FIELDS OF DEFINITION FOR REPRESENTATIONS OF ASSOCIATIVE ALGEBRAS

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Abstract. We examine situations, where representations of a finite-dimensional $F$-algebra $A$ defined over a separable extension field $K/F$, have a unique minimal field of definition. Here the base field $F$ is assumed to be a $C_1$-field. In particular, $F$ could be a finite field or $k(t)$ or $k((t))$, where $k$ is algebraically closed.

We show that a unique minimal field of definition exists if (a) $K/F$ is an algebraic extension or (b) $A$ is of finite representation type. Moreover, in these situations the minimal field of definition is a finite extension of $F$. This is not the case if $A$ is of infinite representation type or $F$ fails to be $C_1$. As a consequence, we compute the essential dimension of the functor of representations of a finite group, generalizing a theorem of N. Karpenko, J. Pevtsova and the second author.

1. Introduction

Notational conventions. Throughout this paper $F$ will denote a base field and $A$ a finite-dimensional associative algebra over $F$. If $K/F$ is a field extension (not necessarily algebraic), we will denote the tensor product $K \otimes_F A$ by $A_K$. Let $M$ be an $A_K$-module. Unless otherwise specified, we will always assume that $M$ is finitely generated (or equivalently, finite-dimensional as a $K$-vector space). If $L/K$ is a field extension, we will write $M_L$ for $L \otimes_K M$.

An intermediate field $F \subset K_0 \subset K$ is called a field of definition for $M$ if there exists a $K_0$-module $M_0$ such that $M \cong (M_0)_K$. In this case we will also say that $M$ descends to $K_0$.

Minimal fields of definition. A field of definition $K_0$ of $M$ is said to be minimal if whenever $M$ descends to a field $L$ with $F \subset L \subset K$, we have $K_0 \subset L$.

Minimal fields of definition do not always exist. For example, let $F = \mathbb{Q}$ and $A$ be the quaternion algebra

$$A = \mathbb{Q}\{i,j,k\}/(i^2 = j^2 = k^2 = ijk = -1).$$

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Then $A_K$ has a two dimensional module $M$ given by

$$i \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad j \mapsto \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

over any field $K$ of characteristic 0 having two elements $a$ and $b$ such that $a^2 + b^2 = -1$. Examples of such fields include $\mathbb{C}$, $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-5})$. If we take $K$ to be “the generic field” of this type, i.e., the field of fractions of $\mathbb{Q}[a,b]/(a^2+b^2+1)$, then $M$ has no minimal field of definition; see Proposition 6.3(b).

$C_1$-fields. Such examples arise because of the existence of noncommutative division rings of finite dimension over $F$. So, it makes sense to develop a theory over fields for which these do not exist. We say that $F$ is a $C_1$-field if any homogeneous polynomial $f_1(x_1, \ldots, x_n)$ of degree $d < n$ with coefficients in $F$ has a non-trivial solution in $F^n$. Examples of $C_1$-fields include finite fields, $k(t)$, and $k((t))$, where $k$ is algebraically closed. An algebraic extension of a $C_1$-field is again $C_1$. Over a $C_1$-field every central division algebra is commutative. For a detailed discussion of this class of fields, including proofs of the above assertions, we refer the reader to [GS, Section 6.2]. Our first main result is as follows.

**Theorem 1.1.** Let $F$ be a $C_1$-field, $A$ be a finite-dimensional $F$-algebra, $K/F$ be a separable algebraic field extension and $M$ be an $A_K$-module. Then $M$ has a minimal field of definition $F \subset K_0 \subset K$ such that $[K_0 : F] < \infty$.

To illustrate Theorem 1.1, let us consider a simple case, where char($F$) = 0, $A := FG$ is the group algebra of a finite group $G$, and $M$ is absolutely irreducible $KG$-module. Denote the character of $G$ associated to $M$ by $\chi : G \to K$. We claim that in this case the minimal field of definition is $F(\chi)$, the field generated over $F$ by the character values $\chi(g)$, as $g$ ranges over $G$. Indeed, it is clear that $F(\chi)$ has to be contained in any field of definition $F \subset K_0 \subset K$ of $M$. Thus to prove the above assertion, we only need to show that $M$ descends to $F(\chi)$. The minimal degree of a finite field extension $L/F(\chi)$, such that $M$ is defined over $L$ (i.e., there exists an $LG$-module with character $\chi$), is the Schur index $s_M$; cf. [CR, Definition 41.4]. Thus it suffices to show that $s_M = 1$. By [CR, Theorem (70.15)], $s_M$ is the index of the endomorphism algebra $\text{End}_A(M)$ of $M$, which is a central simple algebra over $F(\chi)$. Since $F$ is a $C_1$-field, and $F(\chi)$ is a finite extension of $F$, $F(\chi)$ is also a $C_1$-field. Hence, the index of every central simple algebra over $F(\chi)$ is 1. In particular, $s_M = 1$, and $M$ descends to $F(\chi)$, as claimed.

Algebras of finite representation type. A finite-dimensional $F$-algebra $A$ is said to be of finite representation type if there are only finitely many indecomposable finitely generated $A$-modules (up to isomorphism).

Our next result shows that for algebras of finite representation type Theorem 1.1 remains valid even if the field extension $K/F$ is not assumed to be algebraic.

**Theorem 1.2.** Let $F$ be a $C_1$-field, $A$ be a finite-dimensional $F$-algebra of finite representation type, $K/F$ be a field extension, and $M$ be an $A_K$-module. Assume further that $F$ is perfectly closed in $K$. Then $M$ has a minimal field of definition $F \subset K_0 \subset K$ such that $[K_0 : F] < \infty$. 

Essential dimension. Given the $A_K$-module $M$, the essential dimension $\text{ed}(M)$ of $M$ over $F$ is defined as the minimal value of the transcendence degree $\text{trdeg}(K_0/F)$, where the minimum is taken over all fields of definition $F \subset K_0 \subset K$. The integer $\text{ed}(M)$ may be viewed as a measure of the complexity of $M$. Note that $\text{ed}(M)$ is well-defined, irrespective of whether $M$ has a minimal field of definition or not. We also remark that this number implicitly depends on the base field $F$, which is assumed to be fixed throughout. As a consequence of Theorem 1.2, we will deduce the following.

**Theorem 1.3.** Let $F$ be a $C_1$-field, $A$ be finite-dimensional $F$-algebra of finite representation type, $K/F$ be a field extension, and $M$ be an $A_K$-module. Then $\text{ed}(M) = 0$.

Both Theorem 1.2 and 1.3 fail if we do not require $F$ to be a $C_1$-field; see Section 6.

The essential dimension of the functor of $A$-modules. We will also be interested in the essential dimension $\text{ed}(\text{Mod}_A)$ of the functor $\text{Mod}_A$ from the category of field extensions of $F$ to the category of sets, which associates to a field $K$, the set of isomorphism classes of $A_K$-modules. By definition,

$$\text{ed}(\text{Mod}_A) := \sup \text{ed}(M),$$

where the supremum is taken over all field extensions $K/F$ and all finitely generated $A_K$-modules $M$. The value of $\text{ed}(\text{Mod}_A)$ may be viewed as a measure the complexity of the representation theory of $A$. For generalities on the notion of essential dimension we refer the reader to [BF, Re$_1$, Re$_2$, Me$_1$, Me$_2$]. Essential dimensions of representations of finite groups and finite-dimensional algebras are studied in [KRP] and [BDH, Section 3].

Note that while $\text{ed}(M) < \infty$, for any given $A_K$-module $M$ (see Lemma 2.1), $\text{ed}(\text{Mod}_A)$ may be infinite. In particular, in the case, where $A = FG$ is the group algebra of a finite group $G$ over a field $F$, it is shown in [KRP, Theorem 14.1] that $\text{ed}(\text{Mod}_A) = \infty$, provided that $F$ is a field of characteristic $p > 0$ and $G$ has a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Our final main result is the following amplification of [KRP, Theorem 14.1].

**Theorem 1.4.** Let $G$ be a finite group and $F$ be a field of characteristic $p$. Then the following conditions are equivalent:

1. The $p$-Sylow subgroup of $G$ is cyclic,
2. $\text{ed}(\text{Mod}_{FG}) = 0$,
3. $\text{ed}(\text{Mod}_{FG}) < \infty$.

2. Preliminaries on fields of definition

**Lemma 2.1.** Let $A$ be a finite-dimensional $F$-algebra, $K/F$ be a field extension and $M$ be an $A_K$-module. Then $M$ descends to an intermediate subfield $F \subset E \subset K$, where $E/F$ is finitely generated.

**Proof.** Suppose $a_1, \ldots, a_r$ generate $A$ as an $F$-algebra. Choose an $F$-vector space basis for $M$. Then the $A$-module structure of $M$ is completely determined by the matrices representing multiplication by $a_1, \ldots, a_r$ in this basis. Each of these matrices has $n^2$ entries in $K$, where $n = \dim_F(M)$. Let $E \subset K$ be the field extension of $F$ obtained by adjoining these these $rn^2$ entries to $F$. Then $M$ descends to $E$. \qed
Next we recall the classical theorem of Noether and Deuring. For a proof, see [CR, (29.7)] or [BP, Lemma 5.1].

**Theorem 2.2.** (Noether-Deuring Theorem) Let $K/E$ be a field extension, $A$ be a finite-dimensional $E$-algebra, and $M_1$, $M_2$ and $M$ be $A$-modules. If $K \otimes_E M_1$ and $K \otimes_E M_2$ are isomorphic as $A_K$-modules, then $M_1$ and $M_2$ are isomorphic as $A$-modules. \hfill $\Box$

**Lemma 2.3.** Let $F$ be a field, $A$ be a finite-dimensional $F$-algebra, $F \subset E \subset K$ be a field extension, $N$ be $A_E$-module, and $M = N_K$ and $F \subset E_0 \subset E$ be an intermediate field. Then

(a) $M$ descends to $E_0$ if and only if $N$ descends to $E_0$.

(b) If $F \subset E_{\text{min}} \subset K$ is a minimal field of definition for $M$, then $E_{\text{min}}$ is a minimal field of definition for $N$.

**Proof.** (a) If $N$ descends to $E_0$, then clearly so does $M$. Conversely, suppose $M$ descends to $E_0$. That is, there exists an $E_0$-module $N_0$ such that $K \otimes_{E_0} N_0 \simeq M$ as an $A_K$-module. Consider the $A_E$-modules $N_1 := E \otimes_{E_0} N_0$ and $N_2 := N$. Both become isomorphic to $M_K$ over $K$. By Theorem 2.2, $N_1 \simeq N_2$ as $A_E$-modules. In other words, $N$ descends to $E_0$, as desired.

(b) Since $E$ is a field of definition for $M$, we have $E_{\text{min}} \subset E$. By part (a), $E_{\text{min}}$ is a field of definition for $N$, and part (b) follows. \hfill $\Box$

We finally come to the main result of this section.

**Proposition 2.4.** Suppose $F$ is a $C_1$-field, $A$ is a finite-dimensional $F$-algebra, $K/F$ is a field extension, $M$ is a finitely generated $A_K$-module, and $F \subset K_0 \subset K$ is an intermediate field, such that $[K_0 : F] < \infty$.

If $M^n$ is defined over $K_0$ for some positive integer $n$, then so is $M$.

**Proof.** Set $\text{End}_{A_K}^s(M)$ to be the quotient of $\text{End}_{A_K}(M)$ by its Jacobson radical. By our assumption $M^n \simeq K \otimes_{K_0} N$ for some $A_{K_0}$-module $N$. By Fitting’s Lemma,

$$\text{End}_{A_K}^s(M^n) \simeq M_n(D),$$

where $D$ is a finite-dimensional division algebra over some finite field extension $K'$ of $K$.

On the other hand,\hfill (2.5)

$$M_n(D) \simeq \text{End}_{A_K}^s(M^n) \simeq \text{End}_{A_K}^s(K \otimes_{K_0} N) \simeq K \otimes_{K_0} \text{End}_{A_{K_0}}^s(N).$$

We conclude that $\text{End}_{A_{K_0}}^s(N)$ is a simple algebra over $K_0$, i.e.,\hfill (2.6)

$$\text{End}_{A_{K_0}}^s(N) \simeq M_m(D_0)$$

over $K_0$, for some integer $m \geq 0$ and some finite-dimensional central division algebra $D_0$ over a field $K_0'$ such that $K_0'$ is a finite extension of $K_0$. Now recall that we are assuming that $F$ is a $C_1$-field and

$$F \subset K_0 \subset K_0'$$

are finite field extensions. Hence, $K_0'$ is also a $C_1$-field, and thus every finite-dimensional division algebra over $K_0'$ is commutative. In particular, $D_0 = K_0'$ is a field, and

$$M_n(D) \simeq K \otimes_{K_0} \text{End}_{A_{K_0}}^s(N) \simeq K \otimes_{K_0} M_m(K_0').$$
Since $M_n(D)$ is a simple algebra, we conclude that $K \otimes_{K_0} K'_0$ is a field. Moreover, the index of $M_n(K \otimes_{K_0} K'_0)$ is 1; hence, $D = K'$ is commutative, $K \otimes_{K_0} K'_0 = K'$, and $m = n$.

Now (2.6) tells us that $N \simeq M_0^n$ as an $A_{K_0}$-module, for some indecomposable $A_{K_0}$-module $M_0$. Since $K \otimes_{K_0} N \simeq M^n$, by the Krull-Schmidt theorem $K \otimes_{K_0} M_0 \simeq M$. Thus $M$ descends to $K_0$, as claimed. \hfill \Box

3. Proof of Theorem 1.1

We begin with a simple criterion for the existence of a minimal field of definition.

**Lemma 3.1.** Let $A$ be a finite-dimensional $F$-algebra, and $K/F$ be a field extension, and $M$ be an $A_K$-module, satisfying conditions (a) and (b) below. Then $M$ has a minimal field of definition.

(a) Suppose $M$ descends to an intermediate field $F \subset L \subset K$, i.e., $M \simeq K \otimes_L N$ for some $A_L$-module $N$. Then $N$ further descends to a subfield $F \subset E \subset L$, where $[E : F] < \infty$.

(b) Suppose $M$ descends to an intermediate field $F \subset E \subset K$ such that $[E : F] < \infty$. That is, $M \simeq K \otimes_E N$ for some $A_E$-module $N$. Then $N$ has a minimal field of definition $E_{\min} \subset E$.

**Proof.** Condition (a) implies that $M$ is defined over some $F \subset E \subset K$ with $[E : F] < \infty$. Let the $A_F$-module $N$ and the field $E_{\min} \subset E$ be as in (b).

We claim that $E_{\min}$ is independent of the choice of $E$. That is, suppose $F \subset E' \subset K$ is another field of definition of $M$ with $[E' : F] < \infty$, $M := K \otimes_{E'} N'$ for some $A_{E'}$-module $N'$. Let $E_{\min}' \subset E'$ be the minimal field of definition of $N'$, so that $N' := E' \otimes_{E_{\min}'} N'_{\min}$. Then our claim asserts that $E_{\min} = E_{\min}'$. If we can prove this claim, then clearly $E_{\min}$ is the minimal field of definition for $M$. Our proof of the claim will proceed in two steps.

First assume $E \subset E'$. By Lemma 2.3(b), $E_{\min}'$ is a minimal field of definition for $N$. By uniqueness of the minimal field of definition for $N$, $E_{\min} = E_{\min}'$.

Now suppose $F \subset E \subset K$ and $F \subset E' \subset K$ are fields of definition for $M$ such that $[E : F] < \infty$ and $[E' : F] < \infty$. Let $E''$ be the composite of $E$ and $E'$ in $K$ and $E_{\min}''$ be the minimal field of definition of $N_{E''} \simeq N'_{E''}$. (Note that $N_{E''}$ and $N'_{E''}$ become isomorphic over $K$; hence, by Theorem 2.2, they are isomorphic over $E''$.) Then, $[E'' : F] < \infty$, and $E, E' \subset E''$. As we just showed, $E_{\min} = E_{\min}''$ and $E_{\min}' = E_{\min}''$. Thus $E_{\min} = E_{\min}'$, as desired. \hfill \Box

We now proceed with the proof of Theorem 1.1.

**Reduction 3.2.** For the purpose of proving Theorem 1.1, we may assume without loss of generality that

(a) $K$ is a finite extension of $F$.

(b) $K$ is a Galois extension of $F$.

**Proof.** (a) follows from Lemma 3.1. Indeed, we are assuming that Theorem 1.1 holds whenever $K$ is a finite extension of $F$. That is, condition (b) of Lemma 3.1 holds. On the other hand, condition (a) of Lemma 3.1 follows from Lemma 2.1.
(b) By part (a), we may assume that $K/F$ is finite. Let $L$ be the normal closure of $K$ over $F$. Then $L/F$ is finite Galois. Lemma 2.3(b) now tells us that if $M_L := L \otimes_K M$ has a minimal field of definition then so does $M$. □

**Lemma 3.3.** Let $F$ be a $C_1$-field, $A$ be a finite-dimensional $F$-algebra, $K/F$ be a finite Galois extension, and $M$ be an $A_K$-module. The Galois group $G := \text{Gal}(K/F)$ acts on the set of isomorphism classes of $A_K$-modules via

$$g: N \to ^gN := K \otimes_g N.$$ 

Let $G_M$ be the stabilizer of $M$ under this action. Then the fixed field $K^{G_M}$ of $G_M$ is the minimal field of definition for $M$.

**Proof.** Suppose $M$ is defined over $K_0$, where $F \subset K_0 \subset K$. Then clearly $^gM \simeq M$ for every $g \in \text{Gal}(K/K_0)$. Hence, $\text{Gal}(K/K_0) \subset G_M$ and consequently, $K^{G_M} \subset K_0$. This shows that $K^{G_M}$ is contained in every field of definition of $M$.

It remains to show that $M$ descends to $K_0 := K^{G_M}$, Write $M = M_1^{d_1} \oplus \cdots \oplus M_r^{d_r}$, where $M_1, \ldots, M_r$ are distinct indecomposables. The condition that $^gM \simeq M$ for any $g \in G_M$ equivalent to the following: if $M_j \simeq ^gM_i$ for some $g \in \text{Gal}(K/K_0)$, then $d_i = d_j$. Grouping $G_M$-conjugate indecomposables together, we see that $M \simeq S_1 \oplus \cdots \oplus S_m$, where each $S_1, \ldots, S_m$ is the $G_M$-orbit sum of one of the indecomposable modules $M_i$. (Here the orbit sums $S_1, \ldots, S_m$ may not be distinct.) It thus suffices to show that each orbit sum is defined over $K_0$.

Consider a typical $G_M$-orbit sum $S := M_1 \oplus \cdots \oplus M_s$, where we renumber the indecomposable factors of $M$ so that $M_1, \ldots, M_s$ are the $G_M$-translates of $M_1$. Let $H$ be the stabilizer of $M_1$ in $G_M$. That is,

$$H := \{ h \in G_M \mid ^hM_1 \simeq M_1 \}.$$ 

Let $K_1 := K^H$. Then

$$K \otimes_{K_1} (M_1)_{\downarrow K_1} = \bigoplus_{h \in H} ^hM_1 = M_1^{[H]}.$$ 

In particular, this tells us that $M_1^{[H]}$ descends to $K_1$. By Proposition 2.4, so does $M_1$. In other words, $M_1 \simeq K \otimes_{K_1} N_1$ for some $K_1$-module $N$. We claim that

$$(3.4) \quad K \otimes_{K_0} (N_1)_{\downarrow K_0} \simeq S.$$ 

If we can prove this claim, then $S$ descends to $K_0$, and we are done.

To prove the claim, note that on the one hand,

$$(3.5) \quad K \otimes_{K_0} (M_1)_{\downarrow K_0} = \prod_{g \in G_M} ^gM_1 = S^{[H]}.$$ 

On the other hand, since $M_1 \simeq K \otimes_{K_1} N_1$, we have

$$(M_1)_{\downarrow K_0} \simeq ((M_1)_{\downarrow K_1})_{\downarrow K_0} \simeq (N_1^{[H]})_{\downarrow K_0},$$ 

and thus

$$(3.6) \quad K \otimes_{K_0} (M_1)_{\downarrow K_0} = (K \otimes_{K_0} ((N_1)_{\downarrow K_0})^{[H]}) \simeq (K \otimes_{K_0} (N_1)_{\downarrow K_0))^{[H]}.$$ 

Comparing (3.5) and (3.6), we obtain

$$(K \otimes_{K_0} (N_1)_{\downarrow K_0})^{[H]} \simeq S^{[H]}.$$
The desired isomorphism (3.4) follows from this by the Krull-Schmidt theorem.

4. Algebras of finite representation type

A finite-dimensional $F$-algebra $A$ is said to be of finite representation type if there are only finitely many indecomposable finitely generated $A$-modules (up to isomorphism).

**Theorem 4.1.** Let $F$ be a $C_1$-field, $A$ be finite-dimensional $F$-algebra of finite representation type, and $K/F$ be a field extension (not necessarily algebraic) such that $F$ is perfectly closed in $K$. (That is, for every subextension $F \subset E \subset K$ with $[E : F] < \infty$, $E$ is separable over $F$.) Suppose $M$ is an indecomposable $A_K$-module. Then

(a) $M$ descends to an intermediate subfield $F \subset E \subset K$ such that $[E : F] < \infty$.

(b) $M$ is a direct summand of $K \otimes_F N$ for some indecomposable $A_F$-module $N$.

**Proof.** (a) Consider the $A$-module $M_{|F}$. Generally speaking this module is not finitely generated over $A$. Nevertheless, since $A$ has finite representation type, thanks to a theorem of Tachikawa [Ta, Corollary 9.5], $M_{|F}$ can be written as a direct sum of finitely generated indecomposable $A$-modules. Denote one of these modules by $N$. That is,

$$M_{|F} \cong N \oplus N',$$

for some $A$-module $N'$ (not necessarily finitely generated).

Let us now take a closer look at $N$. By Fitting’s lemma, $E := \text{End}_A^a(N)$ is a finite-dimensional division algebra over $F$. Since $F$ is a $C_1$-field, $E$ is a field extension of $F$. Now set $F' := E \cap K$ and $m = [F' : F]$. Since $F$ is perfectly closed in $K$, $F'$ is finite and separable over $F$. Thus

$$\text{End}_A^a(F' \otimes_F N) \cong F' \otimes_F \text{End}_A^a(N) \cong E \times \cdots \times E.$$

This tells us that over $F'$, $N$ decomposes into a direct sum of $m$ indecomposables,

$$F' \otimes_F N = N_1 \oplus \cdots \oplus N_m.$$  

By the definition of $F'$, $K \otimes_{F'} E$ is a field. Hence, each indecomposable $A_{F'}$-module $N_i$ remains indecomposable over $K$.

Tensoring both sides of (4.2) with $K$, we obtain an isomorphism of $A_K$-modules

$$K \otimes M_{|F} \cong (K \otimes_F N) \oplus (K \otimes_F N')$$

$$= \bigoplus_{i=1}^m K \otimes_{F'} N_i \oplus (K \otimes_F N')$$

$$= (K \otimes_F N_1) \oplus N'',$$

where $N'' := (\bigoplus_{i=2}^m K \otimes_{F'} N_i) \oplus (K \otimes_F N')$. Note that

$$K \otimes M_{|F'} \cong \bigoplus_B M,$$

where $B$ is a basis of $K$ as an $F'$-vector space. As we mentioned above, $K \otimes_{F'} N_1$ is an indecomposable $A_K$-module. Since $K \otimes_{F'} N_1$ is finitely generated and is contained in
\( \bigoplus_B M \), it lies in the direct sum of finitely many copies of \( M \), say, in \( M^r := M \oplus \cdots \oplus M \) (\( r \) copies). Thus we have maps

\[
K \otimes_F N_1 \hookrightarrow M^r \hookrightarrow \bigoplus_B M \rightarrow K \otimes_F N_1
\]

whose composite is the identity, and so \( K \otimes_F N_1 \) is isomorphic to a direct summand of \( M^r \). By the Krull-Schmidt Theorem, \( K \otimes_F N_1 \cong M \). In particular, \( M \) descends to \( F' \), as claimed.

(b) By (4.3), \( N \) is an indecomposable \( A \)-module, and \( N_1 \) is a direct summand of \( F' \otimes_F N \). Hence, \( M \cong K \otimes_F N_1 \) is a direct summand of \( K \otimes_F N \), as desired.

**Corollary 4.4.** Let \( F \) be a \( C_1 \)-field, \( A \) be finite-dimensional \( F \)-algebra of finite representation type, and \( K/F \) be a field extension such that \( F \) is perfectly closed in \( K \). Then \( A_K \) is also of finite representation type.

**Proof.** By our assumption \( A \) has finitely many indecomposable modules \( N^{(1)}, \ldots, N^{(d)} \). By Theorem 4.1(b) every indecomposable \( A_K \)-module is isomorphic to a direct summand of \( K \otimes_F N^{(i)} \) for some \( i \). By the Krull-Schmidt Theorem, each \( K \otimes_F N^{(i)} \) has finitely many direct summands (up to isomorphism), and the corollary follows. \( \Box \)

5. **Proof of Theorems 1.2 and 1.3**

We will deduce Theorem 1.2 from Lemma 3.1. \( M \) satisfies condition (b) of Lemma 3.1 by Theorem 1.1. It thus remains to show that \( M \) satisfies condition (a) of Lemma 3.1. For notational simplicity, we may assume that \( K = L \) and \( M = N \). That is, we want to show that \( M \) descends to some intermediate field \( F \subset E \subset K \) with \([E:F] < \infty\). Note that in the case, where \( M \) is indecomposable, this is precisely the content of Theorem 4.1(a).

In general, write \( M = M_1 \oplus \cdots \oplus M_r \) as a direct product of (not necessarily distinct) indecomposables. By Theorem 4.1(a), each \( M_i \) descends to an intermediate field \( F \subset K_i \subset K \) such that \([K_i:F] < \infty\). Let \( E \) be the compositum of \( K_1, \ldots, K_r \) inside \( K \). Then \([E:F] < \infty\), and \( M \) descends to \( E \). This completes the proof of Theorem 1.2. \( \Box \)

We now proceed with the proof of Theorem 1.3. Denote the perfect closure of \( F \) in \( K \) by \( F^{pf} \). By Theorem 1.2, \( M \) descends to an intermediate field \( F^{pf} \subset K_0 \subset K \) such that \([K_0:F^{pf}] < \infty\). Hence, \( K_0 \) is algebraic over \( F \), and consequently, \( \text{ed}(M) \leq \text{trdeg}_F(K_0) = 0 \), as desired. \( \Box \)

6. **An example**

In this section we will show by example that both Theorem 1.2 and 1.3 fail if we do not require \( F \) to be a \( C_1 \)-field. Let \( F = \mathbb{Q} \) and \( A \) be the quaternion algebra

\[
A = \mathbb{Q}\{x, y\}/(x^2 = y^2 = -1, \ xy = -yx).
\]

and \( K/F \) be any field having two elements \( a \) and \( b \) satisfying \( a^2 + b^2 = -1 \). Then \( A \) has a two dimensional \( A_K \)-module \( M \) given by

\[
x \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad y \mapsto \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}.
\]
Lemma 6.2. The following conditions on an intermediate field \( Q \subset E \subset K \) are equivalent:

(a) \( \varphi \) descends to \( E \),
(b) \( A \) splits over \( E \),
(c) there exist elements \( a_0, b_0 \in E \) such that \( a_0^2 + b_0^2 = -1 \).

Proof. (a) \( \implies \) (b). Suppose \( M \) descends to an \( A_E \)-module \( N \). Since \( A_E := E \otimes Q A \) is a central simple 4-dimensional algebra over \( E \), the homomorphism of algebras given by

\[
A_E \to \text{End}_E(N) \simeq M_2(E)
\]

is an isomorphism. In other words, \( E \) splits \( A \).

(b) \( \implies \) (a). Conversely, suppose \( E \) splits \( A \). Then the representation of \( A \to \text{End}_K(M) \) factors as follows:

\[
A \to E \otimes Q A \simeq M_2(E) \to M_2(K).
\]

This shows that \( \varphi \) descends to \( E \).

The equivalence of (b) and (c) a special case of Hilbert’s criterion for the splitting of a quaternion algebra; see the equivalence of conditions (1) and (7) in [Lam, Theorem III.2.7] as well as Remark (B) on [Lam, p. 59]. \( \square \)

Proposition 6.3. Let \( a \) and \( b \) be independent variables over \( Q \), \( E \) be the field of fractions of \( \overline{Q}[a,b]/(a^2+b^2+1) \), and \( M \) be the 2-dimensional \( A_E \)-module given by (6.1). Then

(a) \( \text{ed}(M) = 1 \),
(b) \( M \) does not have a minimal field of definition.

Proof. (a) The assertion of part (a), follows from [KRP, Example 6.1]. For the sake of completeness, we will give an independent proof.

Suppose \( M \) descends to an intermediate subfield \( Q \subset E_0 \subset E \). Since \( \text{trdeg}_Q(E) = 1 \), \( \text{trdeg}_Q(E_0) = 0 \) or \( 1 \). Our goal is to show that \( \text{trdeg}_Q(E_0) \neq 0 \). Assume the contrary, i.e., \( E_0 \) is algebraic over \( Q \). Note that \( E \) is the function field of the conic curve \( a^2+b^2+c^2 = 0 \) in \( \mathbb{P}^2 \). Since this curve is absolutely irreducible, \( Q \) is algebraically closed in \( E \). Thus the only possibility for \( E_0 \) is \( E_0 = Q \). On the other hand, \( M \) does not descend to \( Q \) by Lemma 6.2, a contradiction.

(b) Suppose \( M \) descends to \( E_1 \subset E \). Our goal is to show that \( M \) descends to a proper subfield \( E_3 \subset E_1 \). By Lemma 6.2(c) there exist \( a_1 \) and \( b_1 \) in \( E_1 \) such that \( a_1^2 + b_1^2 = -1 \). If \( Q(a_1,b_1) \) is properly contained in \( E_1 \), then we are done. Thus we may assume without loss of generality that \( E_1 = Q(a_1,b_1) \). Set \( E_3 := Q(a_3,b_3) \) where \( a_3 := a_1^3 - 3a_1b_1^2 \) and \( b_3 = 3a_1^2b_1 - b_1^3 \). We claim that (i) \( A \) splits over \( E_3 \), and (ii) \( E_3 \subsetneq E_1 \).

In order to establish (i) and (ii), let us consider the following diagram.
Here as usual, $i$ is a primitive 4th root of 1. It is easy to see that $E_1(i) = \mathbb{Q}(i)(a_1, b_1) = \mathbb{Q}(i)(z)$ is a purely transcendental extension of $\mathbb{Q}(i)$, where $z = a_1 + b_1i$ and $\frac{1}{z} = -a_1 + b_1i$.

Similarly $E_2(i) = \mathbb{Q}(i)(z^3)$, where $z^3 = a_3 + b_3i$ and $\frac{1}{z^3} = -a_3 + b_3i$. In particular, this shows $a_3^2 + b_3^2 = -1$, thus proving (i). Moreover, since $z$ is transcendental over $\mathbb{Q}(i)$, we have $[E_1(i) : E_3(i)] = [\mathbb{Q}(i)(z) : \mathbb{Q}(i)(z^3)] = 3$ and thus
\[
[E_1 : E_3] = \frac{[E_3(i) : E_3] \cdot [E_1(i) : E_3(i)]}{[E_1(i) : E_1]} = \frac{2 \cdot 3}{2} = 3.
\]
This proved (ii).

**Remark 6.4.** Write $z^n = a_n + b_ni$ for suitable $a_n, b_n \in E_1$ and set $[E_1 : E_n] = n$. We showed above that $[E_1 : E_3] = 3$ and thus $E_3 \nsubseteq E_1$. The same argument yields $[E_1 : E_n] = n$ for any positive integer $n$.

7. **Proof of Theorem 1.4**

We shall actually prove a stronger, more natural theorem, about blocks of finite group algebras. Theorem 1.4 will follow from the fact that $p$-Sylow $p$ of a finite group $G$ are cyclic if and only if every block over a field $F$ of characteristic $p$ has cyclic defect.

**Theorem 7.1.** Let $B$ be a block of a finite group algebra $FG$, where $F$ is a field of characteristic $p$. Then the following are equivalent:

1. $B$ has cyclic defect,
2. $\text{ed}(\text{Mod}_B) = 0$,
3. $\text{ed}(\text{Mod}_B) < \infty$.

The implication (1) $\implies$ (2) is a direct consequence of Theorem 1.3. The implication (2) $\implies$ (3) is obvious.

The remainder of this section will be devoted to proving that (3) $\implies$ (1). We shall show that if $B$ has non-cyclic defect, then $\text{ed}(\text{Mod}_B) = \infty$. Let $K$ be an extension field of $F$, let $e$ be the block idempotent of $B$, let $D$ be a defect group of $B$, and let $N = \Phi(D)$, the Frattini subgroup of $D$. If $D$ is not cyclic, $D/N$ is elementary abelian of rank $r \geq 2$, with basis the images of elements $g_1, \ldots, g_r \in D$. Since $D$ is a defect group of $B$, any $KD$-module $M$ is a summand of $\text{Res}_{G,D}(e, \text{Ind}_{D,G}(M))$.

Now let $n > 0$, and let $K = F(t_{1,1}, \ldots, t_{n,r})$ be a function field in $nr$ indeterminates, and let $M_i$ ($1 \leq i \leq n$) be the two dimensional $KD$-module $g_j \mapsto \begin{pmatrix} 1 & t_{i,j} \\ 0 & 1 \end{pmatrix}$.

Then $J^2(KD)$ is in the kernel of $M_i$, so $M_i$ is really a module for $KD/J^2(KD)$, which has a basis $1, (g_1 - 1), \ldots, (g_r - 1)$. The last $r$ elements of this list form a basis for $J(KD)/J^2(KD)$, and we form a vector space $V$ with basis $(g_1 - 1), \ldots, (g_r - 1)$. The kernel of $M_i$ as a module for $KD/J^2(KD)$ is the codimension one subspace $H_i$ of $J(KD)/J^2(KD) \cong V$.
given by

\[(7.2) \quad H_i := \{ \lambda_j (g_j - 1) \mid \sum_j t_{i,j} \lambda_j = 0 \}.\]

By the Mackey decomposition theorem, the module \(M'_i = \text{Res}_{G,D}(e. \text{Ind}_{D,G}(M_i))\) is a direct sum of at least one copy of \(M_i\), some conjugates of \(M_i\) by elements of \(N_G(D)\), and some modules of the form \(\text{Ind}_{D \cap D,D,D}(D,G,D,D)^g M\). It follows that the Jordan canonical form of elements of \(V\) on \(M'_i\) is constant, except on a set \(S_i\), which is a finite union of hyperplanes \(N_G(D)\)-conjugates of \(H_i\) and linear subspaces of smaller dimension.

Now let \(M := \bigoplus_i M_i\). Our goal is to show that

\[\text{ed}(e. \text{Ind}_{D,G}(M)) \geq n(r - 1).\]

This will imply that \(\text{ed}(\text{Mod}_B) \geq n(r - 1)\) for every \(n > 0\) and thus \(\text{ed}(\text{Mod}_B) = \infty\), as desired.

Note that \(e. \text{Ind}_{D,G}(M)\) is a module whose restriction to \(D\) is \(\bigoplus_i M'_i\). If \(e. \text{Ind}_{D,G}(M)\) descends to an intermediate subfield \(F \subset K_0 \subset K\), then so does the set \(\bigcup_i S_i \subset V\) and its natural image in \(\mathbb{P}(V) = \mathbb{P}^{r-1}\), which we will denote by \(S\). To complete the proof of Theorem 7.1, it remains to show that if \(S\) descends to \(K_0\), then

\[(7.3) \quad \text{trdeg}_F(K_0) \geq n(r - 1).\]

**Lemma 7.4.** Let \(S \subset \mathbb{P}^{r-1}\) be a projective variety defined over a field \(K\). Assume that a hyperplane \(H\) given by \(a_1 x_1 + a_2 x_2 + \cdots + a_r x_r = 0\) is an irreducible component of \(S\) for some \(a_1, \ldots, a_r \in K\) (not all zero). Suppose \(S\) descends to a subfield \(K_0 \subset K\). Then each ratio \(a_j/a_i\) is algebraic over \(K_0\), as long as \(a_i \neq 0\).

To deduce the inequality (7.3) from Lemma 7.4, recall that in our case \(S\) is the union of the hyperplanes \(H_1, \ldots, H_n\), a finite number of other hyperplanes (translates of \(H_1, \ldots, H_n\) by elements of \(N_G(D)\)) and lower-dimensional linear subspaces of \(\mathbb{P}(V) = \mathbb{P}^{r-1}\). In the basis \((g_1 - 1), \ldots, (g_r - 1)\) of \(V\), \(H_i\) is given by \(t_{i,1} x_1 + t_{i,2} x_2 + \cdots + t_{i,r} x_r = 0\); see (7.2). Thus by Lemma 7.4, the elements \(t_{i,j} / t_{i,1}\) are algebraic over \(K_0\) for every \(i = 1, \ldots, n\) and every \(j = 2, \ldots, r\). In other words, if \(K_1\) is the algebraic closure of \(K_0\) in \(K\), then each \(t_{i,j} / t_{i,1} \in K_1\), and thus \(\text{trdeg}_F(K_0) = \text{trdeg}_F(K_1) \geq n(r - 1)\), as desired.

**Proof of Lemma 7.4.** We may assume without loss of generality that \(K_0\) is algebraically closed. To reduce to this case, we replace \(K_0\) by its algebraic closure \(\overline{K_0}\) and \(K\) by a compositum of \(\overline{K_0}\) and \(K_0\). If we know that each \(a_{i,j}\) is algebraic over \(\overline{K_0}\) (or equivalently, is contained in \(\overline{K_0}\)), then \(a_{i,j}\) is algebraic over \(K_0\).

Now assume that \(K_0\) is algebraically closed. Since \(S\) is defined over \(K_0\), every irreducible component of \(S\) is defined over \(K_0\). In particular, \(H\) is defined over \(K_0\). That is, the point \((a_1 : \cdots : a_r)\) of the dual projective space \(\overline{\mathbb{P}}^{r-1}\) is defined over \(K_0\). Equivalently, \(a_j / a_i \in K_0\) whenever \(a_i \neq 0\). This completes the proof of the claim and thus of Lemma 7.4 and Theorem 7.1. \(\square\)

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