ON A RATIONALITY PROBLEM FOR FIELDS OF CROSS-RATIOS

ZINOVY REICHSTEIN

ABSTRACT. Let k be a field, $n \geqslant 5$ be an integer, $x_1, \dots, x_n$ be independent variables and $L_n = k(x_1, \dots, x_n)$. The symmetric group Σ_n acts on L_n by permuting the variables, and the projective linear group PGL_2 acts by applying (the same) fractional linear transformation to each varaible. The fixed field $K_n = L_n^{\mathrm{PGL}_2}$ is called "the field of cross-ratios". Let $S \subset \Sigma_n$ be a subgroup. The Noether Problem asks whether the field extension L_n^S/k is rational, and the Noether Problem for cross-ratios asks whether K_n^S/k is rational. In an effort to relate these two problems, H. Tsunogai posed the following question: Is L_n^S rational over K_n^S? He answered this question in several situations, in particular, in the case where $S = \Sigma_n$. In this paper we extend his results by recasting the problem in terms of Galois cohomology. Our main theorem asserts that the following conditions on a subgroup $S \subset \Sigma_n$ are equivalent: (a) L_n^S is rational over K_n^S, (b) L_n^S is unirational over K_n^S, (c) S has an orbit of odd order in $\{1, \dots, n\}$.

1. INTRODUCTION

Let k be a base field, $n \geqslant 5$ be an integer, $x_1, \dots, x_n$ be independent variables, and

$$L_n = k(x_1, \dots, x_n).$$

The group PGL_2 acts on L_n via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x_i \to \frac{ax_i + b}{cx_i + d}$$

for $i = 1, \dots, n$. The field of invariants $K_n = L_n^{\mathrm{PGL}_2}$ is generated over k by the $n-3$ cross-ratios

$$[x_1, x_2, x_3, x_i] = \frac{(x_i - x_1)(x_3 - x_2)}{(x_i - x_2)(x_3 - x_1)}\,, \quad i = 4, \dots, n.$$

For this reason we will refer to K_n as the field of cross-ratios. The natural action of the symmetric group Σ_n on L_n induced by permuting the variables descends to a faithful action on K_n. Suppose S is a subgroup of Σ_n.

The Noether problem asks whether the fixed field L_n^S is rational (respectively, stably rational or retract rational) over k. The Noether Problem for cross-ratios is whether or not K_n^S is rational (respectively, stably rational or retract rational) over k. In an effort to relate these two problems, H. Tsunogai [Tsu17] posed the following question:

Question 1. Is L_n^S is rational over K_n^S?

2010 *Mathematics Subject Classification.* 14E08, 12G05, 16H05, 16K50.

Key words and phrases. Rationality, Galois cohomology, the Noether problem, quaternion algebras, Galois algebras, Brauer group.

Partially supported by National Sciences and Engineering Research Council of Canada Discovery grant 253424-2017.

He answered this question in several situations (see [Tsu17, Theorems 1, 2, 3]) in particular, in the case, where $S = \Sigma_n$. Our main theorem generalizes his results as follows.

Theorem 2. *Let S be a subgroup of the symmetric group Σ_n, where $n \geqslant 5$. Then the following conditions are equivalent:*

(a) *L_n^S is rational over K_n^S,*
(a) *L_n^S is unirational over K_n^S,*
(c) *S has an orbit of odd order in $\{1, \dots, n\}$.*

The remainder of this note will be devoted to proving Theorem 2.

2. Recasting the problem in the language of Galois cohomology

Let G be the subgroup of $(\mathrm{GL}_2)^n = \mathrm{GL}_2 \times \cdots \times \mathrm{GL}_2$ consisting of n-tuples $(g_1, \dots, g_n)$ such that $\overline{g_1} = \cdots = \overline{g_n}$. Here $\overline{g}$ denotes the image of $g \in \mathrm{GL}_2$ in PGL_2. In other words, $(g_1, \dots, g_n) \in (\mathrm{GL}_2)^n$ lies in G if and only if $g_1, \dots, g_n$ are scalar multiples of each other. The symmetric group Σ_n acts naturally on $(\mathrm{GL}_2)^n$ by permuting the entries; G is invariant under this action. For any subgroup $S \subset \Sigma_n$, we will denote the semidirect product $G \rtimes S$ by G_S. This gives rise to the natural split exact sequence

$$1 \longrightarrow G \xrightarrow{i} G_S \xrightarrow{\phi} S \longrightarrow 1 \tag{1}$$

We will also be interested in another exact sequence,

$$1 \longrightarrow (\mathbb{G}_m^n) \rtimes S \xrightarrow{\alpha} G_S \xrightarrow{\beta} \mathrm{PGL}_2 \longrightarrow 1\ , \tag{2}$$

where map $G \to \mathrm{PGL}_2$ sends $(g_1, \dots, g_n) \in G$ to $\overline{g_1} = \cdots = \overline{g_n}$.

Consider the natural linear action of G_S on the $2n$-dimensional affine space $V = (\mathbb{A}^2)^n$ defined as follows: $(g_1, \dots, g_n) \in G$ acts on $(\mathbb{A}^2)^n$ by

$$(g_1, \dots, g_n) :\ (v_1, \dots, v_n) \mapsto (g_1 v_1, \dots, g_n v_n)$$

and $\sigma \in S \subset \Sigma_n$ by

$$\sigma :\ (v_1, \dots, v_n) \mapsto (v_{\sigma(1)}, \dots, v_{\sigma(n)}).$$

One readily checks that this action is generically free. (Recall that our standing assumption is that $n \geqslant 5$.) That is, V has a dense G-invariant Zariski open subset V_0, such that the stabilizer of v in G_S is trivial for every $v \in V_0$. After passing to a smaller G_S-invariant open subset, we may assume that V_0 is the total space of a G_S-torsor $T_S \colon V_0 \to Z_S$ for some k-variety Z_S; see [BF03, Theorem 4.7]. We thus obtain the following diagram:

$$\begin{tikzcd} & V_0 \arrow[d, "(\mathbb{G}_m^n \rtimes S)\text{-torsor}"'] \arrow[dd, bend left, "T_S\text{, a } G_S\text{-torsor}"] \\ & Y_S \arrow[d, "t_S\text{, a } \mathrm{PGL}_2\text{-torsor}"'] \\ \eta \arrow[r] & Z_S \end{tikzcd}$$

where $Y_S = V_0/(\mathbb{G}_m^n \rtimes S)$. The function fields $k(Z_S)$ and $k(Y_S)$ are naturally isomorphic to L_n^S and K_n^S, respectively. When we pass to the generic point η of Z_S, T_S gives rise to a G_S-torsor

$(V_0)_\eta \to \mathrm{Spec}(K_n^S)$ and t_S to a PGL_2-torsor $(Y_S)_\eta \to \mathrm{Spec}(K_n^S)$, respectively. By abuse of notation we will continue to denote these torsors by T_S and t_S.

Now let K is an arbitrary field. Recall that G_S-torsors over $\mathrm{Spec}(K)$ are classified by the Galois cohomology set $H^1(K, G_S)$, and PGL_2-torsors are classified by $H^1(K, \mathrm{PGL}_2)$; see [Se97, §I.5.2]. We will denote the classes of T_S and t_S by $[T_S] \in H^1(K_n^S, G_S)$ and $[t_S] \in H^1(K_n^S, \mathrm{PGL}_2)$, respectively. The exact sequences (1) and (2) of algebraic groups give rise to exact sequences of Galois cohomology sets

$$H^1(K, G) \xrightarrow{i_1} H^1(K, G_S) \xrightarrow{\phi_1} H^1(K, S) \tag{3}$$

and

$$H^1(K, \mathbb{G}_m^n \rtimes S) \xrightarrow{\alpha_1} H^1(K, G_S) \xrightarrow{\beta_1} H^1(K, \mathrm{PGL}_2) \tag{4}$$

for any field K. If $K = K_n^S$, then by our construction $[t_S] = \beta_1([T_S])$. The following proposition recasts Question 1 in the language of Galois cohomology.

Proposition 3. *The following conditions on a subgroup $S \subset \Sigma_n$ are equivalent:*

(a) L_n^S is rational over K_n^S,

(b) L_n^S is unirational over K_n^S,

(c) $[t_S]$ is the trivial class in $H^1(K_n^S, \mathrm{PGL}_2)$,

(d) $\beta_1 \colon H^1(K, G_S) \to H^1(K, \mathrm{PGL}_2)$ is the trivial for every field K containing k,

(e) $\alpha_1 \colon H^1(K, \mathbb{G}_m^n \rtimes S) \to H^1(K, G_S)$ is surjective for every field K containing k,

(f) $\phi_1 \colon H^1(K, G_S) \to H^1(K, S)$ is bijective for every field K containing k,

(g) $H^1(K, {}_\tau G) = 1$ for every $\tau \in H^1(K, S)$.

In part (g), ${}_\tau G$ denotes the twist of G by τ via the natural permutation action of S on G. For generalities on the twisting operation, see [Se97, Section I.5.3] or [B10, Section II.5]. Note in particular that ${}_\tau G$ is an algebraic group over K; it does not descend to k in general.

Remark 4. The Galois cohomology set $H^1(K, \mathrm{PGL}_2)$ is in a natural (i.e., functorial in K) bijective correspondence with the set of isomorphism classes of quaternion algebras over K; see [Se03, §I.2 and I.3]. Thus condition (c) amounts to saying that a certain quaternion algebra over K_n^S is split.

We defer the proof of Proposition 3 to Section 4.

3. Generalities on Galois cohomology

Suppose $i \colon A \to B$ is a morphism of algebraic groups over k, and K is a field containing k. Following the notational conventions of the previous section, we will denote the induced map $H^d(K, A) \to H^d(K, B)$ of cohomology sets by i_d. Here $d = 0$ or 1.

The following lemma will be used in the proof of Proposition 3.

Lemma 5. *Consider the exact sequence*

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "\pi"] & C \arrow[r] \arrow[l, bend left, "s"] & 1 \end{tikzcd} \tag{5}$$

of smooth algebraic groups over a field k. Then

(a) The map $\pi_1\colon H^1(K,B)\to H^1(K,C)$ is surjective for every field K/k.

(b) π_1 is injective if and only if $H^1(K, {}_{s_1(\gamma)}A)=1$ for every $\gamma\in H^1(K,C)$.

Proof. (a) is clear, since $s_1\colon H^1(K,C)\to H^1(K,B)$ is a section for π_1. To prove (b), twist the exact sequence (5) by $\tau=s_1(\gamma)$ to obtain a new exact sequence

$$\begin{tikzcd} 1 \arrow[r] & {}_\tau A \arrow[r] & {}_\tau B \arrow[r, "{}_\tau\pi"] & {}_\tau C \arrow[r] \arrow[l, bend left, "{}_\tau s"] & 1 \end{tikzcd}$$

of algebraic groups over K and consider the associated long exact sequence

$$\begin{tikzcd} H^0(K, {}_\tau B) \arrow[r, "{}_\tau\pi_0"] & H^0(K, {}_\tau C) \arrow[r, "\delta"] \arrow[l, bend left, "{}_\tau s_0"] & H^1(K, {}_\tau A) \arrow[r, "{}_\tau i_1"] & H^1(K, {}_\tau B) \arrow[r, "{}_\tau\pi_1"] & H^1(K, {}_\tau C) \arrow[l, bend left, "{}_\tau s_1"] \end{tikzcd} \tag{6}$$

in cohomology. Note that ${}_\tau C$ is naturally isomorphic to ${}_\gamma C$. By [Se97, Corollary I.5.5.2], the fiber of γ under π_1 is in bijective correspondence with the set of $H^0(K, {}_\tau C)$-orbits in $H^1(K, {}_\tau A)$. We are interested in the case, where π_1 is injective, i.e., this fiber is trivial for every $\gamma\in H^1(K,C)$.

Since ${}_\tau s_0$ is a section for ${}_\tau\pi_0$, we see that ${}_\tau\pi_0$ is surjective. Thus the connecting map δ in the long exact sequence (6) sends every element of $H^0(K, {}_\tau C)$ to the trivial element of $H^1(K, {}_\tau A)$. Consequently, $H^0(K, {}_\tau C)$ acts trivially on $H^1(K, {}_\tau A)$. We conclude that the fiber of γ under π_1 is in bijective correspondence with $H^1(K, {}_\tau A)$. In particular, π_1 is injective if and only if $H^1(K, {}_\tau A)=1$ for every γ, as claimed. □

Corollary 6. *Consider the split exact sequence*

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{G}_m^n \arrow[r, "i"] & \mathbb{G}_m^n \rtimes S \arrow[r, "\pi"] & S \arrow[r] \arrow[l, bend left, "s"] & 1 \end{tikzcd}, \tag{7}$$

where S is a subgroup of Σ_n, and $\mathbb{G}_m^n\rtimes S$ is the semidirect product with respect to the natural (permutation) action of S on $\mathbb{G}_m^n$. Then π and s induce mutually inverse bijections $\pi_1\colon H^1(K,\mathbb{G}_m^n\rtimes S)\to H^1(K,S)$ and $s_1\colon H^1(K,S)\to H^1(K,\mathbb{G}_m\rtimes S)$ for every field K containing k.

Proof. By Lemma 5 it suffices to show that

$$H^1(K, {}_\gamma(\mathbb{G}_m^n)) = 1 \text{ for every } \gamma\in H^1(K,S). \tag{8}$$

Here S acts on $\mathbb{G}_m^n$ by permuting the n copies of $\mathbb{G}_m$. Thus the twisted group ${}_\gamma(\mathbb{G}_m^n)$ is a quasi-trivial torus, and (8) follows from the Feddeev-Shapiro Lemma [Se97, Section I.2.5]. □

4. Proof of Proposition 3

The implication (a) $\Longrightarrow$ (b) is obvious.

(b) $\Longrightarrow$ (c): If L_n^S is unirational over K_n^S, then t_S has a rational section, and (c) follows.

(c) $\Longrightarrow$ (a): If $[t_S]=1$ is the trivial class in $H^1(K_n^S,\mathrm{PGL}_2)$, then Y_S is birationally isomorphic to $Z_S\times\mathrm{PGL}_2$ over Z_S. Since the group variety of PGL_2 is rational over k, this tells us that Y_S is rational over Z_S. Equivalently, $k(Y_S)=L_n^S$ is rational over $k(Z_S)=K_n^S$.

(c) $\Longrightarrow$ (d): By [Se03, Example I.5.4], $[T_S]$ is a versal G_S-torsor. This implies that if $[t_S] = \beta_1([T_S])$ is trivial in $H^1(K_n^S, \mathrm{PGL}_2)$, then the image of every element of $H^1(K, G_S)$ under β_1 is trivial in $H^1(K, \mathrm{PGL}_2)$ for every infinite field K containing k, as desired. It remains to consider the case, where K is a finite field. By Wedderburn's "little theorem" every quaternion algebra over a finite field K is split. In view of Remark 4, this translates to $H^2(K, \mathrm{PGL}_2) = \{1\}$. We conclude that the map $H^1(K, G_S) \to H^1(K, \mathrm{PGL}_2)$ is trivial for every field K containing k.

(d) $\Longrightarrow$ (c) is obvious.

(d) $\Longleftrightarrow$ (e): Follows from the fact that the sequence (4) is exact.

(e) $\Longleftrightarrow$ (f): Consider the group homomorphisms $S \xrightarrow{s} \mathbb{G}_m^n \rtimes S \xrightarrow{\alpha} G_S \xrightarrow{\phi} S$, whose composition is the identity map $S \to S$. Note that here ϕ, α and s are the same as in (1), (2), and (7), respectively. Let us examine the induced sequence

$$\begin{tikzcd} H^1(K,S) \arrow[r, "\simeq", "s_1"'] \arrow[rrr, bend right, "\mathrm{id}"'] & H^1(K, \mathbb{G}_m^n \rtimes S) \arrow[r, "\alpha_1"] & H^1(K, G_S) \arrow[r, "\phi_1"] & H^1(K,S) \end{tikzcd}$$

in cohomology. By Corollary 6, s_1 is an isomorphism. Thus α_1 is surjective if and only of ϕ_1 is bijective, as claimed.

(f) $\Longleftrightarrow$ (g): Immediate from Lemma 5, applied to the exact sequence (1). $\square$

5. Reduction to the case, where S is a 2-group

Lemma 7. *Let P be a subgroup of S. Assume that the index $d = [S : P]$ is odd. Then $[t_S]$ is trivial in $H^1(K_n^S, \mathrm{PGL}_2)$ if and only if $[t_P]$ is trivial in $H^1(K_n^P, \mathrm{PGL}_2)$.*

Proof. The diagram

$$\begin{tikzcd} & V_0 \arrow[dl] \arrow[dr] \arrow[ddl, bend right, "T_P"'] \arrow[ddr, bend left, "T_S"] & \\ Y_P \arrow[rr, "\deg d"'] \arrow[d, "t_P"] & & Y_S \arrow[d, "t_S"'] \\ Z_P \arrow[rr, "\deg d"'] & & Z_S \end{tikzcd}$$

shows that $[t_P]$ is the image of $[t_S]$ under the restriction map

$$r : H^1(K_n^S, \mathrm{PGL}_2) \to H^1(K_n^P, \mathrm{PGL}_2).$$

By Proposition 3 it suffices to show that r has trivial kernel. By Remark 4 elements of the Galois cohomology set $H^1(K, \mathrm{PGL}_2)$ can be identified with quaternion algebras over K (up to K-isomorphism). The map r sends a quaternion algebra A over K_n^S to the quaternion algebra $A \otimes_{K_n^S} K_n^P$ over K_n^P. Since K_n^P / K_n^S is a field extension of odd degree, $A \otimes_{K_n^S} K_n^P$ is split if and only if A is split. Thus r has trivial kernel, as claimed. $\square$

Combining Lemma 7 with the equivalence of (a), (b), (c) in Proposition 3, we see that for the purpose of proving Theorem 2, S may be replaced its 2-Sylow subgroup P. Note that S has an orbit of odd order in $\{1, 2, \dots, n\}$ if and only if P has an orbit of odd order in $\{1, 2, \dots, n\}$ if and only if P has a fixed point. By the equivalence of parts (a), (b) and (g) in Proposition 3, in order to complete the proof of Theorem 2, it suffices to establish the following.

Proposition 8. *Let S be a 2-subgroup of Σ_n. Then the following conditions are equivalent.*

(i) $H^1(K, {}_\tau G) = 1$ *for every* $\tau \in H^1(K, S)$.

(ii) S *has a fixed point in* $\{1, \dots, n\}$.

6. Conclusion of the proof of Theorem 2

In this section we will complete the proof of Theorem 2 by establishing Proposition 8.

Denote the orbits of S in $\{1, \dots, n\}$ by $\mathcal{O}_1, \dots, \mathcal{O}_t$ where $\mathcal{O}_i \simeq S/S_i$ as a G-set. Here S_i is the stabilizer of a point in $\mathcal{O}_i$. The groups $S_1, \dots, S_t$ are uniquely determined by the embedding $S \hookrightarrow \Sigma_n$ up to conjugacy and reordering. Note that $S_1, \dots, S_t$ may not be distinct.

Recall that elements of $\tau \in H^1(K, S)$ are in a natural bijective correspondence with S-Galois algebras L/K. Here by an S-Galois algebra L/K we mean an étale algebra (i.e., a direct sum of finite separable field extensions of K) equipped with a faithful action of S such that $\dim_K(L) = |S|$ and $L^S = K$; see [Se03, Example 2.2]. To an S-Galois algebra L/K one can naturally associate the étale K-algebra

$$E = L^{S_1} \times \cdots \times L^{S_t}$$

of degree n.

Now observe that the group G (defined at the beginning of Section 2) admits the following alternative description. Consider the natural surjective map $f\colon \operatorname{GL}_2 \times \mathbb{G}^m \to G$ given by

$$(g, t_1, \dots, t_n) \to (gt_1, gt_2, \dots, gt_n).$$

The kernel of f is $\Delta = \{(tI_2, t, \dots, t) \,|t \in \mathbb{G}_m\} \simeq \mathbb{G}_m$. Thus f induces an isomorphism $G \simeq (\operatorname{GL}_2 \times \mathbb{G}_m^n)/\Delta$ of algebraic groups. Moreover, this isomorphism is S-equivariant with respect to the natural actions of S on G (described at the beginning of section 2) and $(\operatorname{GL}_2 \times \mathbb{G}_m^n)/\Delta$ (via permuting the n components of $\mathbb{G}_m$). The twisted forms ${}_\tau G$ of G and the Galois cohomology sets $H^1(K, {}_\tau G)$ are explicitly described in [FR18]. In particular, if $\tau \in H^1(K, S)$ corresponds to the n-dimensional étale K-algebra E as above, then ${}_\tau G \simeq (\operatorname{GL}_2 \times R_{E/K}(\mathbb{G}_m))/\Delta_K$, where $R_{E/K}$ denotes Weil restriction and $\Delta_K \simeq {}_\tau\Delta$ is $\mathbb{G}_m$ (over K), diagonally embedded into $\operatorname{GL}_2 \times R_{E/K}(\mathbb{G}_m)$; see [FR18, Section 4]. Moreover,

$$H^1(K, {}_\tau G) \simeq \{\text{isom. classes of quaternion } K\text{-algebras } A \text{ such that } A \text{ is split by } E \otimes_k K\};$$

see [FR18, Lemma 5.1]. Note that A is split by $E \otimes_k K$ if and only if A is split by $L^{S_i} \otimes_k K$ for every $i = 1, \dots, t$. This explicit description of $H^1(K, {}_\tau G)$ reduces Proposition 8 to the following equivalent form.

Proposition 9. *Let $S \subset \Sigma_n$ be a 2-group. Then the following conditions on S are equivalent.*

(a) There exists a field a field extension K/k, a quaternion division algebra A/K and S-Galois algebra L/K such that A splits over L^{S_i} for every $i = 1, \dots, t$.

(b) S does not have a fixed point in $\{1, \dots, n\}$.

To prove (a) $\Longrightarrow$ (b), assume that S has a fixed point. That is, one of the orbits of S in $\{1, \dots, n\}$, say $\mathcal{O}_1$, consists of a single point. Equivalently, $S_1 = S$. Clearly a quaternion division algebra over K, cannot split over $L^{S_1} = L^S = K$.

To prove (b) $\implies$ (a), assume S does not have a fixed point in $\{1, 2, \dots, n\}$, i.e., $|\mathcal{O}|_i = |S/S_i| \geq 2$ for every $i = 1, \dots, t$. Since S is a 2-group, each S_i is contained in a maximal proper subgroup of S. That is, for each $i = 1, \dots, t$, there exists a subgroup

$$S_i \subseteq H_i \subsetneq S \text{ such that } [S : H_i] = 2.$$

Note that H_i, being a subgroup of index 2, is normal in S.

Now let M/F be an S-Galois field extension. For example, we can let S act on $M = k(x_1, \dots, x_n)$ by permuting the variables and set $F = M^S$. Since H_i is normal in S, M^{H_i}/F is a Galois extension of degree 2 for each $i = 1, \dots, t$. By a theorem of M. Van den Bergh and A. Schofield [VdB-S94, Theorem 3.8], there exists a field extension K/F and a quaternion division algebra A/K such that A contains $M^{H_i} \otimes_F K$ as a maximal subfield for each $i = 1, \dots t$. Now consider the S-Galois algebra $L = M \otimes_F K$ over K. For each $i = 1, \dots, t$, L^{S_i} contains $L^{H_i} = M^{H_i} \otimes_F K$. Hence, each L^{S_i}, splits A, as desired.

This completes the proof of Proposition 9 and thus of Proposition 8 and Theorem 2. □

Acknowledgments

The author is grateful to Skip Garibaldi for helpful comments.

References

[B10] G. Berhuy, *An introduction to Galois cohomology and its applications*, London Mathematical Society Lecture Note Series, 377, Cambridge University Press, Cambridge, 2010. MR2723693

[BF03] G. Berhuy and G. Favi, Essential dimension: a functorial point of view (after A. Merkurjev), Doc. Math. **8** (2003), 279–330. MR2029168

[FR18] M. Florence, Z. Reichstein, The rationality problem for forms of moduli spaces of $\overline{M}_{0,n}$, Bull. London Math. Soc., **50**, no. 1 (2018), 148–158.

[Se97] J.-P. Serre, *Galois cohomology*, Springer-Verlag, Berlin, 1997.

[Se03] J.-P. Serre, Cohomological invariants, Witt invariants, and trace forms, notes by Skip Garibaldi, in *Cohomological invariants in Galois cohomology*, 1–100, Univ. Lecture Ser., 28, Amer. Math. Soc., Providence, RI, 2003. MR1999384

[Tsu17] H. Tsunogai, Toward Noether's problem for the fields of cross-ratios, Tokyo J. Math. **39** (2017), no. 3, 901–922. MR3634298

[VdB-S94] M. Van den Bergh and A. Schofield, Division algebra coproducts of index n, Trans. Amer. Math. Soc. **341** (1994), no. 2, 505–517. MR1033236

Department of Mathematics, University of British Columbia, BC, Canada V6T 1Z2
E-mail address: reichst@math.ubc.ca