]; however, since it is not stated there in the exact form we need, we supply a short explanation. Let \( x \in D \) be an
element of trace zero and norm 1. Then the system
\[
\begin{align*}
\text{Nrd}(x_1) &= \cdots = \text{Nrd}(x_p) \\
\text{Trd}(x_1 \cdots x_p) &= 0
\end{align*}
\]
has a nontrivial solution in \(D\), namely \((x_1, \ldots, x_p) = (x, 1, \ldots, 1)\). (Here, as usual, \(\text{Nrd}\) and \(\text{Trd}\) denote, respectively, the reduced norm and the reduced trace in \(D\).) On the other hand, by [8, Proposition 3.3 and Lemma 5.3], if \(X_D\) has a smooth \(A\)-fixed point then the system (3) has only the trivial solution \((x_1, \ldots, x_p) = (0, \ldots, 0)\). This shows that \(X_D\) does not have a smooth \(A\)-fixed point. \(\square\)

We now observe that in order to prove Theorem 4 it is enough to establish the following:

**Proposition 7.** There exists a division algebra \(D\) of degree \(p\) with center \(F\) such that

(i) \(F\) is a purely transcendental extension of \(k\), and

(ii) there exists an element \(a \in D\) such that \(\text{Trd}(a) = 0\) and \(\text{Nrd}(a) = 1\).

Indeed, suppose \(D\) is a division algebra satisfying the conditions of Proposition 7. Let \(X = X_D\) be a smooth projective \(\text{PGL}_p\)-variety representing the class of \(D\) in \(H^1(K, \text{PGL}_p)\); such a model exists by [10, Proposition 2.2]. We now check that \(X = X_D\) has properties (a) - (c) claimed in the statement of Theorem 4:

(a) \(X\) is not split; otherwise \(D \simeq \text{M}_p(K)\) would not be a division algebra.

(b) By Lemma 5, we may assume \(H = A\), and by Proposition 6, \(A\) acts on \(X\) without fixed points.

(c) \(k(X)^{\text{PGL}_p} = F\) is purely transcendental over \(k\) by Proposition 7(i). \(\square\)

4. CONCLUSION OF THE PROOF

Our strategy for proving Proposition 7 will be to find an element \(a\) of norm 1 and trace 0 in a suitable field extension \(L/K\) of degree \(p\), then embed this field extension into a division algebra.

**Lemma 8.** For any \(n \geq 3\) there exists a field extension \(L/K\) of degree \(n\) such that

(i) \(K\) is a purely transcendental extension of \(k\) of transcendence degree 1 and

(ii) \(\text{Tr}_{L/K}(a) = 0\) and \(\text{N}_{L/K}(a) = 1\) for some \(a \in L\). Here \(\text{Tr}_{L/K}(a)\) and \(\text{N}_{L/K}(a)\) are the trace and the norm of \(a\) in \(L/K\).

**Proof.** Consider the polynomial
\[
P(s, t) = s^n + ts + (-1)^n \in k[t, s],
\]
where \(t\) and \(s\) are independent commuting variables over \(k\). Since we can write \(P = P_0t + P_1\), where \(P_0 = s\) and \(P_1 = s^n + (-1)^n\) are relatively prime in \(k[s]\), we conclude that \(P\) is irreducible in \(k[t, s]\), and hence, in \(k(t)[s]\).
Now let \( K = k(t), L = K[s]/(P(t,s)) \) and let \( a \) be the image of \( s \) in \( L \). Then condition (i) is clearly satisfied. Moreover, since \( L/K \) is a field extension of degree \( n \) and \( P \) is the minimal polynomial of \( a \) over \( K \), \(-1\)^{n-1} \( N_{L/K}(a) \) are, respectively, the coefficient of \( s^{n-1} \) and the constant term of \( P \). Thus \( \text{Tr}_{L/K}(a) = 0 \) and \( N_{L/K}(a) = 1 \), as claimed. □

We are now ready to prove Proposition 7. Let \( L/K \) be as in Lemma 8, with \( n = p \). It is sufficient to show that there exists a division algebra \( D \) with center \( F = K(\lambda_1, \ldots, \lambda_r) \) and maximal subfield \( L(\lambda_1, \ldots, \lambda_r) \), where \( \lambda_1, \ldots, \lambda_r \) are algebraically independent variables over \( K \). Then \( D \) is the algebra we want: \( F \) is a purely transcendental extension of \( k \) and an element \( a \in D \) with desired properties can be found in \( L \subset D \).

To show that such a \( D \) exists, we appeal to a result of Fein, Saltman and Schacher [2, Corollary 5.4]. Let \( G \) be a finite group, \( H \) be a subgroup of \( G \) and \( q \) be a prime dividing \( |G| \). Following [2], we define \( m_q(G, H) \) to be the maximal value of \( |T| \), taken over all \( q \)-subgroups \( T \) of \( G \) which are contained in \( \bigcup_{g \in G} gHg^{-1} \).

Returning to the setting of Lemma 8, let \( E \) be the Galois closure of \( L \) over \( K \), \( G = \text{Gal}(E/K) \) and \( H = \text{Gal}(E/L) \). [2, Corollary 5.4] guarantees the existence of \( D \) if \( m_q(G, H) = |H_q| \) for every \( q \) dividing \( |L : K| \); here \( H_q \) is a Sylow \( q \)-subgroup of \( H \). In our case \( |L : K| = p \), so we only need to check that \( m_p(G, H) = |H_p| \).

Note that \( E \) is the splitting field and \( G \) is the Galois group of the irreducible polynomial (4) over \( K = k(t) \), with \( n = p \). Thus \( G \) is naturally a subgroup of \( S_p \) and consequently \( |G| \) is not divisible by \( p^2 \). On the other hand, \( |G : H| = |L : K| = p \). We conclude that \( |H| \) is not divisible by \( p \), i.e., \( |H_p| = 1 \). Moreover, the order of every element of \( \bigcup_{g \in G} gHg^{-1} \) is prime to \( p \); thus \( m_p(G, H) = 1 \). To sum up, \( m_p(G, H) = 1 = |H_p| \), and [2, Corollary 5.4] applies.

This completes the proof of Proposition 7 and thus of Theorem 4. □

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REFERENCES


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