ON THE NUMBER OF GENERATORS OF AN ALGEBRA OVER A COMMUTATIVE RING

URIYA A. FIRST, ZINOVY REICHSTEIN, AND BEN WILLIAMS

ABSTRACT. A theorem of O. Forster says that if R is a noetherian ring of Krull dimension d, then every projective R-module of rank n can be generated by $d+n$ elements. S. Chase and R. Swan subsequently showed that this bound is sharp: there exist examples that cannot be generated by fewer than $d+n$ elements. We view projective R-modules as *R-forms* of the non-unital R-algebra where the product of any two elements is 0. The first two authors generalized Forster's theorem to forms of other algebras (not necessarily commutative, associative or unital); A. Shukla and the third author then showed that this generalized Forster bound is optimal for étale algebras.

In this paper, we prove new upper and lower bound on the number of generators of an R-form of a k-algebra, where k is an infinite field and R has finite transcendence degree d over k. In particular, we show that, contrary to expectations, for most types of algebras, the generalized Forster bound is far from optimal. Our results are particularly detailed in the case of Azumaya algebras. Our proofs are based on reinterpreting the problem as a question about approximating the classifying stack BG, where G is the automorphism group of the algebra in question, by algebraic spaces of a certain form.

1. INTRODUCTION

Throughout this paper k will denote a base field, A a finite-dimensional k-algebra, R a k-ring, and B a finitely generated R-algebra. By a *ring* we will always mean a commutative associative unital ring. In contrast, an *algebra* will not be required to be commutative, associative, or unital. We write $\mathrm{gen}_R(B)$ for the smallest possible number of generators of B as an R-algebra.

Recall that an *R-form* of A is an R-algebra B such that there exists a faithfully flat R-ring S for which $A \otimes_k S \simeq B \otimes_R S$ as S-algebras. Many important classes of algebras arise as forms of particular k-algebras. For example:

(i) R-forms of $A = k^n$ with the zero multiplication are projective R-modules of rank n.

(ii) R-forms of $A = k^n$ with componentwise multiplication are finite étale R-algebras of rank n.

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(iii) R-forms of $A = \mathrm{M}_s(k)$ are Azumaya algebras of degree s over R.
(iv) R-forms of the split octonion k-algebra are octonion algebras over R.

O. Forster [For64] showed that a projective module of rank n over a noetherian ring R can be generated by $\dim R + n$ elements, where $\dim R$ denotes the Krull dimension of R; see also [Swa67]. In [FR17, Theorem 1.2 and Proposition 4.1], the first two authors extended this result to algebras as follows.

Theorem 1.1 (Generalized Forster bound)**.** *Assume k is infinite. Let A be a finite-dimensional k-algebra. If R is a noetherian k-ring and B is an R-form of A, then* $\mathrm{gen}_R(B) \leqslant \dim R + \mathrm{gen}_k(A)$.

In case (i) above, Theorem 1.1 specializes to Forster's bound, since in this case R-forms of A are projective R-modules of rank n. Forster's bound was shown to be optimal by S. Chase and R. Swan [Swa62, Theorem 4]. More precisely, they showed that for every $n \geqslant 1$ and $d \geqslant 0$ there exists a d-dimensional rings R of finite type (i.e. finitely generated) over the field of real numbers and a projective R-module B of rank n such that $\mathrm{gen}_R(B) = d + n$.

In case (ii), Theorem 1.1 shows that $\mathrm{gen}(B) \leqslant \dim R + 1$ for every finite étale algebra B over R. This upper bound was recently shown to be optimal by A. Shukla and the third author [SW19]. Specifically, for every pair of integers $n \geqslant 2$ and $d \geqslant 0$, they constructed a finitely generated ring R over the field of real numbers, of dimension d, and a finite étale R-algebra B of rank n such that $\mathrm{gen}_R(B) = d + 1$.*

Question 1.2. Is the generalized Forster bound of Theorem 1.1 optimal for every finite-dimensional k-algebra A?

In this paper we will show that the answer to this question is "no". In particular, Theorem 1.1 tells us that

(iii) $\mathrm{gen}_R(B) \leqslant \dim R + 2$ for every Azumaya algebra B over R and
(iv) $\mathrm{gen}_R(B) \leqslant \dim R + 3$ for every octonion algebra B over R;

see [FR17, Corollary 4.2]. We will show that both of these bounds are far from optimal.

In the sequel it will be more natural for us to use the transcendence degree to measure the size of a k-ring R, rather than the Krull dimension. Note that in the most interesting case, where R is of finite type over k, they are the same: $\mathrm{trdeg}_k R = \dim R$. Our first main result is the following upper bound on the number of generators.

Theorem 1.3. *Assume that k is an infinite field, A is an n-dimensional k-algebra, and R is a k-ring of transcendence degree d. Let $\overline{k}$ be an algebraic closure of k and set $\overline{A} = A \otimes_k \overline{k}$. If the maximal dimension of a proper $\overline{k}$-subalgebra of $\overline{A}$ is $n_{\max}$, then*

$$\mathrm{gen}_R(B) \leqslant \left\lfloor \frac{d}{n - n_{\max}} \right\rfloor + n_{\max} + 1$$

for any R-form B of A.

*A similar example was independently found by M. Ojanguren (2017, unpublished).

If $n_{\max} \leqslant n-2$, as in examples (iii) and (iv) above, then Theorem 1.3 yields an asymptotically stronger upper bound on $\mathrm{gen}_R(B)$ than Theorem 1.1 (for large d).

Our second main result is a lower bound on $\mathrm{gen}_R(B)$ for certain rings R and certain R-forms of A. While this lower bound does not quite match the upper bound of Theorem 1.3, it is not far off in the sense that both are linear in $d = \mathrm{trdeg}_k R$.

Theorem 1.4. *Let k be a field of characteristic 0 and let A be an n-dimensional k-algebra such that the group scheme $G = \mathrm{Aut}_k(A)$ is not unipotent. Then there exists a positive integer ρ_G with the following property: For any positive integer d, there exist a k-ring R of finite type and an R-form B of A such that $\mathrm{trdeg}_k R = d$ and*

$$\mathrm{gen}_R(B) \geqslant \left\lfloor \frac{d + 2\dim G - \rho_G}{2n} \right\rfloor + 1.$$

Here ρ_G depends only on G and not on d, R or B. Moreover, if the dimension of G is greater than the dimension of its unipotent radical, then we may take $\rho_G = 4$.

Theorems 1.3 and 1.4 remain valid in the more general setting where we allow A and B to be *multialgebras*, rather than algebras. The same is true for most of the other results of this paper, where the structure of the algebra is not specified. Here, by a *multialgebra* over a ring R, we mean a module B together with a (not necessarily finite) collection of R-multilinear maps $\{m_i : B^{n_i} \to B\}_{i \in I}$, where each n_i is a non-negative integer.* For example, a usual binary (algebra) product on B can be specified by an R-bilinear map $B^2 \to B$, an involution by an R-linear map $B^1 \to B$, and a unit element in B by a map $B^0 \to B$ (which is always multilinear). A multialgebra structure on B naturally induces one on the tensor product $B \otimes_R S$ for any R-ring S. Subalgebras, forms and generating sets of multialgebras are defined in the obvious manner. Viewing B as a multialgebra is useful even in the case where B is an R-algebra in the usual sense, with a unit element. Here we may view B as a unital algebra or a non-unital algebra, and the multialgebra structure of B allows us to distinguish between these two settings. (See also Remark 1.6 below.) Aside from usual algebras, unital and non-unital, the multialgebras of greatest interest to us will be algebras with involution.

Our third main result sharpens the upper and lower bounds of Theorems 1.3 and 1.4 in the case of Azumaya algebras.

Theorem 1.5. *Let k be an infinite field.*

(a) Suppose R is a k-ring of transcendence degree d, and B is an Azumaya R-algebra of degree s. Then $\mathrm{gen}_R(B) \leqslant \left\lfloor \frac{d}{s-1} \right\rfloor + 2$.

(b) On the other hand, if $\mathrm{char}(k) = 0$, then for any $s \geqslant 3$ and $d \geqslant 0$ there exists a finitely generated k-ring R of transcendence degree d and an Azumaya R-algebra B of degree s such that $\mathrm{gen}_R(B) \geqslant \left\lfloor \frac{d}{2(s-1)} \right\rfloor + 2$.

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*This definition of multialgebra is more restrictive than the one used in [FR17].

A leading role in this paper will be played by the automorphism group $G = \mathrm{Aut}_k(A)$ and by the varieties U_r and Z_r of r-tuples $(a_1, \dots, a_r)$ of elements of A that do and do not generate A, respectively. It is well known that the category of R-forms of A is equivalent to the category of G-torsors over $\operatorname{Spec} R$, or equivalently, to the category of R-points of the classifying stack BG. On the other hand, the pairs $(B, (b_1, \dots, b_r))$, consisting of an R-form B of A and generators $b_1, \dots, b_r$ of the R-algebra B, are in a natural bijective correspondence with R-points of the quotient stack U_r/G; see Proposition 6.2. In fact, U_r/G is an algebraic space; see Remark 5.3. There is an evident morphism $U_r/G \to BG$ sending the pair $(B, (b_1, \dots, b_r))$ to B. An R-algebra B can be generated by r elements if and only if the R-point of BG corresponding to B can be lifted to U_r/G. Informally speaking, much of this paper revolves around the following question: how well does U_r/G approximate BG? The upper bounds on $\mathrm{gen}_R(B)$ in Theorems 1.3 and 1.5(a) are proved by constructing explicit liftings when $\mathrm{trdeg}_k R$ is suitably small. On the other hand, the lower bounds of Theorems 1.4 and 1.5(b) are proved by exhibiting topological obstructions to the existence of a such a lifting.

This paper is structured as follows. In Section 2, we collect notational conventions used throughout the paper. Sections 3 and 4 are devoted to general results on G-torsors, where G is an arbitrary linear algebraic group. In particular, in Section 4 we introduce the notion of a *d-versal torsor*, extending the notion of versality from [Ser03, Section 5] (see also [DR15]) for torsors over a field to torsors over a ring. In Section 5, we study the varieties U_r and Z_r of r-tuples of generators and non-generators in A. In Section 6, we prove an upper bound on $\mathrm{gen}_R(B)$ for any R-form B of A in terms of codimension of Z_r; see Theorem 6.1. All of the upper bounds on the number of generators in this paper are deduced from this theorem. We use it to give a new short proof of the generalized Forster bound (Theorem 1.1) in Section 7 and to prove Theorem 1.3 in Section 8. In Sections 9 and 10, we use Theorem 6.1 to prove new upper bounds on the number of generators of an Azumaya algebra (Theorem 1.4(a)), of an Azumaya algebra with involutions (Proposition 9.2) and of an octonion algebra (Proposition 10.1). Note that our upper bounds on the number of generators for these specific classes of algebras are stronger than those given by Theorem 1.3. Like the bound of Theorem 1.3, they are of the form

$$\mathrm{gen}_R(B) \leqslant \frac{d}{n - n_{\max}} + O(1),$$

but the additive constant is smaller. Section 11 is devoted to the topological preliminaries for the proofs of Theorems 1.4 and 1.5(b). In Section 12, we prove a version of the Lefschetz principle that will allow us to assume that k is a subfield of $\mathbb{C}$ in these proofs, thus opening the door to the use of non-algebraic techniques such as Betti cohomology. In Section 13, we prove Theorem 13.4, which is a general device for constructing examples of R-forms B of a k-algebra A requiring many generators. We use this device to prove Theorem 1.4 in Section 14 and Theorem 1.5(b) in Sections 15 and 16.

Remark 1.6. Let R be a k-ring and B be an R-algebra (in the usual sense) with a unit element 1 (e.g., an étale algebra, an Azumaya algebra, an octonion algebra, etc.). Unless otherwise specified, we will view B as a unital algebra, and similarly for algebras with involution. Note, however, that if B_0 is B viewed as a non-unital algebra and B_1 is B viewed as a unital algebra, then we often have

$$\mathrm{gen}_R(B_0) = \mathrm{gen}_R(B_1). \tag{1.1}$$

More specifically, it is easy to see that equality (1.1) holds if B is an R-form of any finite-dimensional k-algebra A that has an identity element 1 and satisfies the following condition: there is no map of $\overline{k}$-algebras: $A \otimes_k \overline{k} \to \overline{k}$. That is, the algebra $A \otimes_k \overline{k}$ does not have an augmentation.

In particular, (1.1) holds if B is an Azumaya algebra or an octonion algebra over R, as in Theorem 1.5 or Proposition 10.1. On the other hand, if $B = R$ (viewed as a rank 1 étale algebra over R), then (1.1) fails: $\mathrm{gen}_R(B_0) = 1$, whereas $\mathrm{gen}_R(B_1) = 0$.

Remark 1.7. We prove the lower bounds of Theorems 1.3 and 1.5(b) in characteristic 0 only. The main reason is that our arguments rely on the Affine Lefschetz Hyperplane Theorem (Theorem 11.1) which requires the characteristic-0 assumption. Since we are forced to work in characteristic 0, we liberally use non-algebraic techniques in our proofs of Theorems 1.4 and 1.5(b). In particular, we work with Betti cohomology of complex algebraic varieties and treat the complex points of the algebraic space U_r/G as a differentiable manifold. It is likely that the use of these non-algebraic techniques can be avoided by replacing Betti cohomology with another oriented (as axiomatized in [Pan03] and [Pan09]), bounded-below, $\mathbb{A}^1$-invariant cohomology theory of smooth varieties, for instance ℓ-adic étale cohomology. If one can prove an Affine Lefschetz Hyperplane Theorem for such a cohomology theory, then one should be able to establish lower bounds similar to Theorems 1.4 and 1.5(b) in positive characteristic.

Remark 1.8. The results of Forster [For64] and the first two authors [FR17] do not make any assumption on R beside it being noetherian and bound the number of generators of R-forms of A in terms of the Krull dimension of R, whereas Theorem 1.3 requires that R contain an infinite field k and bounds the number of generators in terms of $\mathrm{trdeg}_k R$. We do not know whether the assumptions of Theorem 1.3 can be relaxed to match those of [For64] and [FR17]. In particular, we do now know whether Theorem 1.5(a) remains valid for Azumaya algebras over an arbitrary noetherian ring R of Krull dimension d, even if the transcendence degree of R over a subfield k is greater than d or if R does not contain a field.

2. Notational conventions

Throughout this paper:

- k will denote a field. All schemes, varieties, algebraic groups, groups actions and morphisms are assumed to be defined over k.

By a variety (or equivalently, a k-variety) we shall mean a finite-type separated and reduced scheme over k.

- A will denote a finite-dimensional k-algebra, or more generally, a finite-dimensional k-multialgebra.

- n will denote $\dim A$, if not otherwise indicated.
- G will denote $\mathrm{Aut}(A)$, the automorphism group scheme of A defined over k.

By definition, G is a closed subgroup of $\mathrm{GL}(A) \cong \mathrm{GL}_n$. In positive characteristic G need not be smooth.

- R will denote a k-ring, i.e., a commutative associative unital k-algebra.
- B will denote an R-form of A. This means that there exists a faithfully flat R-ring S for which $A \otimes_k S \simeq B \otimes_R S$ as S-algebras.
- $\mathrm{gen}_R(B)$ will denote the minimal number of elements which generate B as an R-algebra (or multialgebra).
- V_1 will denote the affine space $\mathbb{A}(A)$ underlying our finite-dimensional k-algebra A. More generally, V_r will denote the r-fold direct product $V_1 \times_k \ldots \times_k V_1$ for any $r \geqslant 1$. If R is a k-ring, then R-points of V_r are in a natural bijection with r-tuples of elements in $A_R = A \otimes_k R$.
- Z_r will denote the closed subscheme of V_r consisting of r-tuples $a_1, \ldots, a_r \in A$ which do not generate A.

More precisely, Z_r is defined as follows. Denote the multialgebra operations on A by $\{m_i : A^{r_i} \to A\}_{i \in I}$ and recall that $n = \dim_k A$. Let W_r denote the collection of $\{m_i\}_{i\in I}$-monomials in the letters $x_1, \ldots, x_r$. That is, $W_r = \bigcup_{t=1}^{\infty} W_{r,t}$, where $W_{r,1} = \{x_1, \ldots, x_r\}$ and for any $t > 1$, $W_{r,t}$ is the set of formal expressions obtained by substituting elements from $W_{r,t-1}$ in the $\{m_i\}_{i\in I}$ in every possible way. Each $w \in W_r$ determines a morphism $w : V_r \to V_1$, given by substitution. By definition, $\overline{a} = (a_1, \ldots, a_r) \in V_r(k) = A^r$ generates A if and only if there exists $\overline{w} = (w_1, \ldots, w_n) \in (W_r)^n$ such that $w_1(\overline{a}), \ldots, w_n(\overline{a})$ form a k-basis of A. For every n-tuple of monomials $\overline{w} = (w_1, \ldots, w_n) \in (W_r)^n$, denote by $f_{\overline{w}}$ the composition of $(w_1, \ldots, w_n) : V_r \to V_n \cong \mathrm{M}_{n\times n}$ with the determinant morphism $\mathrm{M}_{n\times n} \to \mathbb{A}^1_k$, and let $Z_{\overline{w}} = f_{\overline{w}}^{-1}(0)$. The variety of non-generators Z_r is defined as the closed subscheme of V_r cut by the equations $f_{\overline{w}} = 0$ as $\overline{w}$ ranges over W_r.*

- $U_r = V_r - Z_r$, the open subscheme of V_r consisting of r-tuples $a_1, \ldots, a_r \in A$ which generate A.

The group scheme G acts on V_r from the left by $g \cdot (a_1, \ldots, a_r) = (g(a_1), \ldots, g(a_r))$, and the action restricts to Z_r and U_r.

All G-torsors in this paper are *left* G-torsors $\pi : T \to X$ such that π admits a section after base-changing along some faithfully flat quasi-compact morphism $X' \to X$ (that is, we consider left G-torsors for the fpqc topology).

- For a G-torsor $T \to \operatorname{Spec} R$ (with $G = \mathrm{Aut}_k(A)$ and R a k-ring), we let ${}^{T}\!A$ denote the twist of A by T.

Recall that ${}^{T}\!A$ is an R-form of A defined as follows. Elements of ${}^{T}\!A$ are R-points of the twisted variety ${}^{T}V_1$ over $\operatorname{Spec} R$, which is the quotient of $T \times V_1$ by the diagonal action of

*Note that Z_r may not be reduced, and the natural G-action on Z_r may not restrict to the associated reduced scheme Z_r^{red}. Indeed, set $r = 1$ and consider the example of the non-unital algebra $A = k[\varepsilon]/(\varepsilon^2)$.

G. When A is an algebra (in the usual sense), the R-algebra operations on ${}^{T}A$ are given by twisting each of the following operations on A by T:

- ∘ addition, $+\colon V_1 \times V_1 \to V_1$, given by $(a_1, a_2) \mapsto a_1 + a_2$;
- ∘ scalar multiplication $\mathbb{A}^1 \times V_1 \to V_1$ given by $(c, a) \mapsto ca$; and
- ∘ algebra multiplication $\cdot\colon V_1 \times V_1 \to V_1$ given by $(a_1, a_2) \mapsto a_1 \cdot a_2$.

Here G acts trivially on $\mathbb{A}^1$. For details, see [Gir71, Section III.2.3]. The same construction works for multialgebras.

- $\mathrm{H}^i(X)$ denotes the singular cohomology of a topological space X with coefficients in the ring $\mathbb{Z}$.
- $\mathrm{H}^*(X)$ denotes the graded ring $\oplus_{i=0}^{\infty}\mathrm{H}^i(X)$.

If different coefficients are required, they will be specified, e.g., $\mathrm{H}^i(X;\mathbb{Q})$.

- If Y is a variety over a field k and k is equipped with an embedding $k \hookrightarrow \mathbb{C}$, then $Y(\mathbb{C})$ will denote the topological space underlying the associated analytic space of $Y_{\mathbb{C}}$, [Ser56, Sec. 2]. If Y is such a variety, then notation $\mathrm{H}^i(Y)$ is taken to mean $\mathrm{H}^i(Y(\mathbb{C}))$.
- If Γ is a topological group, then $E\Gamma \to B\Gamma$ will denote a universal principal Γ-bundle, and if Γ acts on a space X, then $\mathrm{H}^i_{\Gamma}(X)$ denotes the Borel equivariant cohomology group $\mathrm{H}^i((X \times E\Gamma)/\Gamma)$, [Hsi75, Chapter III].

The notation $\mathrm{H}^i_G(Y)$ will be used in place of $\mathrm{H}^i_{G(\mathbb{C})}(Y(\mathbb{C}))$ when G is a linear algebraic group acting on a variety Y over a field $k \hookrightarrow \mathbb{C}$.

- The notation $\mathbb{G}_m$ will be reserved for the multiplicative group-scheme GL_1.

3. Torsors and the variety of maximal-rank matrices

Denote the affine space of $a \times b$-matrices over k by $\mathrm{M}_{a\times b}$. If $b \geqslant a$, we will denote the Zariski open subvariety of $a \times b$ matrices of rank a by $\mathrm{M}^0_{a\times b}$. We regard GL_n as an algebraic group over k. The purpose of this section is to prove the following result.

Proposition 3.1. *Let G be a closed subgroup scheme of GL_n, R be a k-ring, and $\tau\colon T \to \operatorname{Spec} R$ be a G-torsor. Then there exists an integer $m \geqslant 1$ and a G-equivariant k-morphism $F\colon T \to \mathrm{M}^0_{n\times nm}$. Here G acts on $\mathrm{M}^0_{n\times nm}$ via multiplication on the left.*

Our proof of Proposition 3.1 will rely on the following two lemmas.

Lemma 3.2 (cf. [RT20, Lemma 4.1(a)])**.** *Let G be a closed subgroup scheme of GL_n, R be a local k-ring, and $\tau\colon T \to \operatorname{Spec} R$ be a G-torsor. Then there exists a G-equivariant k-morphism $f\colon T \to \mathrm{GL}_n$.*

Proof. Consider the GL_n-torsor $\tau'\colon T' = T \times^G \mathrm{GL}_n \to \operatorname{Spec} R$. By Hilbert's Theorem 90, τ' is split. Thus T' is GL_n-equivariantly isomorphic to $\mathrm{GL}_n \times_k \operatorname{Spec} R$. Projecting to the first component, we obtain a GL_n-equivariant morphism $T' \to \mathrm{GL}_n$. Composing this morphism with the natural G-equivariant map $T \to T'$, we obtain a desired G-equivariant morphism $T \to \mathrm{GL}_n$. □

Lemma 3.3. *Let R be a k-ring, G be a closed subgroup scheme of GL_n, $\tau\colon T \to \operatorname{Spec} R$ be a G-torsor, P be a prime ideal of R, R_P be the localization of R at P, and let $\tau_P : T_P \to \operatorname{Spec} R_P$ denote the pullback of τ under $\operatorname{Spec} R_P \to \operatorname{Spec} R$. Then there exists a G-equivariant morphism $F_P\colon T \to \mathrm{M}_{n\times n}$ mapping T_P to $\mathrm{GL}_n = \mathrm{M}^0_{n\times n}$.*

Proof. Applying Lemma 3.2 to the G-torsor $\tau_P\colon T_P \to \operatorname{Spec} R_P$, we obtain a G-equivariant morphism $f_P\colon T_P \to \mathrm{GL}_n = \mathrm{M}^0_{n\times n}$. Our goal is to extend this G-equivariant morphism to $F_P\colon T \to \mathrm{M}_{n\times n}$.

Since G and $\operatorname{Spec} R$ are affine schemes, so are T and T_P, and we may consider their coordinate rings. Let $\{x_{ij}\}_{i,j\in\{1,\ldots,n\}}$ denote the coordinate functions on $\mathrm{M}_{n\times n}$. Then $k[\mathrm{M}_{n\times n}]$ is the polynomial k-algebra on the x_{ij} and $k[\mathrm{GL}_n] = k[\mathrm{M}_{n\times n}][\det(x_{ij})^{-1}]$. Let $f_P^*\colon k[\mathrm{GL}_n] \to k[T_P] = k[T] \otimes_R R_P$ be the homomorphism of k-algebras induced by f_P. Then $f_P^*(x_{ij}) = \alpha_{ij}^{-1}\varphi_{ij}$ for some $\varphi_{ij} \in k[T]$, $\alpha_{ij} \in R - P$. After writing each fraction $\alpha_{ij}^{-1}\varphi_{ij}$ under a common denominator $\alpha = \prod_{i,j}\alpha_{ij} \in R - P$, we may assume without loss of generality that all the α_{ij} are equal and simply denote them by α.

Viewing T and GL_n as sheaves on the large fpqc site of $\operatorname{Spec} k$, and the f_{ij} (resp. α) as morphisms from T (resp. $\operatorname{Spec} R$) to $\mathcal{O}_{\operatorname{Spec} k}$, we see that, on the level of sections, f_P is given by

$$f_P\colon t \mapsto \begin{bmatrix} \alpha(\tau(t))^{-1}\varphi_{11}(t) & \ldots & \alpha(\tau(t))^{-1}\varphi_{1n}(t) \\ \vdots & & \vdots \\ \alpha(\tau(t))^{-1}\varphi_{n1}(t) & \ldots & \alpha(\tau(t))^{-1}\varphi_{nn}(t) \end{bmatrix}.$$

In the case where R is an integral domain, we can now define F_P by clearing denominators in the above matrix, i.e., set $F_P = \alpha \cdot f_P$, thus completing the proof. Since we are not assuming that R is an integral domain, we will proceed more cautiously and define

$$F_P\colon t \mapsto \beta(\tau(t)) \begin{bmatrix} \varphi_{11}(t) & \ldots & \varphi_{1n}(t) \\ \vdots & & \vdots \\ \varphi_{n1}(t) & \ldots & \varphi_{nn}(t) \end{bmatrix} \tag{3.1}$$

for some $\beta \in R - P$, to be specified later. Since $\beta(\tau(t)) \cdot \det(\varphi_{ij}(t))$ is invertible for every section t of T factoring through T_P, the morphism F_P restricts to a morphism from T_P to GL_n.

It remains to show that for a suitable choice of β, the morphism F_P is G-equivariant. We check this on the level of coordinate rings. The group action $G \times T \to T$ induces a morphism $\Delta\colon k[T] \to k[G] \otimes_k k[T]$ of k-algebras. By abuse of notation we will also write Δ for the homomorphism $k[T_P] \to k[G] \otimes_k k[T_P]$ induced by the G-action on T_P and similarly for the G-action on $\mathrm{M}_{n\times n}$ and GL_n. Writing g_{ij} for the image of $x_{ij} \in k[\mathrm{GL}_n]$ in $k[G]$, we have $\Delta(x_{ij}) = \sum_{s=1}^n g_{is} \otimes x_{sj}$ in $k[G] \otimes_k k[\mathrm{M}_{n\times n}]$. Since f_P is G-equivariant, we have

$$\begin{aligned} \sum_{s=1}^n g_{is} \otimes \alpha^{-1}\varphi_{sj} = \sum_{s=1}^n g_{is} \otimes f_P^*(x_{sj}) &= (\mathrm{id}_{k[G]} \otimes_k f_P^*)\Delta(x_{ij}) \\ &= \Delta(f_P^*(x_{ij})) = \Delta(\alpha^{-1}\varphi_{ij}) = \alpha^{-1}\Delta(\varphi_{ij}) \end{aligned}$$

in $k[G] \otimes k[T_P]$, where the last equality holds because the action map $G \times T_P \to T_P$ is an R_P-morphism. Consequently, there is $\beta \in R - P$ such that

$$\sum_{s=1}^{n} g_{is} \otimes \beta\varphi_{sj} = \beta\Delta(\varphi_{ij}) = \Delta(\beta\varphi_{ij})$$

in $k[G] \otimes k[T]$ for all i, j. This is the β we will use in the definition of F_P; see (3.1). Note that if R is an integral domain, then we may take $\beta = 1$. Formula (3.1) can now be restated as $F_P^*(x_{ij}) = \beta\varphi_{ij}$, and

$$\begin{aligned}(\mathrm{id}_{k[G]} \otimes_k F_P^*)\Delta(x_{ij}) &= (\mathrm{id}_{k[G]} \otimes_k F_P^*)(\sum_{s=1}^{n} g_{is} \otimes x_{sj}) \\ &= \sum_{s=1}^{n} g_{is} \otimes \beta\varphi_{sj} = \Delta(\beta\varphi_{ij}) = \Delta(F_P^*(x_{ij})).\end{aligned}$$

This shows that F_P is G-equivariant, as claimed. □

Proof of Proposition 3.1. For every $P \in \operatorname{Spec} R$, let $F_P : T \to \mathrm{M}_{n\times n}$ be a G-equivariant morphism sending T_P to GL_n, as in Lemma 3.3. Let $V_P = F_P^{-1}(\mathrm{GL}_n)$. Then V_P is a Zariski open subset of T containing T_P. As P ranges over $\operatorname{Spec} R$, the open subsets V_P cover T. Since T is affine, it is quasi-compact, and thus covered by $V_{P_1}, \dots, V_{P_m}$ for some $P_1, \dots, P_m \in \operatorname{Spec} R$. Then $F = (F_{P_1}, \dots, F_{P_m})\colon T \to \prod_{i=1}^{m} \mathrm{M}_{n\times n} = \mathrm{M}_{n\times mn}$ is a G-equivariant morphism whose image is contained in $\mathrm{M}^0_{n\times mn}$. □

4. Versal group actions

Let k be a field, G be a group scheme over k, and $\mathscr{C}$ be a class of k-schemes. A G-scheme is a k-scheme equipped with a G-action $G \times_k X \to X$. We will say that a G-scheme X is *versal for* $\mathscr{C}$ if for every G-torsor $\tau\colon T \to Y$ with $Y \in \mathscr{C}$, there exists a G-equivariant k-morphism $T \to X$. We will be particularly interested in the case where $\mathscr{C}$ is the class of affine k-schemes $Y = \operatorname{Spec} R$ with $\operatorname{trdeg}_k R \leqslant d$. If X is versal with respect to this class of k-schemes, we will say that X is *d-versal*.

Remark 4.1. The notion of versality has been previously studied in the case where $\mathscr{C}$ consists of schemes of the form $\operatorname{Spec} K$ for some field K containing k; see, [Ser03, Section 5] or [DR15]. Our definition of d-versality here is analogous to "weak versality" in [DR15]. For notational simplicity we will use the term "d-versal" instead of "weakly d-versal", even though the latter may be more accurate.

Proposition 4.2. *Let G be an affine algebraic group over an infinite field k, $\rho : G \to \mathrm{GL}(V)$ be a faithful finite-dimensional representation, and Z be a closed G-subscheme of V of dimension $\leqslant \dim V - (d+1)$ for some integer $d \geqslant 0$. Then $U = V - Z$ is a d-versal G-scheme.*

Remark 4.3. It is well known that for any linear group scheme G over k and any $d \geqslant 0$, there exists a faithful representation $G \to \mathrm{GL}(V)$ and a closed G-invariant subvariety

$Z \subseteq V$ of co-dimension $> d$ such that $V - Z$ is the total space of a torsor $(V - Z) \to X$ for some scheme X; see [EG98, Lemma 9]. In fact, we may even assume that X is a quasi-projective variety; see [Tot99, Remark 1.4]. In particular, Proposition 4.2 shows that d-versal G-schemes with a free action of G exist for every $d \geqslant 0$ and every linear algebraic group G.

Proof of Proposition 4.2. Recall that $\rho : G \to \mathrm{GL}(V)$ is a closed embedding [Mil17, Theorem 5.34]. Let R be a k-ring of transcendence degree $\leqslant d$, and let $\tau \colon T \to \operatorname{Spec} R$ be a G-torsor. Our goal is to establish the existence of a G-equivariant morphism $f : T \to U$.

Write $n = \dim V$ and let GL_n, $\mathrm{M}_{n\times m}$ and $\mathrm{M}^0_{n\times m}$ be as in Section 3. We begin by identifying V with $\mathrm{M}_{n\times 1}$, $\mathrm{GL}(V)$ with GL_n and G with a closed subgroup of GL_n via ρ. Observe that $\mathrm{GL}(V) = \mathrm{GL}_n$ acts on $\mathrm{M}_{n\times mn}$ by multiplication on the left, and hence, so does $G \subset \mathrm{GL}_n$, whereas GL_{mn} acts on $\mathrm{M}_{n\times mn}$ via multiplication on the right. These actions commute, and both of them leave $\mathrm{M}^0_{n\times mn}$ invariant. Note also that $\mathrm{M}^0_{n\times mn}$ is a homogeneous space under the action of GL_{mn} for all $m \geqslant 1$.

Proposition 3.1 tells us that there exists a G-equivariant morphism $F \colon T \to \mathrm{M}^0_{n\times mn}$ for some integer $m \geqslant 1$. Let Z' be the preimage of $Z \subset V$ under the projection map $p\colon \mathrm{M}^0_{n\times mn} \to V$ to the first column. (Recall that each column of $\mathrm{M}_{n\times mn}$ is G-equivariantly, identified with V.) In other words, Z' consists of $n \times mn$ matrices of rank n whose first column lies in Z. Given $s \in \mathrm{GL}_{mn}(k)$, let $r_s : \mathrm{M}^0_{n\times mn} \to \mathrm{M}^0_{n\times mn}$ denote the G-equivariant morphism given by multiplying with s on the right, and let sT denote T, regarded as an $\mathrm{M}^0_{n\times mn}$-scheme via $r_s \circ F$.

Claim. Let X and Y be closed G-subschemes of $\mathrm{M}^0_{n\times mn}$ defined over k. Assume $\dim X + \dim Y < mn^2 + \dim G$. Then there exists a non-empty Zariski open subset $W_{X,Y} \subset \mathrm{GL}_{mn}$ such that $sX \cap Y = \emptyset$ for every $s \in W_{X,Y}(k)$.

Note that mn^2 is the dimension of $\mathrm{M}^0_{n\times mn}$. Assume for a moment that this claim is established. Let $Y := Z'$ and X be the intersection of the scheme theoretic image of $F : T \to \mathrm{M}_{n\times mn}$ (see [Sta20, Tag 01R7]) with $\mathrm{M}^0_{n\times mn}$. Then

$$\dim X = \operatorname{trdeg}_k k(X) \leqslant \dim \operatorname{trdeg}_k k(T) = \operatorname{trdeg}_k R + \dim G \leqslant d + \dim G,$$

where $k(X)$ denotes the ring of rational functions on X (which is a field if X is integral), and similarly for $k(T)$. On the other hand,

$$\dim Z' = \dim Z + n(mn-1) < (\dim V - d) + n(mn-1) = n - d + n(mn-1) = mn^2 - d.$$

Thus, $\dim X + \dim Y < mn^2 + \dim G$, and the claim applies to this choice of X and Y. Since GL_{mn} is rational, and k is an infinite field, $W_{X,Y}$ has a k-point $s \in W_{X,Y}(k)$. After replacing F by $r_s \circ F$, we may assume that the image of T in $\mathrm{M}^0_{n\times mn}$ is disjoint from Z'. Composing F with the projection $p\colon \mathrm{M}^0_{n\times mn} \to V$ to the first column, we obtain a desired G-equivariant morphism $f = p \circ F \colon T \to U$.

It thus remains to prove the claim. Consider the natural projection $\pi\colon I \to \mathrm{GL}_{mn}$, where $I \subset \mathrm{GL}_{mn} \times X \times Y$ is the incidence scheme consisting of triples (g, x, y) such that $x \cdot g = y$. We take $W_{X,Y}$ to be the complement of the Zariski closure of $\pi(I)$ in GL_{mn}. By definition, $W_{X,Y}$ is a Zariski open subset of GL_{mn} defined over k. We only need to show that it is non-empty, or equivalently, that π is not dominant. Since π is quasi-compact, by [Gro65, Proposition 2.3.7], we may pass to the algebraic closure of k in order to check this, and thus assume that k is algebraically closed.

It is harmless to replace G with the identity connected component of G^{red}, so assume G is reduced an connected. Since k is algebraically closed and G is reduced, X^{red} and Y^{red} are G-varieties [Bri17, Proposition 2.5.1(1)], so we may replace X and Y with their reductions. We may also assume that X and Y are irreducible. Indeed, if $X_1, \ldots, X_\alpha$ are the irreducible components of X, and $Y_1, \ldots, Y_\beta$ are the irreducible components of Y, then

$$W_{X,Y} = \bigcap_{i=1}^{\alpha} \bigcap_{j=1}^{\beta} W_{X_i,Y_j}.$$

As $\mathrm{M}^0_{n\times mn}$ is irreducible, it is enough to show that each W_{X_i,Y_j} is nonempty. Note that each X_i and Y_j is a G-variety because G is connected.

Now that k is algebraically closed and X and Y are irreducible varieties, we may apply the Kleiman Transversality Theorem, see [Kle74, Corollary 4(i)] or [Har77, Theorem 10.8], which tells us that there exists an open dense subset W' of GL_{mn} such that for every $s \in W'(k)$, the intersection $sX \cap Y$ in $\mathrm{M}^0_{n\times mn}$ is either

(i) empty, or
(ii) equidimensional of dimension $\dim X + \dim Y - \dim \mathrm{M}^0_{n\times mn} < \dim G$.

Since the GL_{mn}-action and the G-action on $\mathrm{M}^0_{n\times mn}$ commute, sX and Y are both G-subvarieties of $\mathrm{M}^0_{n\times mn}$, and so is their intersection $sX \cap Y$. On the other hand, since the G-action on $\mathrm{M}^0_{n\times mn}$ (via multiplication on the left) has trivial stabilizers, every G-invariant subvariety of $\mathrm{M}_{n\times mn}$ has dimension $\geqslant \dim G$. This shows that (ii) is impossible. Thus, $sX \cap Y = \emptyset$ for every $s \in W'(k)$, which means that $W' \subseteq W_{X,Y}$. Since W' is dense in GL_{mn}, we have $W_{X,Y} \neq \emptyset$, as desired. □

5. Preliminaries on varieties of generators and non-generators

Let A be a finite-dimensional (multi)algebra over a field k. Recall from Section 2 that we denote the affine space of ordered r-tuples of elements in A by V_r, the closed subscheme of r-tuples not generating A by $Z_r \subset V_r$ and the open subscheme of r-tuples of generators by $U_r = V_r - Z_r$. In this section, we will discuss some properties of these varieties that will be of interest to us in the sequel. We write A_R for $A \otimes_k R$.

Lemma 5.1. *Let R be a k-ring. Then $U_r(R)$ is the set of r-tuples $(b_1, \ldots, b_r) \in V_r(R) = A_R^r$ which generate A_R as an R-algebra.*

Proof. Let $\overline{b} = (b_1, \ldots, b_r) \in V_r(R)$ and let $t : \operatorname{Spec} R \to V_r$ be the corresponding morphism.

Suppose that $\overline{b}$ generates A_R. Let $P \in \operatorname{Spec} R$, let $A(P) = A \otimes_k \operatorname{Frac}(R/P)$ and let $b_i(P)$ denote the image of b_i in $A(P)$. Then $\overline{b}(P) := (b_1(P), \ldots, b_r(P))$ generates $A(P)$. By the construction of Z_r in Section 2, there is $\overline{w} \in (W_r)^n$ such that $f_{\overline{w}}(\overline{b}(P)) \neq 0$ in $\operatorname{Frac}(R/P)$. This means that the image of t does not meet the P-fiber of Z_r. As this holds for all $P \in \operatorname{Spec} R$, the image of $t : \operatorname{Spec} R \to V_r$ is contained $V_r - Z_r = U_r$ and $\overline{b} \in U_r(R)$.

Conversely, assume $\bar{b} \in U_r(R)$. Then for every $P \in \operatorname{Spec} R$, we have $t(P) \notin Z_r$. This means that there is $\overline{w} \in (W_r)^n$ such that $f_{\overline{w}}(\bar{b}(P)) \neq 0$, hence $\bar{b}(P)$ generates $A(P)$. This holds for all $P \in \operatorname{Spec} R$, so by [FR17, Lemma 2.1], $\bar{b}$ generates A_R. □

Recall from Section 2 that the group scheme $G = \operatorname{Aut}_k(A)$ acts on V_r from the left, and the action restricts to Z_r and U_r.

Lemma 5.2. *The action of G on U_r is scheme-theoretically free, i.e., the action map*

$$\Phi : G \times U_r \longrightarrow U_r \times U_r, \tag{5.1}$$

given by $\Phi(g, (a_1, \ldots, a_r)) = ((g(a_1), \ldots, g(a_r)), (a_1, \ldots, a_r))$ is a closed embedding.

Proof. By [Gro64, Cor. 18.12.6], showing that Φ is a closed embedding is equivalent to showing that Φ is a proper monomorphism. To see that Φ is a monomorphism, it suffices to test that

$$\Phi(R) : G(R) \times U_r(R) \to U_r(R) \times U_r(R) \tag{5.2}$$

is injective for every k-ring R. Here, thanks to Lemma 5.1, $U_r(R)$ is the set of all r-tuples $(b_1, \ldots, b_r)$ of generators for A_R. The action of $G(R) = \operatorname{Aut}_R(A_R)$ on $U_r(R)$ is free because $g \in G(R)$ can fix $(b_1, \ldots, b_r) \in U_r(R)$ if and only if g fixes the R-algebra generated by the b_i, i.e., the entire algebra A_R. On the other hand, the element g fixes the entire algebra A_R if and only if g is the identity automorphism. It follows that for any R, the map $\Phi(R)$ in (5.2) is an injection, as claimed.

It remains to show that Φ is a proper map. We use the valuative criterion for properness. Suppose R is a valuation ring over k with fraction field K, and let φ, ψ be morphisms as in the following diagram. We need to show the existence of the dotted arrow.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\varphi"] \arrow[d] & G \times U_r \arrow[d] \\ \operatorname{Spec} R \arrow[r, "\psi"] \arrow[ur, dashed] & U_r \times U_r. \end{tikzcd}$$

We remark that $A_R \subset A_K$, and $U_r(R) \subset U_r(K)$. The arrow φ represents an automorphism $g \in G(K)$ and an r-tuple $(a_1, \ldots, a_r) \in U_r(K)$; the arrow ψ represents a pair of r-tuples in $U_r(R)$, and commutativity of the diagram implies that these r-tuples are $(ga_1, \ldots, ga_r)$ and $(a_1, \ldots, a_r)$. In particular, both $(ga_1, \ldots, ga_r)$ and $(a_1, \ldots, a_r)$ lie in $U_r(R)$. In order to produce the lift, we must show that g is defined over R. In other words, if $e_1, \ldots, e_n$ is a k-basis of A, we claim that $g(e_i) \in A_R$ for every $i = 1, \ldots, n$. This will prove that $g \in \operatorname{GL}_n(R) \cap G(K) = G(R)$.

To prove the claim, note that since $(a_1, \ldots, a_r) \in U_r(R)$, it is possible to find polynomials $f_1, \ldots, f_n$ in r non-commuting variables with coefficients in R* such that

$$f_i(a_1, \ldots, a_r) = e_i \quad \forall i \in \{1, \ldots, n\}.$$

*When A is a multialgebra, the f_i should be taken to be formal linear combinations of members of W_r from Section 2.

Now consider the result of applying g to e_i. Since g is known to lie in $G(K)$, $g(e_i)$ is a priori an element of A_K, but using the polynomials f_i, we see that

$$g(e_i) = g(f_i(a_1, \dots, a_r)) = f_i(ga_1, \dots, ga_r)$$

actually lies in A_R, as claimed. This allows us to produce a lift, so the valuative criterion for properness is satisfied. ☐

Remark 5.3. Regard U_r/G as a sheaf on the large fppf site of $\operatorname{Spec} k$. Then U_r/G is an algebraic space. Indeed, since G is flat of finite type over k, we may regard U_r/G as an Artin stack, and because the action is free, the G-stabilizer of every section of U_r is trivial.

6. An upper bound on the number of generators

The purpose of this section is to prove the following theorem, from which we will derive Theorems 1.3 and 1.5(a). It highlights the important role that the varieties Z_r and U_r play in this setting. The notation is as in Section 2.

Theorem 6.1. *Let A be a finite-dimensional algebra over an infinite field k, and $c_A(r)$ be the codimension of Z_r in V_r, i.e., $c_A(r) = r \dim A - \dim Z_r$. Let R be a k-ring of transcendence degree d. If $c_A(r) > d$, then $\operatorname{gen}_R(B) \leqslant r$ for any R-form B of A.*

Recall from the Introduction that an R-form of A is an R-algebra B for which there exists a faithfully flat R-ring S such that $A \otimes_k S \simeq B \otimes_R S$ as S-algebras. It is well known that the category of G-torsors over $\operatorname{Spec} R$ is equivalent to the category of R-forms of A, functorially in R. The equivalence is given by sending an R-algebra B to $\operatorname{Hom}_{R\text{-alg}}(A_R, B)$, regarded as an R-scheme and endowed with the *left* G_R-action given by $g \cdot \psi = \psi \circ g^{-1}$ on sections. Conversely, a G-torsor $T \to \operatorname{Spec} R$ corresponds to the twisted algebra ${}^T\!A$ defined in Section 2.

Proposition 6.2. *Let R be a k-ring, let B be an R-form of A, and let $T \to \operatorname{Spec} R$ be its associated G-torsor. Then there is a natural (in R) bijective correspondence between the following sets:*

(a) G-equivariant k-morphisms from T to U_r, and
(b) r-tuples $(b_1, \dots, b_r) \in B^r$ generating B as an R-algebra.

Theorem 6.1 is an immediate consequence of this proposition. Indeed, suppose that $c_A(r) > d$. Then Proposition 4.2, with $V = V_r$ and $Z = Z_r$, tells us that U_r is d-versal. Let R be a k-ring of transcendence degree $\leqslant d$, B be an R-form of A and $T \to \operatorname{Spec} R$ be the G-torsor associated to B. Since U_r is d-versal, there exists a G-equivariant morphism $T \to U_r$, and by Proposition 6.2, this means that B can be generated by r elements, i.e., $\operatorname{gen}_R(B) \leqslant r$.

Our proof of Proposition 6.2 will rely on the following lemmas.

Lemma 6.3. *Let X and Y be k-schemes, G a group scheme over k, and*

$$\begin{tikzcd} V \arrow[d, "\alpha"'] & W \arrow[l, "\pi"'] \arrow[d, "\beta"] \\ X & Y \arrow[l, "\overline{\pi}"] \end{tikzcd}$$

be a cartesian diagram over k, where α and β are G-torsors and π is G-equivariant. Then there is a natural (in the morphism $X \to Y$) bijection between sections $\overline{f}\colon X \to Y$ of $\overline{\pi}$ and G-equivariant sections $f\colon V \to W$ of π such that $\overline{f} \circ \alpha = \beta \circ f$ whenever f corresponds to $\overline{f}$.

Proof. It is enough to prove the corresponding statement for sheaves over the (large) fpqc site of $\operatorname{Spec} k$. To that end, it is enough to show that for every k-scheme S, there is a unique (and hence natural in S, X, Y) bijection between $G(S)$-equivariant $\pi(S)$-sections $f : V(S) \to W(S)$ and $\overline{\pi}(S)$-sections $\overline{f} : X(S) \to Y(S)$ such that $\overline{f} \circ \alpha(S) = \beta(S) \circ f$ whenever f corresponds to $\overline{f}$. That is, it is enough to prove the set-theoretic analogue of the lemma, which is straightforward. □

Lemma 6.4. *Let R be a k-ring and $T \to \operatorname{Spec} R$ be a G-torsor. Then there is a natural isomorphism between the twisted R-(multi)algebra $B = {}^T A$ and the R-(multi)algebra $\operatorname{Mor}_G(T, V_1)$ of G-equivariant morphisms $T \to V_1$. Here "natural" means that if S is an R-ring, then this isomorphism is compatible with the base change from R to S.*

The (multi)algebra structure on $\operatorname{Mor}_G(T, V_1)$ is induced from the (multi)algebra structure on A. For example, to add G-equivariant morphisms $\varphi_1\colon T \to V_1$ and $\varphi_2\colon T \to V_1$, we compose $(\varphi_1, \varphi_2)\colon T \to V_1 \times V_1$ with the addition map $+\colon V_1 \times V_1 \to V_1$.

Proof. Consider the cartesian diagram

$$\begin{tikzcd} T \arrow[d, "\alpha"'] & T \times V_r \arrow[l, "\pi_r"'] \arrow[d, "\beta_r"] \\ \operatorname{Spec} R & {}^T V_r \arrow[l, "\overline{\pi_r}"] \end{tikzcd}$$

where $\pi_r\colon T \times V_r \to T$ is projection to the first component. Now set $r = 1$. A G-equivariant morphism $\varphi\colon T \to V_1$ gives rise to a G-equivariant section $(\mathrm{id}, \varphi)\colon T \to T \times V_1$ of π_1. The latter descends to a section $b\colon \operatorname{Spec} R \to {}^T V_1$, which we view as an element of B. This gives us a map $F\colon \operatorname{Mor}_G(T, V_1) \to B$ taking φ to b. By Lemma 6.3, F is an isomorphism of sets.

Using the definition of the (multi)algebra structure on $B = {}^T A$, one readily checks that F is in fact a homomorphism (and thus an isomorphism) of (multi)algebras. For example, to show that F is a homomorphism of additive groups, we consider the diagram:

$$\begin{tikzcd} T \arrow[d, "\alpha"'] & T \times V_2 \arrow[l, "\pi_2"'] \arrow[r, "(\mathrm{id},+)"] \arrow[d, "\beta_2"] & T \times V_1 \arrow[d, "\beta_1"] \\ \operatorname{Spec} R & {}^T V_2 \arrow[l, "\overline{\pi_2}"] \arrow[r, "+"] & {}^T V_1 \end{tikzcd}$$

Note that by definition, $V_2 = V_1 \times V_1$, and thus ${}^T V_2 = {}^T V_1 \times {}^T V_1$. Similar arguments with $+$ replaced by scalar multiplication and (multi)algebra operations in A show that F is a homomorphism of (multi)algebras.

Finally, to show that F is natural in R, we consider a k-ring morphism $R \to S$ and examine the diagram

$$\begin{tikzcd} T_S \arrow[r] \arrow[d, "\alpha_S"'] & T \arrow[d, "\alpha"'] & T \times V_1 \arrow[l, "\pi_1"'] \arrow[d, "\beta_1"] \\ \operatorname{Spec} S \arrow[r] & \operatorname{Spec} R & {}^T V_1 \arrow[l, "\overline{\pi_1},"] \end{tikzcd}$$

where the square on the left is Cartesian. Since ${}^{T_S} V_1 \simeq ({}^T V_1) \times_{\operatorname{Spec} R} \operatorname{Spec} S$, the S-points of the R-scheme ${}^T V_1$ are in a natural bijection with the S-points of the S-scheme ${}^{T_S} V_1$. $\square$

Proof of Proposition 6.2. By Lemma 6.4, an r-tuple $b = (b_1, \ldots, b_r)$ of elements of $B = {}^T A$ may be viewed as an r-tuple of G-equivariant maps $\varphi_1, \ldots, \varphi_r \colon T \to V_1$ or equivalently, as a G-equivariant morphism $\varphi = (\varphi_1, \ldots, \varphi_r) \colon T \to V_r = (V_1)^r$. It remains to prove the following.

Claim. $b_1, \ldots, b_r$ generate B as an R-algebra if and only if $\varphi(T)$ is contained in U_r.

To prove this claim, let S/R be a faithfully flat ring extension which splits T. By functoriality ${}^{T_S} A = ({}^T A) \otimes_R S$. Since S is faithfully flat over R, the elements $b_1, \ldots, b_r$ generate ${}^T A$ over R if and only if they generate ${}^{T_S} A$ over S. In other words, without loss of generality, we may replace R by S and thus assume that T is split over R, i.e., that T has a section $\psi \colon \operatorname{Spec} R \to T$. This induces an isomorphism $G \times \operatorname{Spec} R \to T$ given sectionwise by $(g, x) \mapsto g \cdot \psi(x)$, and thus a set bijection $B = {}^T A \simeq {}^{G \times \operatorname{Spec} R} A = A_R$. Thus, we have an R-algebra isomorphism $\operatorname{Mor}_G(T, V_1) \simeq A_R$ which, after unfolding the definitions, assumes the following simple form: a G-equivariant morphism $\varphi_i \colon T \to V_1$ corresponds to the element $b_i = \varphi_i \circ \psi \colon \operatorname{Spec} R \to V_1$ of A. By Lemma 5.1, $b_1, \ldots, b_r$ generate $B = A$ if and only if the image of

$$b = (b_1, \ldots, b_r) = (\varphi_1 \circ \psi, \ldots, \varphi_r \circ \psi) = \varphi \circ \psi \colon \operatorname{Spec} R \to V_r$$

lies in U_r. Since $G \times_k \operatorname{Spec} R$ is isomorphic to T via $(g, x) \mapsto g \cdot \psi(x)$, $\varphi \colon T \to V_r$ is a G-equivariant map, and U_r is a G-invariant subvariety of V_r, we see that $\varphi(T)$ lies in U_r if and only if $(\varphi \circ \psi)(\operatorname{Spec} R) = b(\operatorname{Spec} R)$ lies in U_r. This completes the proof of the claim and thus of Proposition 6.2. $\square$

7. A new proof of the generalized Forster bound

We now apply Theorem 6.1 to give a short proof of Theorem 1.1, which is conceptually simpler than the proof given in [FR17]. Our proof will rely on the following.

Proposition 7.1. *$c_A(r+1) \geqslant c_A(r) + 1$ whenever $r \geqslant \mathrm{gen}_k(A) - 1$.*

It follows immediately from Proposition 7.1 that $c_A(r) \geqslant r - \mathrm{gen}_k(A) + 1$. Combining this inequality with Theorem 6.1, we deduce Theorem 1.1.

In the course of proving Proposition 7.1 we may assume without loss of generality that k is algebraically closed. It is then harmless to replace V_r and Z_r with their sets of k-points endowed with the Zariski topology, and this shall be our convention in this section.

We say that an r-tuple $(a_1, \ldots, a_r) \in A^r$ *almost generates* A if there exists $a_{r+1} \in A$ such that $a_1, \ldots, a_r, a_{r+1}$ generates A. Denote the set of almost generating r-tuples in A^r by U'_r.

Lemma 7.2. *Let $\psi : V_{r+1} \to V_r$ denote the projection to the first r coordinates.*

(a) Let $\overline{a} = (a_1, \ldots, a_r) \in Z_r$. Then $\dim \psi^{-1}(\overline{a}) < \dim A$ if and only if $\overline{a} \in U'_r$.

(b) U'_r is Zariski open in V_r.

(c) If $r \geqslant \mathrm{gen}_k(A) - 1$, then $Z_r \cap U'_r$ is dense in Z_r.

Proof. (a) is obvious from the definition of U'_r. (b) holds because ψ is flat and finitely presentated, hence open, and $U'_r = \psi(U_{r+1})$.

(c) It is enough to show that if there exists $\overline{a} = (a_1, \ldots, a_r) \in Z_r - \overline{Z_r \cap U'_r}$, then $r < \mathrm{gen}_k(A) - 1$, or equivalently, that $U'_r = \emptyset$. To prove this assertion, we will show by induction on $i = 0, 1, \ldots, r$ that the affine space

$$L_i := \{(x_1, \ldots, x_i, a_{i+1}, \ldots, a_r) \,|\, x_1, \ldots, x_{i+1} \in A\} \text{ is contained in } Z_r - U'_r. \tag{7.1}$$

For $i = r$, (7.1) tells us that $\emptyset = L_r \cap U'_r = V_r \cap U'_r = U'_r$. In other words, $r < \mathrm{gen}_k(A) - 1$, and part (c) follows.

The base case, where $i = 0$, is clear. For the induction step, we assume assume that $L_i \subseteq Z_r - U'_r$ for some $i \in \{0, \ldots, r-1\}$. This means that for every $x_1, \ldots, x_r \in A$, $(x_1, \ldots, x_i, a_{i+1}, \ldots, a_r) \notin U'_r$. Consequently, for every $x_{i+1} \in A$, the $r+1$ elements $x_1, \ldots, x_{i+1}, a_{i+1}, \ldots, a_r$ do not generate A as an k-algebra, and, in particular, $(x_1, \ldots, x_{i+1}, a_{i+2}, \ldots, a_r) \in Z_r$. In other words, $L_{i+1} \subseteq Z_r$. Our goal is to show that $L_{i+1} \cap U'_r = \emptyset$. Assume the contrary. Then by part (b), $L_{i+1} \cap U'_r$ is Zariski dense in L_{i+1}. In particular, $\overline{a}$ lies in the Zariski closure of $Z_r \cap U'_r$, contradiction our assumption that $\overline{a} \notin Z_r - \overline{Z_r \cap U'_r}$. This contradicting tells us that $L_{i+1} \cap U'_r = \emptyset$, completing the proof of (7.1) and thus of part (c). □

Proof of Proposition 7.1. By Lemma 7.2 the fiber of $\psi : Z_{r+1} \to Z_r$ over every point of a dense open subset $Z_r \cap U'_r$ of Z_r is of dimension $< \dim(A)$. Hence, $\dim Z_{r+1} < \dim Z_r + \dim A$ by the Fiber Dimension Theorem. Equivalently, $c_A(r+1) > c_A(r)$. □

Remark 7.3. This argument does not entirely supplant [FR17, Section 3] because the main result of [FR17] is more general than Theorem 1.1.

Example 7.4. Let $A = k^n$, viewed as a k-module (i.e., as a k-algebra with zero multiplication). Here R-forms of A are projective R-modules of rank n and $G = \mathrm{Aut}_k(A) = \mathrm{GL}_n$. Let us write elements of V_1 as column vectors of length n and identify V_r with the space $\mathrm{M}_{n\times r}$ of $n \times r$ matrices. We assume that $r \geqslant n$; indeed, we need at least n elements to generate B. The group $G = \mathrm{GL}_n$ acts on $V_r = \mathrm{M}_{n\times r}$ via left multiplication. Clearly $U_r = \mathrm{M}^0_{n\times r}$ is the open subvariety of $n \times r$ matrices of rank n, and Z_r is the closed subvariety of $\mathrm{M}_{n\times r}$ of matrices of rank $< n$, i.e., of matrices with linearly dependent rows. An easy calculation shows that $\dim Z_r = (n-1) + (n-1)r = (n-1)(r+1)$, and thus $c_A(r) = r \dim A - \dim Z_r = r - n + 1$. We conclude that if $r - n + 1 > \mathrm{trdeg}_k R$, then B is generated by r elements as an R-module. In other words, $\mathrm{gen}(B) \leqslant \mathrm{trdeg}_k R + n$. This way we recover Forster's original bound (with Krull dimension replaced by transcendence degree).

Example 7.5. Let $A = k \times \ldots \times k$ (n times) with componentwise multiplication. Here R-forms of A are étale algebras of rank n over R and $G = \mathrm{Aut}_k(A)$ is the symmetric group S_n permuting the n factors of k. Once again, we will write elements of A as column vectors of length n, and elements of V_r as $n \times r$-matrices. Then S_n acts on $V_r = \mathrm{M}_{n\times r}$ by permuting the rows, U_r consists of matrices with distinct rows, and Z_r consists of matrices whose rows are not distinct. From this we readily see that $\dim Z_r = nr - r$ or equivalently, $c_A(r) = rn - \dim Z_r = r$. Theorem 6.1 tells us that if $r > \mathrm{trdeg}_k R$, then every étale R-algebra B is generated by r elements. Equivalently, $\mathrm{gen}_R(B) \leqslant \mathrm{trdeg}_k R + 1$. This is the same as the upper bound of Theorem 1.1 (again, with Krull dimension replaced by transcendence degree).

8. Proof of Theorem 1.3

Let $\overline{k}$ be a an algebraic closure of k and $n_{\max}$ be the largest possible dimension of a proper $\overline{k}$-subalgebra of $A \otimes_k \overline{k}$. Theorem 1.3 is an immediate consequence of Theorem 6.1 in combination with part (b) of the following lemma.

Lemma 8.1. *(a)* $\dim Z_r \geqslant n_{max} r$. *(b) If* $r > n_{max}$*, then* $\dim Z_r \leqslant n_{max}(n + r - n_{max})$.

Proof. Once again, passing from k to $\overline{k}$ does not change $\dim Z_r$, so we assume that k is algebraically closed. We will not distinguish between varieties and their sets of k-points.

(a) Let A_{max} be a proper subalgebra of A of maximal dimension n_{max}. If $a_1, \ldots, a_r \in A_{max}$, then the subalgebra generated by $a_1, \ldots, a_r$ is contained in A_{max} and hence the r-tuple $(a_1, \ldots, a_r)$ lies in Z_r. Thus $\dim Z_r \geqslant \dim(\underbrace{A_{max} \times \ldots \times A_{max}}_{r \text{ times}}) = rn_{max}$.

(b) Any r-tuple $a_1, \ldots, a_r \in A$ which spans a linear subspace of dimension $> n_{\max}$ in A automatically generates A as a k-algebra. Thus,

$$Z_r^{\mathrm{red}} \subset \mathrm{Rank}_{r,0}(A) \cup \mathrm{Rank}_{r,1}(A) \cup \ldots \cup \mathrm{Rank}_{r,n_{\max}}(A), \tag{8.1}$$

where here, if V is an n-dimensional k-vector space, then $\mathrm{Rank}_{r,s}(V)$ stands for the subvariety of V^r consisting of r-tuples $(v_1, \ldots, v_r)$ such that $\dim \mathrm{Span}_k\{v_1, \ldots, v_r\} = s$. In our case, $V = A$, but from this point on the algebra structure of A will play no role in the proof. To compute the dimension of $\mathrm{Rank}_{r,s}(V)$, let $\mathrm{Gr}(V,s)$ denote the Grassmannian of s-dimensional subspaces of V and consider the natural morphism

$$\mathrm{Rank}_{r,s}(V) \to \mathrm{Gr}(V,s)$$

sending $(v_1,\dots,v_r)$ to $\mathrm{Span}_k\{v_1,\dots,v_r\}$. The fibres of this map are readily seen to be irreducible of dimension rs; hence, $\mathrm{Rank}_{r,s}(V)$ is irreducible of dimension

$$\dim \mathrm{Rank}_{r,s}(V) = \dim \mathrm{Gr}(V,s) + rs = (n-s)s + rs = (n-s+r)s.$$

In view of (8.1), it remains to show that the maximal value of the function

$$f(x) = (n-x+r)x$$

on the interval $[0, n_{\max}]$ is attained for $x = n_{\max}$. Since $n > n_{\max}$ and $r > n_{\max}$, we have

$$f'(x) = n - 2x + r = (n-x) + (r-x) > 0$$

for any $x \in [0, n_{\max}]$, and the desired conclusion follows. This completes the proof of Lemma 8.1 and thus of Theorem 1.3. □

Remark 8.2. If $nr - \dim Z_r > d$, then by Lemma 8.1(a), $(n - n_{max})r > d$ or equivalently, $r > \dfrac{d}{n-n_{max}}$. This shows that the best upper bound we can hope to deduce from Theorem 6.1 is

$$\mathrm{gen}(B) < \frac{d}{n-n_{max}} + c.$$

Here, B is an arbitrary R-form of A, R is a k-ring of transcendence degree d, and c is a constant which depends only on A and not on d. In Theorem 1.3, $c = n_{max}$. In Sections 9 and 10 we will find a smaller constant term c for specific types of algebras; however, the linear term $\dfrac{d}{n-n_{max}}$ in the upper bound of Theorem 1.3 cannot be sharpened by this method.

9. Upper bounds on the number of generators for Azumaya algebras

Throughout this section, M_s denotes the affine k-space of $s \times s$ matrices.

9.1. **Proof of Theorem 1.5(a).** Our proof will rely on Theorem 6.1 with $A = \mathrm{M}_s(k)$, the algebra of $s \times s$ matrices over k. Recall that Azumaya algebras of degree s over R are precisely the R-forms of $\mathrm{M}_s(k)$. By definition $Z_r \subset \mathrm{M}_s \times \ldots \times \mathrm{M}_s$ (r times) consists of r-tuples $(a_1,\dots,a_r)$ of $s\times s$ matrices which do not generate $\mathrm{M}_s(k)$ as a k-algebra. Our goal now is to compute the dimension of Z_r. In order to do so, we may pass to the algebraic closure of k and thus assume that k is algebraically closed. Under this assumption, we may apply Burnside's theorem: Z_r^{red} is the union of the $s-1$ subvarieties $X_1,\dots,X_{s-1}$ of M_s^r determined by

$$X_i(k) = \left\{ (a_1,...,a_r) \;\middle|\; \begin{array}{c} a_1,\dots,a_r \in \mathrm{M}_s(k) \text{ have a common} \\ i\text{-dimensional invariant subspace} \end{array} \right\}. \tag{9.1}$$

For simple proofs of Burnside's theorem, see [Lam98] or [LR04].

Proposition 9.1. *(a) For every $i = 1, \dots, s-1$, X_i is a closed irreducible subvariety of M_s^r of dimension $rs^2 - (r-1)i(s-i)$.*

(b) $\dim Z_r = rs^2 - (r-1)(s-1)$. *Equivalently,* $c_{\mathrm{M}_s(k)}(r) = (r-1)(s-1)$.

Assume for a moment that Proposition 9.1 is established. Then setting $r = \left\lfloor \dfrac{d}{s-1} \right\rfloor + 2$, we obtain

$$c_{\mathrm{M}_s(k)}(r) = (r-1)(s-1) = (\left\lfloor \frac{d}{s-1} \right\rfloor + 1)(s-1) > \frac{d}{s-1} \cdot (s-1) = d.$$

Theorem 1.5(a) now follows from Theorem 6.1. It remains to prove Proposition 9.1. The case where $r = 1$ is obvious, so we will assume that $r \geqslant 2$.

(a) Again, we treat varieties as their sets of k-points. Consider the incidence variety

$$Y_i = \{(a_1, ..., a_r, W) \mid a_1(W), \dots, a_r(W) \subset W\} \subset \mathrm{M}_s \times \dots \times \mathrm{M}_s \times \mathrm{Gr}(i, s),$$

equipped with the natural projections

$$\begin{tikzcd} & Y_i \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ V_r = \mathrm{M}_s^r & & \mathrm{Gr}(i, s). \end{tikzcd}$$

Here, $\mathrm{Gr}(i, s)$ denotes the Grassmannian of i-dimensional subspaces in k^s. Note that π_2 is surjective. Since $\mathrm{Gr}(i, s)$ is projective, Y_i is closed in $V_r \times \mathrm{Gr}(i, s)$, and X_i is the image of Y_i under π_1, we conclude that X_i is closed in V_r.

The fibre of $W \in \mathrm{Gr}(i, s)$ under π_2 consists of the r-tuples of matrices $(a_1, \dots, a_r)$ such that $a_i(W) \subset W$ for each a_i. This means that in a suitable basis of k^s, the matrices $a_1, \dots, a_r$ are all block upper-triangular of the form

$$\begin{bmatrix} A_{i\times i} & B_{i\times(s-i)} \\ 0_{(s-i)\times i} & C_{(s-i)\times(s-i)} \end{bmatrix}$$

where $A_{i\times i}$ is an $i\times i$-matrix, $B_{i\times(s-i)}$ is an $i\times(s-i)$-matrix, $C_{(s-i)\times(s-i)}$ is an $(s-i)\times(s-i)$-matrix, and $0_{(s-i)\times i}$ is the $(s-i)\times i$ zero matrix. In particular, every fibre of π_2 is an affine space of dimension $r(s^2 - i(s-i))$. By the Fiber Dimension Theorem, Y_i irreducible of dimension $r(s^2 - i(s-i)) + \dim \mathrm{Gr}(i, s) = rs^2 - (r-1)i(s-i)$.

Now recall that $X_i \subset V_r$, is the image of Y_i under π_1. Since Y_i is irreducible, so is X_i. It remains to show that $\dim X_i = \dim Y_i$, i.e., the fibre of π_2 over $(a_1, \dots, a_r) \in X_i$ in general position is finite. This fibre consists of i-dimensional subspaces, invariant under each of the matrices $a_1, \dots, a_r$. If $(a_1, \dots, a_r) \in X_i$ is in general position, we may assume that a_1 has distinct eigenvalues and hence, a_1 alone has only finitely many invariant subspaces in k^n. This shows that the fibre $\pi_2^{-1}(a_1, \dots, a_r)$ is finite, and $\dim X_i = \dim Y_i = rs^2 - (r-1)i(s-i)$.

(b) It follows from part (a) that the components of Z_r of the largest dimension are X_1 and X_{s-1}. Thus $\dim Z_r = \dim X_1 = \dim X_{s-1} = rs^2 - (r-1)(s-1)$ and $c_{\mathrm{M}_s(k)}(r) =$

$r \dim \mathrm{M}_s - \dim Z_r = (r-1)(s-1)$, as claimed. This completes the proof of Proposition 9.1 and thus of Theorem 1.5(a). $\square$

9.2. **Azumaya algebras with involution.** In the following proposition we consider R-algebras with involution B, and $\mathrm{gen}_R(B)$ denotes the number of generators of B when regarded as an algebra with involution.

Proposition 9.2. *Assume k is infinite. Let R be a k-ring of transcendence degree d. Then:*

(a) If B is an Azumaya algebra of degree s with orthogonal involution over R, then

$$\mathrm{gen}_R(B) \leqslant \left\lfloor \frac{d+(s-2)}{2s-3} \right\rfloor + 1, \text{ if } s \neq 4, \text{ and}$$
$$\mathrm{gen}_R(B) \leqslant \left\lfloor \frac{d+1}{4} \right\rfloor + 1, \text{ if } s = 4.$$

(b) If B is an Azumaya algebra of even degree s with symplectic involution over R, then

$$\mathrm{gen}_R(B) \leqslant \left\lfloor \frac{d+(s-1)}{2s-3} \right\rfloor + 1, \text{ if } s \geqslant 8,$$
$$\mathrm{gen}_R(B) \leqslant \left\lfloor \frac{d+6}{9} \right\rfloor + 1, \text{ if } s = 6,$$
$$\mathrm{gen}_R(B) \leqslant \left\lfloor \frac{d+3}{4} \right\rfloor + 1, \text{ if } s = 4, \text{ and}$$
$$\mathrm{gen}_R(B) \leqslant d+2, \text{ if } s = 2.$$

Proof. (a) Recall that Azumaya algebras with an orthogonal involution are forms of $A = (\mathrm{M}_s(k), t)$, where t is the matrix transposition involution. By [NTW19, Propositions 4.6 and 4.8],

$$\dim Z_r = \begin{cases} r(s^2-2s+3)+(s-2), & \text{if } s \neq 4, \text{ and} \\ 12r+1, & \text{if } s = 4. \end{cases}$$

Substituting these formulas into the inequality of Theorem 6.1, we deduce part (a).

(b) is proved by the same argument with t replaced by the standard symplectic involution of M_s (where s is even). In this case the formula for $\dim Z_r$ is given by [NTW19, Propositions 4.5 and 4.10]. $\square$

10. Upper bounds on the number of generators for octonion algebras

Proposition 10.1. *Let k be a field and A be the split octonion algebra over k. Then:*

(a) The scheme Z_r of r-tuples not generating A (see Section 2) is irreducible of dimension $6r+5$ for any $r \geqslant 3$.

(b) If k is infinite, R is a k-ring of transcendence degree d, and B is an octonion R-algebra, then $\mathrm{gen}(B) \leqslant \left\lfloor \frac{d+1}{2} \right\rfloor + 3.$

An octonion algebra B over R is an R-form of the split octonion algebra A. Thus, part (b) is an immediate consequence of part (a) and Theorem 6.1. To prove part (a), we may assume without loss of generality that our base field k is algebraically closed. Again, we shall freely identify varieties with their sets of k-points. Recall that the automorphism group of A is the exceptional group G_2. Since k is algebraically closed, A has no quaternion division k-subalgebras, and thus [Rac74, Theorem 5] tells us that, up to translations by elements of G_2, there is a unique maximal subalgebra $A_{max} \subset A$. The subalgebra A_{max} is 6-dimensional; it is sometimes called the algebra of *sextonions*; see [Wes06]. We will use the following description of A_{max} from [Rac74]. Choose a k-basis x_i, y_i for the octonion algebra A, where $i = 0, 1, 2, 3$ and x_i, y_i satisfy the relations [Rac74, (14) on p. 165]. Then A_{max} is spanned by $x_0, x_1, x_2, y_0, y_1, y_3$.

Lemma 10.2. *(a) A_{max} is generated by three elements,*

(b) the stabilizer H of A_1 in G_2 is a 9-dimensional parabolic subgroup of G_2.

Proof. (a) It suffices to show that A_{max} is generated by x_1, x_2 and y_1. Indeed, let A_0 be the subalgebra of A_{max} generated by x_1, x_2 and y_1. Then the relations [Rac74, (14) on p. 165] tell us that $x_0 = -x_1y_1$, $y_0 = 1 - x_0$ and $y_3 = x_1x_2$ also lie in B_0. Thus $A_0 = A_{max}$, as desired.

(b) Recall that (x_i, y_i) are, by construction, mutually orthogonal hyperbolic pairs relative to the norm form in A for $i = 0, 1, 2, 3$. This readily implies that

$$A_{max}^{\perp} = L, \text{ where } L = \mathrm{Span}_k(x_2, y_3), \text{ and conversely, } L^{\perp} = A_{max}. \tag{10.1}$$

The multiplication table in [Rac74, (14), p. 165] tells us that $x_2^2 = y_3^2 = x_2y_3 = 0$. This tells us that L is γ-isotropic in the sense of D. Anderson; see [And11, Lemma 3.5]. In view of (10.1), the stabilizer H of A_0 in G_2 equals the stabilizer of L in G_2. By [And11, Proposition A.5], the stabilizer of L is a 9-dimensional parabolic subgroup of G_2. ☐

We are now ready to complete the proof of Proposition 10.1(a). Recall our standing assumptions: k is algebraically closed and $r \geqslant 3$.

Let $Z_r^{\leqslant t}$ be the subvariety of r-tuples $(a_1, \ldots, a_r) \in A^r$ such that $a_1, \ldots, a_r$ generate a k-subalgebra of A of dimension $\leqslant t$. More precisely, we choose a k-vector space basis $e_1, \ldots, e_8$ in A and identify an element $x = x_1e_1 + \ldots + x_8e_8$ of A with the row vector $(x_1, \ldots, x_8) \in k^8$. Then $Z_r^{\leqslant t}$ is the Zariski closed subvariety of A^r cut out by the conditions

$$\operatorname{rank} \begin{bmatrix} p_1(a_1, \ldots, a_r) \\ \vdots \\ p_{t+1}(a_1, \ldots, a_r) \end{bmatrix} \leqslant t \tag{10.2}$$

for any $t+1$ monomials $p_1, \ldots, p_{t+1}$ in r variables. Here $p_1, \ldots, p_{t+1}$ are non-commutative non-associative monomials, $p_i(a_1, \ldots, a_r) \in A$ represents a row of the $(t+1) \times 8$ matrix in (10.2), and each entry of this matrix is a polynomial function in the coordinates of $a_1, \ldots, a_r$.

Now set $Z_r^{(t)} = Z_r^{\leqslant t} - Z^{\leqslant t-1}$ to be the variety of r-tuples $(a_1, \ldots, a_r)$ such that $a_1, \ldots, a_r$ generate a k-algebra of dimension exactly t. Since every proper subalgebra of A is of dimension $\leqslant 6$, we have $Z_r = Z_r^{\leqslant 6}$. Now recall that every proper subalgebra of A is contained in a 6-dimensional subalgebra S, which is a translate of A_{max}, and that by Lemma 10.2(a), A_{max} (and thus S) is generated by 3 (and thus r) elements as a k-algebra. This tells us that $Z_r^{(6)}$ is dense in Z_r.

It remains to show that $Z_r^{(6)}$ is an irreducible variety of dimension $6r+5$. Indeed, consider the natural morphism $f\colon Z_r^{(6)} \to \mathrm{Gr}(A,6)$ taking an r-tuple $(a_1,\dots,a_r)$ to the 6-dimensional subalgebra generated by $a_1,\dots,a_r$. To see that f us a morphism, note that $Z_r^{(6)}$ is covered by open subsets

$$U_{p_1,\dots,p_6} = \{(a_1,\dots,a_r) \,|\, p_1(a_1,\dots,a_r)\wedge\dots\wedge p_6(a_1,\dots,a_r) \neq 0 \ \text{ in } \ \Lambda^6(A)\}$$

indexed by 6-tuples of monomials $p_1,\dots,p_6$ in r variables. On any $U_{p_1,\dots,p_6} \neq \emptyset$, we define

$$\begin{aligned} f_{p_1,\dots,p_6}\colon U_{p_1,\dots,p_6} &\to \mathrm{Gr}(A,6) \\ (a_1,\dots,a_r) &\mapsto p_1(a_1,\dots,a_r)\wedge\dots\wedge p_6(a_1,\dots,a_r) \in \mathbb{P}(\Lambda^6(A)). \end{aligned}$$

Here we identify $\mathrm{Gr}(A,6)$ with the variety of decomposable tensors in $\mathbb{P}(\Lambda^6(A))$ via the Plücker embedding. One readily sees from this definition that every $f_{p_1,\dots,p_6}$ is a morphism. These morphisms agree on the overlaps: the image of $(a_1,\dots,a_r) \in Z_r^{(6)}$ under any of them is the 6-dimensional subalgebra generated by $a_1,\dots,a_r$. Hence, they patch together to a morphism $f\colon Z_r^{(6)} \to \mathrm{Gr}(A,6)$. By Lemma 10.2(b), the image of f is the irreducible projective homogeneous space G_2/H of dimension $14-9=5$. The fibres of f are isomorphic to Zariski open subvarieties of V_r. By Lemma 10.2(a), these open subvarieties are non-empty; hence, they are irreducible of dimension $6r = r\dim A_{max}$. Applying the Fibre Dimension Theorem to f, we conclude that $Z_r^{(6)}$ is irreducible of dimension $6r+5$, as claimed. This completes the proof of Proposition 10.1. $\square$

11. Betti cohomology

This section consists of preliminary material for the rest of the paper. When $k \subset \mathbb{C}$, we make use of the Betti cohomology of a k-variety X, i.e., the singular cohomology of the complex points, $\mathrm{H}^*(X(\mathbb{C});\mathbb{Z})$, which we write simply as $\mathrm{H}^*(X)$. We use Betti cohomology, rather than some other cohomology theory of varieties, because of the following theorem.

Theorem 11.1 (Affine Lefschetz Hyperplane Theorem)**.** *Let $k \subset \mathbb{C}$ be a field with a chosen complex embedding, $\mathbb{A}^N$ be an affine space and $X \subset \mathbb{A}^N$ be a smooth closed subvariety of dimension d. Then there exists a smooth affine hyperplane section $Y \subseteq X$ defined over k such that $\dim Y = d-1$ and $\mathrm{H}^i(X) \to \mathrm{H}^i(Y)$ is an isomorphism if $i < d-1$ and an injection if $i = d-1$.*

When $k = \mathbb{C}$, then this is an immediate consequence of [HT85, Theorem 1.1.3]. In fact, if k is a field with a chosen complex embedding $k \subseteq \mathbb{C}$, then we can still apply the result of [HT85]. This is because there is an open dense set of complex hyperplanes in some projective space that induce the required isomorphism on cohomology, and any open dense subset of an affine space contains a point defined over $\mathbb{Q}$, and therefore over k.

We will encounter the following situation more than once: There will be a ring R and an R-form B of A such that an induced map on Betti cohomology

$$\mathrm{H}^i(BG) \to \mathrm{H}^i(\operatorname{Spec} R)$$

prohibits generation of B by a small number of elements. The Affine Lefschetz Hyperplane Theorem will allow us to replace R by a quotient ring of potentially much smaller transcendence degree without changing the i-th cohomology group. This is the device that allows us to give the bounds on the transcendence degree in Theorems 1.4 and 1.5(b).

What follows in the rest of this section is a description of the properties of Betti cohomology of k-varieties ($k \subset \mathbb{C}$) that we will make use of in the subsequent sections.

If $i\colon Z \to X$ is the inclusion of a closed smooth subscheme of constant codimension c and if $j\colon X - Z \to X$ is the inclusion of the complement, then there is a long exact *Gysin sequence*

$$\longrightarrow \mathrm{H}^*(X-Z) \xrightarrow{\partial} \mathrm{H}^{*-2c}(Z) \xrightarrow{i_*} \mathrm{H}^*(X) \xrightarrow{j^*} \mathrm{H}^*(X-Z) \xrightarrow{\partial} \dots.$$

This sequence has a naturality property which we now explain:

Definition 11.2. Let

(11.1)
$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f"'] & X' \arrow[d, "g"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

be a cartesian square of smooth schemes, where i is a closed embedding. Let N' denote the normal bundle of Z' in X' and N the normal bundle of Z in X. The square is *transversal* if the induced map $N' \to f^*N$ is an isomorphism.

If we are given a transversal square, then there is an induced natural transformation of Gysin sequences.

(11.2)
$$\begin{tikzcd} \arrow[r] & \mathrm{H}^*(X-Z) \arrow[r, "\partial"] \arrow[d] & \mathrm{H}^{*-2c}(Z) \arrow[r, "i_*"] \arrow[d, "f^*"] & \mathrm{H}^*(X) \arrow[r, "j^*"] \arrow[d, "g^*"] & \mathrm{H}^*(X-Z) \arrow[r, "\partial"] \arrow[d] & \dots \\ \arrow[r] & \mathrm{H}^*(X'-Z') \arrow[r, "\partial"] & \mathrm{H}^{*-2c}(Z') \arrow[r, "i'_*"] & \mathrm{H}^*(X') \arrow[r] & \mathrm{H}^*(X'-Z') \arrow[r] & \dots \end{tikzcd}$$

This and other useful properties of this Gysin sequence are listed in [Pan09, Section 2.2].

The most important case for us will be of a transversal square in which $X' \to X$ and $Z \to X$ are both closed embeddings of smooth varieties. In this case, X' and Z will be said to *intersect transversely*, and Z' is their transverse intersection.

Intersecting transversely is a condition on normal bundles. Using the normal exact sequence [Har77, p. 182], we can say the intersection is transverse if

$$\dim T_p X = \dim T_p X' + \dim T_p Z - \dim T_p Z'$$

for all closed points p of the smooth variety Z'.

Remark 11.3. A square is transversal if a certain map of locally free sheaves is an isomorphism. This holds for the square defined by $Z \subseteq X$ and $X' \to X$ if and only if it holds for the square obtained by pulling back along a faithfully flat map $U \to X$.

Lemma 11.4. *Let X be a smooth variety. If $Z \to X$ is a possibly-nonsmooth closed subvariety of codimension d—that is, if each irreducible component of Z is of codimension at least d in X—and if $U = X - Z$, then the pullback map $\mathrm{H}^i(X) \to \mathrm{H}^i(U)$ is an isomorphism when $i < 2d - 1$ and is injective when $i = 2d - 1$.*

Proof. The argument proceeds by induction on the dimension of Z, starting with the case where Z is empty, and by convention of codimension $\dim X + 1$. This case is trivially true.

Suppose the statement is proved for closed subvarieties of dimension less than that of Z.

Let W denote the singular locus of Z. By the induction hypothesis, the pullback map $\mathrm{H}^i(X) \to \mathrm{H}^i(X - W)$ is an isomorphism when $i < 2d-1$ and is injective when $i = 2d-1$, the dimension of W being less than Z.

Therefore it suffices to prove that the pullback map $\mathrm{H}^i(X - W) \to \mathrm{H}^i(U)$ is an isomorphism (resp. is injective) in the range under consideration. Note $U = (X - W) - (Z - W)$. The closed inclusion $Z - W \hookrightarrow X - W$ is a closed embedding of smooth varieties, and is of codimension at least d.

Therefore we have are reduced to the case of smooth varieties. In general, $Z - W$ consists of finitely many connected components of varying dimensions. We may remove each of these components from $X - W$ in nondecreasing order of dimension. Consequently, we may suppose there is only one component, of codimension $\geqslant d$. In this case the Gysin sequence gives the result directly. □

Remark 11.5. If $s : X \to V$ is the zero-section of a rank-m vector bundle on a smooth variety, then the composite of the Gysin map s_* with s^*

$$\mathrm{H}^0(X) \xrightarrow{s_*} \mathrm{H}^{2m}(V) \xrightarrow{s^*} \mathrm{H}^{2m}(X)$$

sends 1_X to the Euler class $e(V)$. We sketch the argument: In the case of a zero-section of a bundle, the definition in [Pan09, Section 2.2] of the Gysin map coincides with a composite of taking a cup-product with the Thom class of the bundle, followed by extension of support:

$$\mathrm{H}^{*-2m}(X) \xrightarrow{\smile \tau} \mathrm{H}^*_X(V) \longrightarrow \mathrm{H}^*(V) .$$

Composing further with s^* gives the definition of Euler class in [Bre93, Definition 12.1].

It will also be important later that $\mathrm{H}^*(\mathbb{P}^n) \cong \mathbb{Z}[\vartheta]/(\vartheta^{n+1})$, where ϑ is the first Chern class of $\mathcal{O}(1)$, so that $|\vartheta| = 2$.

If Γ is a Lie group, then there exists a universal principal Γ-bundle $E\Gamma \to B\Gamma$ where $E\Gamma$ is contractible. If Γ acts on X, then the *Borel equivariant cohomology* ([Hsi75, III.1]) of X is defined by

$$\mathrm{H}^*_\Gamma(X) := \mathrm{H}^*((X \times E\Gamma)/\Gamma).$$

As a special case, if X is a point with trivial Γ-action, then $\mathrm{H}^*_\Gamma(X) = \mathrm{H}^*(B\Gamma)$.

If G is a linear algebraic group defined over $k \subseteq \mathbb{C}$, then we may define a Lie group $G(\mathbb{C})$. Suppose G acts on a k-variety X. We abuse notation and write $\mathrm{H}^*_G(X)$ for $\mathrm{H}^*_{G(\mathbb{C})}(X(\mathbb{C}))$.

For any n, we can find a G-representation V and an open G-invariant subvariety $U \subset V$ with the properties that $U \to U/G$ is a G-torsor, U/G is a variety, and $(V - U) \hookrightarrow V$ is a closed subvariety of codimension exceeding $n/2$; see Remark 4.3. Then

$$\mathrm{H}^i_G(X) \cong \mathrm{H}^i((X \times U)/G) \tag{11.3}$$

for $i \leqslant n$. This allows us to treat the equivariant cohomology $\mathrm{H}^*_G(X)$ in many ways like the ordinary cohomology of a variety.

If G acts on a variety X in such a way that $X \to X/G$ is a G-torsor, then the quotient $(X \times U)/G$ appearing in (11.3) above is isomorphic to $X/G \times U$. We deduce that

$$\mathrm{H}^n_G(X) \cong \mathrm{H}^n(X/G)$$

for all nonnegative integers n.

The equivariant Borel cohomology groups for G are contravariantly functorial for G-equivariant maps. Since there is always a unique G-equivariant map $X \to \operatorname{Spec}\mathbb{C}$, there is a natural map $\mathrm{H}^*_G(\operatorname{Spec}\mathbb{C}) = \mathrm{H}^*(BG) \to \mathrm{H}^*_G(X)$.

We can also construct a Gysin map: if $Z \to X$ is a closed G-invariant embedding of constant codimension c where Z and X are both smooth, then in any range of dimensions we may find some U as above in order to calculate both $\mathrm{H}^*_G(Z)$ and $\mathrm{H}^*_G(X)$. The inclusion $(Z \times U)/G \to (X \times U)/G$ is a closed embedding of smooth varieties, again of codimension c. This means there is a Gysin map

$$\mathrm{H}^{*-2c}_G(Z) \to \mathrm{H}^*_G(X)$$

exactly as in the non-equivariant case.

Lemma 11.6. *Let G be an algebraic group acting freely on a smooth variety X in such a way that $X \to X/G$ is a G-torsor in the category of smooth varieties. Suppose Y and Z are G-invariant closed smooth subvarieties of X that intersect transversely. Then Y/G and Z/G intersect transversely in X/G.*

Proof. By using the observations of Remark 11.3, we can check for the transversality of Y/G and Z/G after faithfully flat pullback to X, but these pullbacks are Y and Z, which intersect transversely. ☐

Lemma 11.7. *Let G be an algebraic group acting on a smooth variety X. Suppose Y and Z are smooth closed G-subvarieties intersecting transversely, and Y is of constant codimension c. Then there is a natural transformation of equivariant Gysin sequences*

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{H}^*_G(X-Y) \arrow[r, "\partial"] \arrow[d] & \mathrm{H}^{*-2c}_G(Y) \arrow[r] \arrow[d] & \mathrm{H}^*_G(X) \arrow[r, "j^*"] \arrow[d] & \mathrm{H}^*_G(X-Y) \arrow[r, "\partial"] \arrow[d] & \dots \\
{} \arrow[r] & \mathrm{H}^*_G(Z-(Y\cap Z)) \arrow[r, "\partial"] & \mathrm{H}^{*-2c}_G(Y\cap Z) \arrow[r] & \mathrm{H}^*_G(Z) \arrow[r] & \mathrm{H}^*_G(Z-(Y\cap Z)) \arrow[r] & \dots
\end{tikzcd}$$

Proof. The groups H^i_G may be calculated by choosing a G-representation V and an open subset U where $U \to U/G$ is a G-torsor and $V - U$ has sufficiently high codimension. Then $\mathrm{H}^i_G(X)$ is the cohomology of $(X \times U)/G$ and similarly for the other varieties. It suffices therefore to prove that $(Y \times U)/G$ and $(Z \times U)/G$ intersect transversely in $(X \times U)/G$, but this is implied by Lemma 11.6. □

Lemma 11.8. *Let G be an algebraic group and H a subgroup. Suppose Z is a G-invariant closed smooth subvariety of a smooth G variety X of constant codimension c. Then there is a natural transformation of equivariant Gysin sequences:*

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{H}^*_G(X-Z) \arrow[r, "\partial"] \arrow[d] & \mathrm{H}^{*-2c}_G(Z) \arrow[r] \arrow[d] & \mathrm{H}^*_G(X) \arrow[r, "j^*"] \arrow[d] & \mathrm{H}^*_G(X-Z) \arrow[r, "\partial"] \arrow[d] & \dots \\
{} \arrow[r] & \mathrm{H}^*_H(X-Z) \arrow[r, "\partial"] & \mathrm{H}^{*-2c}_H(Z) \arrow[r] & \mathrm{H}^*_H(X) \arrow[r] & \mathrm{H}^*_H(X-Z) \arrow[r] & \dots
\end{tikzcd}$$

Proof. It suffices to prove that

$$\begin{tikzcd}
(Z \times U)/H \arrow[r] \arrow[d] & (X \times U)/H \arrow[d] \\
(Z \times U)/G \arrow[r] & (X \times U)/G
\end{tikzcd}$$

is transversal. This can be verified after faithfully flat base change along $(X \times U) \to (X \times U)/G$, whereupon it is isomorphic to

$$\begin{tikzcd}
Z \times U \times (G/H) \arrow[r] \arrow[d] & X \times U \times (G/H) \arrow[d] \\
Z \times U \arrow[r] & X \times W
\end{tikzcd}$$

where the maps are either projections or closed embeddings. This is obviously transversal. □

Lemma 11.9. *Let X be a smooth variety equipped with a G-action. If $Z \to X$ is a possibly-nonsmooth closed G-subvariety of codimension d and if $U = X - Z$, then the pullback map $\mathrm{H}^i_G(X) \to \mathrm{H}^i_G(U)$ is an isomorphism when $i < 2d-1$ and is injective when $i = 2d-1$.*

Proof. Choose a G-representation V and an open subset U such that $U \to U/G$ is a G-torsor and $V - U \hookrightarrow V$ has codimension greater than d. The result now follows by applying Lemma 11.4 to $(Z \times U)/G \to (X \times U)/G$. □

12. The Lefschetz principle

Our proofs of Theorems 1.4 and 1.5 will rely on topological methods. These methods work best if the base field k is equipped with an embedding $k \hookrightarrow \mathbb{C}$. To extend our arguments to an arbitrary base field k of characteristic 0, we will repeatedly use the following version of the Lefschetz principle.

Lemma 12.1. *Let R be a k-ring, and B be an R-algebra. Suppose B is finitely generated as an R-module. For any field extension K/k, set $R_K = R \otimes_k K$ and $B_K = B \otimes_R R_K$. Then*

(a) $\mathrm{gen}_R(B) \geqslant \mathrm{gen}_{R_K}(B_K)$.
(b) Moreover, there exists a subextension $k \subset F \subset K$ *such that* F/k *is finitely generated and* $\mathrm{gen}_{R_F}(B_F) = \mathrm{gen}_{R_K}(B_K)$.
(c) Suppose k *is a finitely generated extension of* $\mathbb{Q}$*. If* $\mathrm{gen}_{R_K}(B_K) < r$ *for some field* K *containing* k*, then*

$$\mathrm{gen}_{R_{\mathbb{C}}}(B_{\mathbb{C}}) < r.$$

Proof. (a) is obvious: if $a_1, \ldots, a_r$ generate A as an R-algebra, then $a_1, \ldots, a_r$ generate B_K as an R_K-algebra.

(b) Suppose $\mathrm{gen}_{R_K}(B_K) = r$. Choose r elements, $b_1, \ldots, b_r \in B_K$ that generate B_K as an R_K-algebra. We claim that these same elements will generate B_F as an R_F-algebra for some extension $k \subset F \subset K$ such that F is finitely generated over k. In particular, the claim tells us that $\mathrm{gen}_{R_F}(B_F) \leqslant r$. The opposite inequality is given by part (a), so if we can prove this claim, then part (b) will follow.

To prove the claim, choose elements $a_1, \ldots, a_d$ which generate B as an R-module and write each element b_i as

$$b_i = r_{i1}a_1 + \ldots + r_{id}a_d.$$

for some coefficients $r_{ij} \in R_K$. Each r_{ij} lies in $R_{F_{ij}}$ for some intermediate subfield $k \subset F_{ij} \subset K$ such that F_{ij} is finitely generated over k. After replacing k by the compositum of F_{ij} in K (which is still finitely generated over k), we may assume without loss of generality that $b_1, \ldots, b_r$ lie in B.

Since $b_1, \ldots, b_r$ generate B_K as an R_K-algebra, we can write each a_i as

$$a_i = s_{i1}M_{i1} + \ldots + s_{it_i}M_{it_i}$$

for some monomials M_{ij} in $b_1, \ldots, b_r$ and some coefficients $s_{ij} \in R_K$. Once again, each s_{ij} lies in some intermediate subfield E_{ij} finitely generated over k. Setting F to be the compositum of E_{ij} in K, we see that the R_F-subalgebra of B_F generated by $b_1, \ldots, b_r$ contains $a_1, \ldots, a_d$. This shows that $b_1, \ldots, b_r$ generate B_F as an R_F-algebra. Hence, $\mathrm{gen}_{R_F}(B_F) \leqslant r$, as desired.

(c) Choose F as in part (b), so that $\mathrm{gen}_{R_F}(B_F) < r$. Since F is a finitely generated extension of $\mathbb{Q}$, it is isomorphic (over k) to a subfield of $\mathbb{C}$. Thus we may assume without loss of generality that $F \subset \mathbb{C}$. By part (a),

$$\mathrm{gen}_{R_{\mathbb{C}}}(B_{\mathbb{C}}) \leqslant \mathrm{gen}_{R_F}(B_F) = \mathrm{gen}_{R_K}(B_K) < r,$$

as desired. □

13. Equivariant Cohomology of U_r

The purpose of this section is to prove Theorem 13.4, which is a general device for constructing examples of forms of an algebra A requiring many generators.

We continue to write V_r for the affine variety whose k-points are A^r, Z_r for the closed subscheme of V_r representing r-tuples that fail to generate A as an k-algebra, and $c_A(r)$ for the codimension of Z_r in V_r. As before, write $U_r = V_r - Z_r$. The automorphism group of the algebra A is denoted G.

Lemma 13.1. *The natural map* $\mathrm{H}^n(BG) \to \mathrm{H}^n_G(U_r)$ *is an isomorphism for all values of* $n < c_A(r)$.

Proof. The inclusion $U_r \to V_r$ induces an isomorphism

$$\mathrm{H}^n(BG) \cong \mathrm{H}^n_G(V_r) \to \mathrm{H}^n_G(U_r)$$

by virtue of Lemma 11.4 and homotopy invariance for $\mathrm{H}^n_G(\cdot)$. □

Suppose B is an R-form of the algebra A, and suppose $(b_1, \dots, b_r) \in B^r$ is a generating r-tuple of elements. Let $T \to X$ be the G-torsor associated to B. There is a G-equivariant classifying map $\varphi_r : T \to U_r$ by Proposition 6.2, and so there is an induced morphism

$$\varphi_r^* : \mathrm{H}^*_G(U_r) \to \mathrm{H}^*_G(T) = \mathrm{H}^*(\operatorname{Spec} R) \tag{13.1}$$

This leads to an argument to rule out generation by r elements.

Proposition 13.2. *Suppose R is a ring and B is an R-form of A with associated G-torsor T. Suppose that B can be generated by r elements. Then there is a commutative diagram of cohomology rings*

$$\begin{tikzcd} & \mathrm{H}^*(BG) \arrow[dl] \arrow[dr] & \\ \mathrm{H}^*_G(U_r) \arrow[rr] & & \mathrm{H}^*_G(T) = \mathrm{H}^*(\operatorname{Spec} R) \end{tikzcd}$$

in which the maps with source $\mathrm{H}^*(BG)$ *are the natural maps for G-equivariant cohomology.*

Proof. If B can be generated by r elements, then there exists G-equivariant map $T \to U_r$. Applying H^*_G gives the diagram. □

We use the following variant of the Jouanolou construction, due to Thomason [Wei89, Proposition 4.4].

Theorem 13.3. *Let X be a regular k-variety. Then there exists a vector bundle E on X and an E-torsor $p : W \to X$ for which W is an affine scheme.*

This result is stated in greater generality in [Wei89]. A regular k-variety is quasi-compact, separated (and therefore quasi-separated) and carries an ample family of line bundles according to [Ill71].

The vector bundle E is to be viewed as a group-scheme on X where the group structure is given by addition in the vector bundle. When the construction is applied to a regular variety X, the result is an affine k-variety Y such that there is a map $p : Y \to X$ that is Zariski-locally isomorphic to the projection $U \times \mathbb{A}^n_k \to U$. In particular, the map $p : Y(\mathbb{C}) \to X(\mathbb{C})$ is a fibre bundle with contractible fibres, and is therefore a homotopy equivalence, [Jam84, Theorem 7.57].

Theorem 13.4. *Let k be a field with a chosen embedding $k \hookrightarrow \mathbb{C}$ and A be a finite-dimensional algebra over k. Suppose the natural map $\mathrm{H}^i(BG) \to \mathrm{H}^i_G(U_r)$ is not injective for some $r \geqslant \operatorname{gen}_k(A)$ (so that $U_r \neq \emptyset$). Then there exists a finite type k-ring R and an R-form B of A so that $\operatorname{trdeg}_k R \leqslant i$ and $\operatorname{gen}_{R_\mathbb{C}}(B_\mathbb{C}) > r$.*

In particular, $\mathrm{gen}_R(B) > r$; see Lemma 12.1(a).

Proof. By Proposition 7.1 there exists an integer $t \geqslant 0$ such that $i < c_A(r+t)$. The algebra B will be defined as ${}^T A$, for a suitable G-torsor $T \to \operatorname{Spec} R$. Our goal is to construct this torsor. We will do so by starting with the G-torsor $U_{r+t} \to U_{r+t}/G$ and modifying it in stages. At each stage, we will produce a G-torsor $Y \to Y/G$ so that Y has the same G-equivariant cohomology as U_{r+t} in degrees less than or equal to i.

The first problem with U_{r+t} is that U_{r+t}/G may not be a scheme, only an algebraic space; see Remark 5.3. To fix this, recall that by Remark 4.3 there exists a linear representation $G \to \mathrm{GL}(\bar{W})$ and a G-invariant open subset $W \subset \bar{W}$ such that $\bar{W} - W$ has codimension at least $i/2$ and W is the total space of a G-torsor $W \to W/G$ in the category of schemes. There is now a torsor $q' : U_{r+t} \times W \to (U_{r+t} \times W)/G$ in the category of schemes; see [EG98, Prop. 23(1)] for instance. The quotient scheme $(U_{r+t} \times W)/G$ is separated [MFK94, Lemma 0.6] and quasi-compact, being the surjective image of a quasi-compact scheme. It is of finite type by [Gro64, Prop. 1.5.4(v)], since $U_r \times W$ is of finite type. The morphism $U_{r+t} \times W \to (U_{r+t} \times W)/G$ is smooth, since G is smooth, and because $U_{r+t} \times W$ is a smooth k-variety, it follows from [Gro67, Prop. 17.7.7] that $(U_{r+t} \times W)/G$ is a smooth k-variety.

The codimension of $\bar{W} - W$ in W is at least $i/2$, so that applying Lemma 11.9 shows there is an isomorphism $\mathrm{H}^j_G(U_{r+t}) \to \mathrm{H}^j_G(W \times U_{r+t})$ whenever $j \leqslant i$.

The next problem is that $(W \times U_{r+t})/G$ is not affine in general. We will "approximate" it by an affine scheme as follows. Write $f : \operatorname{Spec} R \to (W \times U_{r+t})/G$ for an affine vector-bundle torsor, which exists by virtue of the construction in Theorem 13.3, and consider the pullback

$$\begin{tikzcd} T \arrow[r, "F"] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & W \times U_{r+t} \arrow[d] \\ \operatorname{Spec} R \arrow[r, "f"] & (W \times U_{r+t})/G. \end{tikzcd} \tag{13.2}$$

The composite of F with the projection map $W \times U_{r+t} \to U_{r+t}$ is G-equivariant and classifies an R-form B of A, which is generated by $r+t$ elements. The map f is a vector-bundle torsor, and therefore induces an isomorphism on cohomology:

$$\mathrm{H}^*_G(W \times U_{r+t}) \cong \mathrm{H}^*((W \times U_{r+t})/G) \xrightarrow{f^*} \mathrm{H}^*(\operatorname{Spec} R) \cong \mathrm{H}^*_G(T).$$

We now define the R-algebra $B = {}^T A$ by twisting A by T. We claim that $B_{\mathbb{C}}$ cannot be generated by r elements. We remind the reader that we write $\mathrm{H}^*(X)$ for $\mathrm{H}^*(X(\mathbb{C}))$, so that $\mathrm{H}^*(X)$ and $\mathrm{H}^*(X_{\mathbb{C}})$ are identified.

Suppose for the sake of contradiction that $B_{\mathbb{C}}$ can be generated by r elements. Then there is a $G_{\mathbb{C}}$-equivariant map $\varphi_r : T_{\mathbb{C}} \to U_{r,\mathbb{C}}$ representing $B_{\mathbb{C}}$ and a generating r-tuple of elements, by Proposition 6.2. The natural map $\mathrm{H}^i(BG) \to \mathrm{H}^i_G(T)$ does not depend on the chosen generators of $B_{\mathbb{C}}$, only on the isomorphism class of the $R_{\mathbb{C}}$ algebra $B_{\mathbb{C}}$, since the $G_{\mathbb{C}}$-space $T_{\mathbb{C}}$ can be recovered from $B_{\mathbb{C}}$. This leads us to the following commutative diagram

$$\begin{tikzcd} & \mathrm{H}^i(BG) \arrow[dl, "\cong"'] \arrow[d] \arrow[dr] \arrow[drr, bend left] & & \\ \mathrm{H}^i_G(U_{r+t}) \arrow[r, "\cong"] & \mathrm{H}^i_G(W \times U_{r+t}) \arrow[r, "\cong"] & \mathrm{H}^i_G(T) & \mathrm{H}^i_G(U_r) \arrow[l, "\varphi^*"'] \end{tikzcd} \tag{13.3}$$

in which the first three groups in the bottom row are isomorphic. The rightmost triangle in (13.3) is the triangle of Proposition 13.2, and the leftmost diagonal arrow is an isomorphism by Lemma 13.1. The isomorphism $\mathrm{H}^i(BG) \to \mathrm{H}^i_G(T)$ in (13.3) must factor through the non-injective map $\mathrm{H}^i(BG) \to \mathrm{H}^i_G(U_r)$, a contradiction. Therefore, $B_{\mathbb{C}}$ cannot be generated by r elements. This proves the claim.

Note that at this point we do not know what $d = \operatorname{trdeg}_k R = \dim R$ is, since we have no control over t or over $\dim W$ or over the dimension of the fibres of f. If $d \leqslant i$, then we are done. If $d > i$, we embed $\operatorname{Spec} R$ into an affine space $\mathbb{A}^N$ and denote by $\operatorname{Spec} R'$ the intersection of $\operatorname{Spec} R$ with a generic hyperplane . Let $\iota\colon \operatorname{Spec} R' \hookrightarrow \operatorname{Spec} R$ be the inclusion map. Then $\dim R' = \dim R - 1$. By Theorem 11.1, $\iota^* : \mathrm{H}^i(\operatorname{Spec} R) \to \mathrm{H}^i(\operatorname{Spec} R')$ is an isomorphism. Since $\mathrm{H}^i_G(\iota^* T) \hookrightarrow \mathrm{H}^i(\operatorname{Spec} R')$, we can apply the argument above with $T' = \iota^* T$ in place of T. It tells us that $B' = {}^{T'}A$ cannot be generated by r elements. Now replace R by R' and T by T'; we have reduced $\dim R$ by 1 while preserving the property that $\operatorname{gen}_R(B) > r$. Repeating this procedure, we eventually arrive at an example where $\dim R = i$. Since R is of finite type over k throughout, this is equivalent to $\operatorname{trdeg}_k R = i$. $\square$

14. Algebras requiring many generators

Recall that if X is defined over $k \hookrightarrow \mathbb{C}$, then we write $\mathrm{H}^*(X)$ for $\mathrm{H}^*(X(\mathbb{C}))$. Similarly, we write $\mathrm{H}^*(BG)$ in place of $\mathrm{H}^*(BG(\mathbb{C}))$.

In this section, we establish Theorem 1.4 by applying Theorem 13.4. We must first prove that $\mathrm{H}^i(BG)$ is nonzero for many values of i, which we do in Lemma 14.1. Our main tool, singular cohomology of the complex points, can be used only when the base field is embedded in $\mathbb{C}$, but by recourse to the Lefschetz principle in Lemma 12.1, we will be able prove Theorem 1.4 over any field of characteristic 0.

Lemma 14.1. *Let k be a field with a fixed embedding in $\mathbb{C}$. Let G be an affine algebraic group over k that is not unipotent. Then there exists a natural number ρ_G such that for all $i \geqslant 1$, there exists $j \in \{i+1, i+2, \dots, i+\rho_G\}$ such that*

$$\mathrm{H}^j(BG) \not\cong 0.$$

If G is connected, then we may suppose $\rho_G = 4$.

Proof. Since the cohomology depends only on $G(\mathbb{C}) = G_{\mathbb{C}}(\mathbb{C})$, and since $G_{\mathbb{C}}$ is unipotent if and only if G is, there is nothing to be lost by assuming $k = \mathbb{C}$.

The proof proceeds by a sequence of reductions. First, consider the natural exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

of algebraic groups, where N is the unipotent radical and H is reductive; see [Hoc81, Theorem 4.3]. Since N is unipotent, it admits a composition series by subgroups having quotients isomorphic to $\mathbb{G}_a$ [DG70, IV Th. de Lazard 4.1]. It follows that N is contractible as a topological space and so $BG \simeq BH$. Without loss of generality, therefore, we may assume that G is reductive.

If G is finite, then ρ_G exists by [BCR90, Theorem 2.4]. Thus, we may assume that G is infinite, i.e., $\dim G > 0$. We will show that in this case $\mathrm{H}^j(BG; \mathbb{Q}) \not\cong 0$ (and hence, $\mathrm{H}^j(BG; \mathbb{Z}) \not\cong 0$) for some $j \in \{i+1, i+2, i+3, i+4\}$. Let G^0 denote the identity component of G. Since the component group G/G^0 is finite, $\mathrm{H}^j(BG^0; \mathbb{Q}) = \mathrm{H}^j(BG; \mathbb{Q})$. We may therefore assume that G is nontrivial, connected and reductive. Let Z be the centre of G. We will treat the cases where Z is finite and infinite separately.

If Z is finite, then G is a semisimple linear algebraic group. By [CE48, Theorem 21.2], we know that $\tilde{\mathrm{H}}^l(G; \mathbb{Q}) \cong 0$ for values of $l \in \{0, 1, 2\}$ but $\tilde{\mathrm{H}}^3(G; \mathbb{Q}) \not\cong 0$. It is an easy consequence of the Serre spectral sequence that $\tilde{\mathrm{H}}^l(BG; \mathbb{Q}) \cong 0$ for $l \in \{0, 1, 2, 3\}$ but there exists a nonzero element $\alpha \in \tilde{\mathrm{H}}^4(BG; \mathbb{Q})$. We know, however, that $\mathrm{H}^*(BG; \mathbb{Q}) \cong \mathrm{H}^*(BT; \mathbb{Q})^W$, where the latter denotes the Weyl invariants of the cohomology of a maximal torus of a maximal compact subgroup of $G(\mathbb{C})$, by [Mal45, Theorem 11] and [Hsi75, Reduction 2, p36]. Since $\mathrm{H}^*(BT; \mathbb{Q})$ is a polynomial ring in some number of generators, the element α of the subring $\mathrm{H}^*(BT; \mathbb{Q})^W$ cannot be nilpotent. Therefore the elements α^l form a family of nonzero elements in $\mathrm{H}^{4l}(BG; \mathbb{Q})$. In this case, we may take $\rho_G = 4$.

If Z is not finite, then G contains a nontrivial torus Z' such that the quotient $G^{ss} = G/Z'$ is semisimple. In particular, $\tilde{\mathrm{H}}^l(BG^{ss}; \mathbb{Q})$ vanishes for values of $l \in \{0, 1, 2, 3\}$. A Serre spectral sequence argument shows that we can find a nontrivial element $\alpha \in \mathrm{H}^2(BG; \mathbb{Q})$ in this case, and we can repeat the same argument as in the previous case to show that the α^l form a family of nonzero elements in $\mathrm{H}^{2l}(BG; \mathbb{Q})$. In this case, we may take $\rho_G = 2$. $\square$

Remark 14.2. The construction of ρ_G in Lemma 14.1 uses the algebraic group structure of $G_{\mathbb{C}}$. However, since $\mathrm{H}^*(BG(\mathbb{C}))$ depends only on the Lie group $G(\mathbb{C})$ we can choose ρ_G which depends only on $G(\mathbb{C})$ and not on G.

Proof of Theorem 1.4. First of all, we consider the case where the field k is a subfield of $\mathbb{C}$. Set r to be the largest integer satisfying

$$d \geqslant 2rn - 2\dim_{\mathbb{C}} G + \rho_G. \tag{14.1}$$

In other words, set

$$r = \left\lfloor \frac{d + 2\dim G - \rho_G}{2n} \right\rfloor. \tag{14.2}$$

By Lemma 14.1, there exists an integer $d+1-\rho_G \leqslant i \leqslant d$ such that $\mathrm{H}^i(BG) \neq 0$. By (14.2), $i > 2rn - 2\dim_{\mathbb{C}} G$.

We claim that $\mathrm{H}^i_G(U_r) = 0$. Consider the quotient $U_r(\mathbb{C}) \to U_r(\mathbb{C})/G(C)$. This is a principal $G(\mathbb{C})$-bundle over a manifold of dimension $2rn - 2\dim_{\mathbb{C}} G$, because the Lie group $G(\mathbb{C})$ acts properly on $U_r(\mathbb{C})$ by Lemma 5.2. The quotient map $EG(\mathbb{C}) \times_{G(\mathbb{C})} U_r(\mathbb{C}) \to U_r(\mathbb{C})/G(\mathbb{C})$ is a fibre bundle with fibre $EG(\mathbb{C})$, as one can see by using an open cover of the manifold $U_r(\mathbb{C})/G(\mathbb{C})$ trivializing $U_r(\mathbb{C}) \to U_r(\mathbb{C})/G(\mathbb{C})$. This implies that $\mathrm{H}^*_G(U_r) \cong \mathrm{H}^*(U_r(\mathbb{C})/G(\mathbb{C}))$, see [Jam84, Theorem 7.57]. On the other hand, we know that $\mathrm{H}^j(U_r(\mathbb{C})/G(\mathbb{C})) = 0$ whenever $j > 2rn - 2\dim G$, since $U_r(\mathbb{C})/G(\mathbb{C})$ is a manifold of real dimension $2rn - 2\dim_{\mathbb{C}} G$. This proves the claim.

We conclude that the map on equivariant cohomology $\mathrm{H}^i(BG) \to \mathrm{H}^i_G(U_r)$ is not injective. By Theorem 13.4, there exists a k-ring R of finite type and an R-form B of A such that $\mathrm{trdeg}_k R \leqslant i \leqslant d$ and $\mathrm{gen}_{R_{\mathbb{C}}}(B_{\mathbb{C}}) > r$. Substituting in the formula for r from (14.2) and remembering $\mathrm{gen}_{R_{\mathbb{C}}}(B_{\mathbb{C}})$ is an integer, we can rewrite this inequality as

$$\mathrm{gen}_{R_{\mathbb{C}}}(B_{\mathbb{C}}) > \frac{d + 2\dim G - \rho_G}{2n}.$$

If $e := \mathrm{trdeg}_k R < d$, we replace R by the polynomial ring $R' = R[t_1, \ldots, t_{d-e}]$ and B by $B' = B \otimes_R R'$; it is easy to see that $\mathrm{trdeg}_k R' = d$ and

$$\mathrm{gen}_{R'_{\mathbb{C}}}(B'_{\mathbb{C}}) = \mathrm{gen}_{R_{\mathbb{C}}}(B_{\mathbb{C}}) > \frac{d + 2\dim G - \rho_G}{2n}.$$

By Lemma 12.1(a), this completes the proof of Theorem 1.4 in the case where k is a subfield of $\mathbb{C}$.

Now we establish the general case, where k is arbitrary field of characteristic 0. Since the finite-dimensional k-algebra A may be defined by means of finitely many structure constants, there exists a subfield $k_0 \subset k$ and a finite-dimensional k_0-algebra A_0, such that k_0 is finitely generated over $\mathbb{Q}$ and $A_0 \otimes_{k_0} k \cong A$. Since k_0 is finitely generated over $\mathbb{Q}$, we may embed k_0 in $\mathbb{C}$. Using the first part of this proof, we produce a k_0-ring R_0 and a form B_0 of A_0 over R_0 such that $\mathrm{trdeg}_{k_0}(R_0) = d$ and

$$\mathrm{gen}_{(R_0)_{\mathbb{C}}}((B_0)_{\mathbb{C}}) > \frac{d + 2\dim G - \rho_G}{2n}.$$

By Lemma 12.1(c),

$$\mathrm{gen}_R(B) > \frac{d + 2\dim G - \rho_G}{2n},$$

where $R = R_0 \otimes_{k_0} k$ is a finite type k-ring and $B = B_0 \otimes_{k_0} k$ is an R_K form of A. Moreover, $\mathrm{trdeg}_k R = \mathrm{trdeg}_{k_0}(R_0) = r$, where r is as in (14.2), as required. □

Remark 14.3. Theorem 1.4 fails if G is a connected unipotent group scheme. Indeed, in this case every G-torsor $T \to \mathrm{Spec}\, R$ splits, see e.g. [AD07, Corollary 3.2].* This tells us that the only R-form B of A (up to isomorphism) is $B = A \otimes_k R$, and $\mathrm{gen}_R(B) \leqslant \mathrm{gen}_k(A)$ for every R.

Remark 14.4. In [GP03], N. L. Gordeev and V. L. Popov show that over an infinite field k, all linear algebraic groups arise as automorphism groups of simple algebras; see also a recent refinement of this result by J. S. Milne [Mil20]. The algebras in question may not be associative, but they are algebras in the usual sense (and not just multialgebras).

*Here we are assuming that $\mathrm{char}(k) = 0$, as in Theorem 1.4. This is also a standing assumption in [AD07].

Even among the class of finite-dimensional commutative associative and unital algebras it is possible for the automorphism group to be nontrivial and unipotent. In [Pol89, Example 1.7], R. D. Pollack shows that the automorphism group of the finite-dimensional commutative, associative and unital $\mathbb{C}$-algebra

$$A = \frac{\mathbb{C}[x,y]}{(x,y)^5 + (x^2 - y^3, x^3 - y^4)}$$

is unipotent. Note that this group is nontrivial, since $x \mapsto x + y^4$, $y \mapsto y$ gives a non-trivial automorphism.

15. The geometry of matrix tuples

In this section and the next, we return to the study of the specific case where A denotes the algebra of $s \times s$ matrices, $M_s(k)$. We will assume k is field of characteristic 0 and $s \geqslant 3$ throughout. We continue to use the notation V_r to denote the affine variety whose k-points are A^r, the notation Z_r to denote the closed subscheme of r-tuples that do not generate A^r, and the notation U_r for $V_r - Z_r$. Recall from Section 9.1 that

$$Z_r^{\mathrm{red}} = X_1 \cup \ldots \cup X_{s-1},$$

where X_i is the variety of r-tuples $(a_1, \ldots, a_r)$ of $s \times s$-matrices having a common i-dimensional invariant subspace; see (9.1). Proposition 9.1(a) says that X_i is a closed irreducible subvariety of V_r of dimension $rs^2 - (r-1)i(s-i)$. We will be particularly interested in X_1, along with the PGL_s-invariant subvariety T_2 of V_r which we define as the variety of r-tuples $(a_1, \ldots, a_r)$ of $s \times s$-matrices satisfying the following conditions:

- there exists a subspace W having dimension at least 2, invariant under each a_i, $i = 1, \ldots, r$
- $a_1, \ldots, a_r$ commute pairwise when restricted to W.

Our ultimate goal is to prove Theorem 1.5(b). We will do this in the next section by showing that the natural map

$$\mathrm{H}^{(r-1)(s-1)}(B\,\mathrm{PGL}_s) \to \mathrm{H}^{(r-1)(s-1)}_{\mathrm{PGL}_s}(U_r)$$

is not injective, then appealing to Theorem 13.4. Since cohomological tools apply best to smooth varieties and transverse intersections, our main focus in this section will be on proving the following two preparatory results.

Proposition 15.1. *The variety X_1 is smooth away from T_2.*

Proposition 15.2. *Fix an invertible $(s-1)\times(s-1)$ matrix a such that none of the standard basis vectors of k^{s-1} is an eigenvector for a. Fix $(s-1)$ distinct elements $\lambda_2, \ldots, \lambda_s \in k^\times$. Let Y be the affine subspace of $V_r = \mathrm{M}_s^r$ consisting of r-tuples of the form*

$$\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & a & \\ 0 & & & \end{bmatrix}, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ x_{22} & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ x_{2s} & 0 & & \lambda_s \end{bmatrix}, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ x_{32} & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ x_{3s} & 0 & & \lambda_s \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ x_{r2} & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ x_{rs} & 0 & & \lambda_s \end{bmatrix} \right).$$

Then $Y \cap T_2 = \emptyset$, and Y intersects X_1 transversely in $V_r = \mathrm{M}_s^r$.

Note that in the definition of Y the matrix a and the distinct nonzero scalars $\lambda_2, \ldots, \lambda_s$ remain fixed, and each of $x_{22}, x_{23}, \ldots, x_{rs}$ varies over k. (Recall that we're assuming that $s \geqslant 3$ throughout. For $s \leqslant 2$, no such a exists.) Thus Y is isomorphic to the affine space $\mathbb{A}_k^{(s-1)(r-1)}$ linearly embedded into V_r. Here and subsequently

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \vec{e}_s = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

denote the standard basis vectors of k^s. The condition that no standard basis vector of k^{s-1} should be an eigenvector for a is equivalent to requiring that every column of a should have a non-zero element away from the main diagonal.

Since Propositions 15.1 and 15.2 are geometric in nature, we will assume that k is an algebraically closed field of characteristic 0 for the remainder of this section.

Proof of Proposition 15.1. Let $k\langle x_1, \ldots, x_r\rangle$ denote the free unital associative k-algebra in r non-commuting variables $x_1, \ldots, x_r$. Let C be the 2-sided ideal generated by the commutators $[x_i, x_j]$.

Now let $E = V_r \times k^s$ be the trivial vector bundle of rank s over V_r. For every $f \in k\langle x_1, \ldots, x_r\rangle$, we define an endomorphism $L_f \colon E \to E$ by $L_f(v) = f(a_1, \ldots, a_r) \cdot v$ over $(a_1, \ldots, a_r) \in V_r$. Let

$$N = \bigcap_{f \in C} \mathrm{Ker}(L_f)$$

be the intersection of the kernels of L_f, where f ranges over C. It follows from the noetherian property of E that N is, in fact, the kernel of the map

$$L = L_{f_1} \times \ldots \times L_{f_N} \colon E \to E \times_{V_r} E \times_{V_r} \ldots \times_{V_r} E$$

of vector bundles over V_r for finitely many elements $f_1, \ldots, f_N \in C$. Let $(V_r)_{\mathrm{null} \geqslant i}$ (respectively, $(V_r)_{\mathrm{null}=i}$) be the closed (respectively, locally closed) subvariety of V_r where the nullity of L is $\leqslant i$ (respectively, $= i$). Then N is a vector bundle of rank i over $(V_r)_{\mathrm{null}=i}$.

Lemma 15.3. *Denote the fibre of N over $a = (a_1, \ldots, a_r) \in V_r$ by N_a. Then*

(a) N_a is invariant under a_i for each $i = 1, \ldots, r$.

(b) $(V_r)_{\mathrm{null} \geqslant 1} = X_1$ and $(V_r)_{\mathrm{null} \geqslant 2} = T_2$.

Proof. (a) Suppose $f \in C$. Then $fx_i \in C$ for every $i = 1, \ldots, r$. Consequently, if v lies in N_a, then $L_f(a_i v) = L_{fx_i}(v) = 0$ for every $f \in C$. Thus $a_i v$ also lies in N_a.

(b) If a 1-dimensional subspace V of k^s is invariant under a_i for each $i = 1, \ldots, r$, then clearly the restrictions of $a_1, \ldots, a_r$ to V commute pairwise. This shows that $X_1 \subset (V_r)_{\mathrm{null} \geqslant 1}$. To prove the opposite inclusion, suppose $a \in (V_r)_{\mathrm{null} \geqslant 1}$, Then $\dim N_a \geqslant 1$ and the restrictions of $a_1, \ldots, a_r$ to N_a commute pairwise. Since we are assuming that our base field k is algebraically closed, $a_1, \dots, a_r$ have a common eigenvector in N_a. Hence, $a \in X_1$. $\square$

Remark 15.4. It is easy to see that T_2 is, in fact, a proper closed subvariety of X_2; in particular, $\dim T_2 < \dim X_2$.

We are now ready to finish the proof of Proposition 15.1. Let N_1 be the restriction of N to

$$(V_r)_{\mathrm{null}=1} = (V_r)_{\mathrm{null}\geqslant 1} - (V_r)_{\mathrm{null}\geqslant 2} = X_1 - T_2.$$

Then N_1 is a vector bundle of rank 1 over $(X_1 - T_2)$. Its projectivization $\mathbb{P}(N_1)$ is the incidence variety in $(X_1 - T_2) \times \mathbb{P}^{s-1}$ consisting of pairs

$$((a_1, \dots, a_r), L),$$

where L is a 1-dimensional subspace of k^s invariant under $(a_1, \dots, a_r) \in (X_1 - T_2)$. Since N_1 is a vector bundle of rank 1 over $(X_1 - T_2)$, the natural projection $\mathbb{P}(N_1) \to (X_1 - T_2)$ is an isomorphism.

It thus suffices to prove that $\mathbb{P}(N_1)$ is smooth. To prove that $\mathbb{P}(N_1)$ is smooth, we will take advantage of the fact that the natural projection map $N \to \mathbb{P}^{s-1}$ given by $\pi\colon ((a_1, \dots, a_r), L) \mapsto L$ is PGL_s-equivariant, and PGL_s acts transitively on the target $\mathbb{P}^{s-1}$. This tells us that π is flat and the fibres of π over the k-points of $\mathbb{P}^{s-1}$ are pairwise isomorphic. Moreover, the fibre over any closed point $v \in \mathbb{P}^{s-1}$ is an open subvariety of the affine space of all r-tuples of matrices having v as an eigenspace. Hence, it is smooth. Therefore, $\pi\colon \mathbb{P}(N_1) \to \mathbb{P}^{s-1}$ is flat with smooth fibres. We conclude that π is a smooth morphism, and consequently, $\mathbb{P}(N_1)$ is smooth over $\operatorname{Spec} k$, as claimed. $\square$

Proof of Proposition 15.2.

Lemma 15.5. *The scheme-theoretic intersection of the subvarieties Y and X_1 of V_r is a single reduced point*

$$p := \left(\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & a & \\ 0 & & & \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_s \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_s \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_s \end{bmatrix} \right).$$

Proof. We claim that p is the only geometric point in the intersection $X_1 \cap Y$. To see this, let L be an algebraically closed field and let $(a_1, \dots, a_r) \in (X_1 \cap Y)(L)$, where

$$a_1 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & a & \\ 0 & & & \end{bmatrix} \quad \text{and} \quad a_i = \begin{bmatrix} 0 & 0 & \dots & 0 \\ x_{i2} & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ x_{is} & 0 & & \lambda_s \end{bmatrix}$$

for $i = 2, \dots, r$. Here x_{ij} are elements of L. The eigenvectors of a_1 are nonzero multiples of $\vec{e}_1$ and $\begin{bmatrix} 0 \\ v_2 \\ \vdots \\ v_s \end{bmatrix}$, where $\begin{bmatrix} v_2 \\ \vdots \\ v_s \end{bmatrix}$ is an eigenvector of the invertible matrix a. The eigenvectors of a_i for $i \geqslant 2$ are nonzero multiples of $\vec{e}_2, \dots, \vec{e}_s$ and of $\begin{bmatrix} 1 \\ -\lambda_2^{-1} x_{i2} \\ \vdots \\ -\lambda_s^{-1} x_{is} \end{bmatrix}$. We conclude that the matrices $a_1, a_2, \dots a_r$ do not have a common eigenvector unless $x_{22} = x_{23} = \cdots = x_{ns} = 0$. This proves the claim. In the case where $x_{22} = x_{23} = \ldots = x_{ns} = 0$, the only eigenspace shared by $a_1, \dots, a_r$ is $\operatorname{Span}(\vec{e}_1)$.

It remains to show that the scheme-theoretic intersection $X_1 \cap Y$ is reduced, i.e., that it is isomorphic to $\operatorname{Spec} k$. Let $(R, \mathfrak{m})$ denote a local ring containing k. We wish to show that there is a unique morphism $f : \operatorname{Spec} R \to X_1 \cap Y$ given by composing $p\colon \operatorname{Spec} k \to X_1 \cap Y$ with the natural projection $\operatorname{Spec} R \to \operatorname{Spec} k$. Write L for the algebraic closure of $R/\mathfrak{m}$. As we showed above, the composite $\operatorname{Spec} L \to \operatorname{Spec} R \to X_1 \cap Y$ must be the geometric point p. Therefore, $f : \operatorname{Spec} R \to X_1 \cap Y$ must correspond to an R-valued point of the form

$$q = \left(\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & a & \\ 0 & & & \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & 0 \\ x_{22} & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ x_{2s} & 0 & & \lambda_s \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 0 \\ x_{r2} & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ x_{rs} & 0 & & \lambda_s \end{bmatrix} \right)$$

where each x_{ij} lies in $\mathfrak{m}$. The condition that q lies in $X_1(R)$ means that there exists a unimodular vector $\vec{v} := \begin{bmatrix} v_1 \\ \vdots \\ v_s \end{bmatrix} \in R^s$ such that $a_1 \vec{v} = \mu_1 \vec{v}, \dots, a_r \vec{v} = \mu_r \vec{v}$ for some $\mu_1, \dots, \mu_r \in R$. If we reduce modulo $\mathfrak{m}$, then as we noted in the previous paragraph, the only eigenspace shared by $a_1, \dots, a_r$ is $\operatorname{Span}(\vec{e}_1)$. In other words, if we reduce modulo $\mathfrak{m}$, the vector $\vec{v}$ will become a scalar multiple of $\vec{e}_1$. Therefore $v_1 \in R^\times$, while $v_2, \dots, v_s \in \mathfrak{m}$. Then the condition that $\vec{v}$ is an eigenvector of a_1 gives

$$\mu_1 \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & a & \\ 0 & & & \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_s \end{bmatrix} = \begin{bmatrix} 0 \\ w_2 \\ \vdots \\ w_s \end{bmatrix}, \text{ where } \begin{bmatrix} w_2 \\ \vdots \\ w_s \end{bmatrix} = a \begin{bmatrix} v_2 \\ \vdots \\ v_s \end{bmatrix}.$$

Because v_1 is a unit, $\mu_1 = 0$, so that $\vec{w} = 0$. Since a is an invertible $(s-1) \times (s-1)$-matrix, it follows that $v_2 = \cdots = v_s = 0$. Moreover, since $\vec{e}_1$ is a unit multiple of $\vec{v}$, we may replace $\vec{v}$ with $\vec{e}_1$. As in the field case, if $\vec{e}_1$ is an eigenvector of each $a_2, \dots, a_r$, then looking in the first entry implies that the associated eigenvalue is 0, whereupon $x_{ij} = 0$ for all $i = 2, \dots, r$

and $j = 2, \ldots, s$. This proves that q is uniquely determined, so that $X_1 \cap Y$ is reduced, as claimed. $\square$

We are now ready to complete the proof of Proposition 15.2. To show that $Y \cap T_2 = \emptyset$, observe that $T_2 \subset X_1$; see Lemma 15.3(b). By Lemma 15.5, $Y \cap X_1$ is a single point $p = (a_1, a_2, \ldots, a_r)$, where

$$a_1 = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & a & \\ 0 & & & \end{bmatrix} \quad \text{and} \quad a_2 = \ldots = a_r = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_s \end{bmatrix}.$$

Hence, it suffices to show that $p \notin T_2$. Assume the contrary: there exists a subspace $V \subset k^s$ of dimension $\geqslant 2$ which is invariant under a_1 and a_2 and such that a_1 and a_2 commute when restricted to V. Note that $\mathrm{Span}_k(\vec{e}_2, \ldots, \vec{e}_s)$ is invariant under both a_1 and a_2. Hence so is $V' = V \cap \mathrm{Span}_k(\vec{e}_2, \ldots, \vec{e}_n)$. Since $\dim V \geqslant 2$, we conclude that $\dim V' \geqslant 1$. Since $a_1|_{V'}$ and $a_2|_{V'}$ commute, a_1 and a_2 must therefore share an eigenvector in V'. But the only eigenvectors a_1 and a_2 share in k^s are scalar multiples of $\vec{e}_1$, a contradiction. We conclude that $p \notin T_2$.

It remains to show that X_1 and Y intersect transversely at p. Since $p \notin T_2$, the variety X_1 is smooth at p by Proposition 15.1. Since Y is an affine space, Y is also smooth at p. By definition Y is of dimension $(s-1)(r-1)$, and by Proposition 9.1(a), X_1 is of dimension $rn^2 - (s-1)(r-1)$. Their intersection is a single reduced point p, by Lemma 15.5. Since

$$\dim T_p(V_r) = rn^2 = \dim T_p(X_1) + \dim T_p(Y),$$

X_1 and Y intersect transversely in V_r. $\square$

16. Proof of Theorem 1.5(b)

In this section, unless otherwise stated, k denotes a subfield of $\mathbb{C}$. We will show that the natural map

$$\mathrm{H}^{2(s-1)(r-1)}(B\,\mathrm{PGL}_s) \to \mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_s}(U_r)$$

is not injective.

Consider the 1-parameter subgroup $\mathbb{G}_m \hookrightarrow \mathrm{PGL}_s$ given by $\lambda \mapsto \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & I_{s-1} \end{bmatrix}$. The group $\mathbb{G}_m$ acts on M_s by

$$\lambda \cdot \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & \lambda^{-1}a_{12} & \ldots & \lambda^{-1}a_{1n} \\ \lambda a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

Recall that the variety Y is isomorphic to $\mathbb{A}^{(s-1)(r-1)}$, the affine space with linear coordinates $(x_{22}, \ldots, x_{rn})$. It is invariant under $\mathbb{G}_m$, and the $\mathbb{G}_m$-action restricted to Y is given by

$$\lambda\colon (x_{22},\dots,x_{rn}) \mapsto (\lambda x_{22},\dots,\lambda x_{rn}).$$

If $(a_1,\dots,a_r) \in Y$, then $a_1,\dots,a_r$ share the $(s-1)$-dimensional invariant subspace $\mathrm{Span}(\vec{e_2},\dots,\vec{e_s})$. Therefore, $Y \subset X_{s-1}$ and the inclusion $U_r \to V_r$ factors through

$$U_r \longrightarrow V_r - Y \longrightarrow V_r\ .$$

The normal bundle N of $\operatorname{Spec} k \cong X_1 \cap Y$ in $Y \cong \mathbb{A}^{(s-1)(r-1)}$ is trivial of rank $(s-1)(r-1)$. As a $\mathbb{G}_m$-representation, it is $(s-1)(r-1)$ copies of the standard representation of $\mathbb{G}_m$ on $\mathbb{A}^1$. The top Chern class of this representation, $c^{\mathbb{G}_m}_{(s-1)(r-1)}(N)$, is therefore $\vartheta^{(s-1)(r-1)} \in \mathrm{H}^{2(s-1)(r-1)}_{\mathbb{G}_m}(\operatorname{Spec} k)$.

Remark 16.1. We describe four identifications that we use for the rest of the section:

First, because $X_1 \cap Y = \operatorname{Spec} k$, we may identify $\mathrm{H}^*_{\mathbb{G}_m}(X_1 \cap Y) = \mathbb{Z}[\vartheta]$, where $|\vartheta| = 2$. There are two choices for the ring generator ϑ, and we adopt the convention that

Second, we use the isomorphism $i^* : \mathrm{H}^*_{\mathbb{G}_m}(Y) \to \mathrm{H}^*_{\mathbb{G}_m}(X_1 \cap Y)$ to identify $\mathrm{H}^*_{\mathbb{G}_m}(Y) = \mathbb{Z}[\vartheta]$ as well.

Third, because of the isomorphism $V_r \cong \mathbb{A}^{rs^2}_k$, we may also identify $\mathrm{H}^*_{\mathbb{G}_m}(V_r) = \mathbb{Z}[\vartheta]$.

Finally, recall from Proposition 9.1(a) that X_2 is a closed irreducible subvariety of V_r of codimension $2(s-2)(r-1)$. By Remark 15.4, the codimension of T_2 in V_r is at least $2(s-2)(r-1)+1$. By Lemma 11.9, we deduce that for values of $j < 4(s-2)(r-1)+1$, the inclusion of spaces induces an isomorphism

$$\mathrm{H}^j_{\mathbb{G}_m}(V_r - T_2) \xrightarrow{\cong} \mathrm{H}^j_{\mathbb{G}_m}(V_r).$$

Using this, we identify the groups $\mathrm{H}^j_{\mathbb{G}_m}(V_r - T_2)$ with $\mathrm{H}^j_{\mathbb{G}_m}(V_r) = \mathrm{H}^j(B\mathbb{G}_m)$, provided $j < 4(s-2)(r-1)+1$.

Proposition 16.2. *The Gysin map*

$$j_* : \mathrm{H}^0_{\mathbb{G}_m}(X_1 \cap Y) \to \mathrm{H}^{2(s-1)(r-1)}_{\mathbb{G}_m}(Y)$$

on $\mathbb{G}_m$-equivariant cohomology, induced by the inclusion $(X_1 \cap Y) \to Y$, satisfies $j_*(1) = \vartheta^{(s-1)(r-1)}$.

Proof. The inclusion $X_1 \cap Y \to Y$ is isomorphic as a map of varieties with $\mathbb{G}_m$-action to the inclusion $\{0\} \to \mathbb{A}^{(s-1)(r-1)}_k$ where the affine space is given the standard $\mathbb{G}_m$-action.

In order to calculate the $\mathbb{G}_m$-equivariant cohomology groups of the spaces concerned, we compare the map with

$$\{0\} \times (\mathbb{A}^{N+1}_k - \{0\}) \to \mathbb{A}^{(s-1)(r-1)}_k \times (\mathbb{A}^{N+1} - \{0\})$$

where $N > (s-1)(r-1)$ and the $\mathbb{G}_m$ action on $\mathbb{A}^{N+1}_k$ is the standard one. Now we take quotients by the $\mathbb{G}_m$-action to arrive at

$$z : \mathbb{P}^N_k \to \mathbb{P}^{N+(s-1)(r-1)}_k - \mathbb{P}^{(s-1)(r-1)-1}_k,$$

which is the inclusion of $\mathbb{P}^N_k$ as the zero-section of the bundle $E = \mathcal{O}(1)^{(s-1)(r-1)}$.

The Gysin map j_* above fits in a commutative diagram:

$$\begin{tikzcd}
\mathrm{H}^0_{\mathbb{G}_m}(X_1 \cap Y) \arrow[r, "j_*"] \arrow[d, "="] & \mathrm{H}^{2(s-1)(r-1)}_{\mathbb{G}_m}(Y) \arrow[d, "="] \arrow[dr, "="] & \\
\mathrm{H}^0(\mathbb{P}^N_k) \arrow[r, "z_*"] & \mathrm{H}^{2(s-1)(r-1)}(E) \arrow[r, "z^*", "="'] & \mathrm{H}^{2(s-1)(r-1)}(\mathbb{P}^N_k).
\end{tikzcd}$$

The arrows marked "=" represent identifications.

From Remark 11.5, we know that the composite $z^* \circ z_*$ sends 1 to the Euler class of E, which is $\vartheta^{(s-1)(r-1)}$ by the multiplicativity of Euler classes [MS74, p. 156]. □

Lemma 16.3. *The Gysin map*

$$\mathrm{H}^*_{\mathbb{G}_m}(X_1 - T_2) \to \mathrm{H}^{*+2(s-1)(r-1)}_{\mathbb{G}_m}(V_r - T_2)$$

on $\mathbb{G}_m$-equivariant cohomology, induced by the inclusion $(X_1 - T_2) \hookrightarrow (V_r - T_2)$, sends $1_{X_1 - T_2}$ to $\vartheta^{(s-1)(r-1)}$. In particular, $\mathrm{H}^0_{\mathbb{G}_m}(X_1 - T_2) \to \mathrm{H}^{2(s-1)(r-1)}_{\mathbb{G}_m}(V_r - T_2)$ is nonzero.

Proof. There is a commutative square

$$\begin{tikzcd}
\mathbb{Z}[\vartheta] \cong \mathrm{H}^*_{\mathbb{G}_m}(Y \cap X_1) \arrow[d] & \mathrm{H}^*_{\mathbb{G}_m}(X_1 - T_2) \cong \mathbb{Z}[\vartheta] \arrow[l] \arrow[d] \\
\mathbb{Z}[\vartheta] \cong \mathrm{H}^{*+2(s-1)(r-1)}_{\mathbb{G}_m}(Y) & \mathrm{H}^{*+2(s-1)(r-1)}_{\mathbb{G}_m}(V_r - T_2) \arrow[l]
\end{tikzcd}$$

using Lemma 11.7 and the transversality of Y and $X_1 - T_2$ (Proposition 15.2). Lemma 16.2 describes the left vertical map. The upper horizontal map is an isomorphism since the varieties in the source and target are connected. The lower horizontal map is an isomorphism in low dimensions by reference to Remark 16.1. □

Lemma 16.4. *The Gysin map*

$$\mathrm{H}^0_{\mathrm{PGL}_n}(X_1 - T_2) \to \mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_n}(V_r - T_2)$$

on PGL_n-equivariant cohomology, induced by the inclusion $(X_1 - T_2) \hookrightarrow (V_r - T_2)$, is nonzero.

Proof. Using Lemma 11.8 we arrive at a diagram.

$$\begin{tikzcd}
\mathbb{Z} \cong \mathrm{H}^0_{\mathbb{G}_m}(X_1 - T_2) \arrow[d] & \mathrm{H}^0_{\mathrm{PGL}_s}(X_1 - T_2) \cong \mathbb{Z} \arrow[l] \arrow[d] \\
\mathrm{H}^{2(s-1)(r-1)}_{\mathbb{G}_m}(V_r - T_2) & \mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_s}(V_r - T_2) \arrow[l]
\end{tikzcd}$$

in which the upper horizontal arrow takes 1 to 1. Referring to Lemma 16.3 now establishes the result. □

Lemma 16.5. *The morphism* $\mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_s}(V_r) \to \mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_s}(U_r)$ *induced by the open immersion* $U_r \to V_r$ *is not injective.*

Proof. There is a factorization of the inclusion $U_r \to V_r$ as

$$U_r \subset V_r - (T_2 \cup X_1) \subset V_r - T_2 \subset V_r,$$

and each of these inclusions is PGL_s-equivariant. By functoriality, it suffices to show

$$\mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_s}(V_r) \to \mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_s}(V_r - (T_2 \cup X_1))$$

is not injective. Clearly, since $T_2 \subsetneq X_1$, the codimension of T_2 in V_r is strictly greater than the codimension of X_1, i.e., is strictly greater than $(r-1)(s-1)$; see Proposition 9.1(a). By Remark 16.1,

$$\mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_s}(V_r) \to \mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_s}(V_r - T_2)$$

is an isomorphism. It is therefore sufficient to show that

$$\mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_s}(V_r - T_2) \to \mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_s}(V_r - (T_2 \cup X_1))$$

is not injective. This follows from Proposition 16.4 in combination with the Gysin sequence

$$\mathrm{H}^{0}_{\mathrm{PGL}_s}(X_1 - T_2) \to \mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_s}(V_r - T_2) \to \mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_s}(V_r - (T_2 \cup X_1)). \qquad \square$$

Proof of Theorem 1.5(b). First, let us suppose the field k is $\mathbb{Q}$. Set r to be the largest integer satisfying $d \geqslant 2(s-1)(r-1)$. In other words, set

$$r = \left\lfloor \frac{d}{2(s-1)} \right\rfloor + 1. \tag{16.1}$$

We claim that the natural map $\mathrm{H}^{2(s-1)(r-1)}(B\,\mathrm{PGL}_s) \to \mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_n}(U_r)$ is not injective. This follows from the commutative diagram

$$\begin{tikzcd} & \mathrm{H}^{2(s-1)(r-1)}(B\,\mathrm{PGL}_s) \arrow[dl, "\cong"'] \arrow[dr] & \\ \mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_s}(V_r) \arrow[rr] & & \mathrm{H}^{2(s-1)(r-1)}_{\mathrm{PGL}_n}(U_r) \end{tikzcd}$$

along with Lemma 16.5.

We now apply Theorem 13.4 with $i = 2(s-1)(r-1)$ to produce a $\mathbb{Q}$-ring R' and an Azumaya R'-algebra B' such that $\mathrm{trdeg}_{\mathbb{Q}} R' \leqslant i = 2(s-1)(r-1)$ and $\mathrm{gen}_{R'_{\mathbb{C}}}(B'_{\mathbb{C}}) \geqslant r+1$. Recall that by our choice of r, we have $i \leqslant d$. After replacing R' by the polynomial ring $R = R'[x_1, \ldots, x_{d-i}]$ and B' by $B = B' \otimes_{R'} R$, we arrive an an Azumaya algebra B over R such that $\mathrm{trdeg}_{\mathbb{Q}} R = d$ and $\mathrm{gen}_{R_{\mathbb{C}}}(B_{\mathbb{C}}) = \mathrm{gen}_{R'_{\mathbb{C}}}(B'_{\mathbb{C}}) \geqslant r+1$. In particular, $\mathrm{gen}_R(B) \geqslant r+1$; see Lemma 12.1. This completes the proof of Theorem 1.5(b) in the case, where $k = \mathbb{Q}$.

In the general case, where k is an arbitrary field of characteristic 0 and not necessarily embedded in $\mathbb{C}$, we invoke the Lefschetz principle of Lemma 12.1. Consider the ring R

and Azumaya algebra B produced in the case of $\mathbb{Q}$. Set $R_k = R \otimes_{\mathbb{Q}} k$ and $B_k = R \otimes_{\mathbb{Q}} k$. Clearly $\operatorname{trdeg}_k(R_k) = \operatorname{trdeg}_{\mathbb{Q}} R = d$. Moreover,

$$\operatorname{gen}_{R_k}(B_k) \geqslant \operatorname{gen}_{R_{\mathbb{C}}}(B_{\mathbb{C}}) \geqslant r+1 = \left\lfloor \frac{d}{2(s-1)} \right\rfloor + 2$$

by Lemma 12.1(c). $\square$

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Department of Mathematics, Faculty of Natural Sciences, University of Haifa, Mount Carmel, Haifa, 3498838, Israel
Email address: ufirst@univ.haifa.ac.il

Department of Mathematics, University of British Columbia, Vancouver BC V6T 1Z2, Canada
Email address: reichst@math.ubc.ca
Email address: tbjw@math.ubc.ca