FIELDS OF DEFINITION FOR REPRESENTATIONS OF ASSOCIATIVE ALGEBRAS

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ABSTRACT. We examine situations, where representations of a finite-dimensional $F$-algebra $A$ defined over a separable extension field $K/F$, have a unique minimal field of definition. Here the base field $F$ is assumed to be a field of dimension $\leq 1$. In particular, $F$ could be a finite field or $k(t)$ or $k((t))$, where $k$ is algebraically closed.

We show that a unique minimal field of definition exists if (a) $K/F$ is an algebraic extension or (b) $A$ is of finite representation type. Moreover, in these situations the minimal field of definition is a finite extension of $F$. This is not the case if $A$ is of infinite representation type or $F$ fails to be of dimension $\leq 1$. As a consequence, we compute the essential dimension of the functor of representations of a finite group, generalizing a theorem of N. Karpenko, J. Pevtsova and the second author.

1. Introduction

Notational conventions. Throughout this paper $F$ will denote a base field and $A$ a finite-dimensional associative algebra over $F$. If $K/F$ is a field extension (not necessarily algebraic), we will denote the tensor product $K \otimes_F A$ by $A_K$. Let $M$ be an $A_K$-module. Unless otherwise specified, we will always assume that $M$ is finitely generated (or equivalently, finite-dimensional as a $K$-vector space). If $L/K$ is a field extension, we will write $M_L$ for $L \otimes_K M$.

An intermediate field $F \subset K_0 \subset K$ is called a field of definition for $M$ if there exists a $K_0$-module $M_0$ such that $M \cong (M_0)_K$. In this case we will also say that $M$ descends to $K_0$.

Minimal fields of definition. A field of definition $K_0$ of $M$ is said to be minimal if whenever $M$ descends to a field $L$ with $F \subset L \subset K$, we have $K_0 \subset L$.

Minimal fields of definition do not always exist. For example, let $F = \mathbb{Q}$ and $A$ be the quaternion algebra

$A = \mathbb{Q}\{i, j, k\}/(i^2 = j^2 = k^2 = ij = -1)$.

Then $A_K$ has a two dimensional module $M$ given by

$$i \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad j \mapsto \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
over any field $K$ of characteristic 0 having two elements $a$ and $b$ such that $a^2 + b^2 = -1$. Examples of such fields include $\mathbb{C}$, $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-5})$. If we take $K$ to be “the generic field” of this type, i.e., the field of fractions of $\mathbb{Q}[a,b]/(a^2+b^2+1)$, then $M$ has no minimal field of definition; see Proposition 6.3(b).

**Fields of dimension $\leq 1$.** Such examples arise because of the existence of noncommutative finite-dimensional division algebras over $F$. So, it makes sense to develop a theory over those fields $F$ over which these division algebras do not exist. More precisely, we require that

\[(1.1) \quad \text{Br}(E) = 0 \text{ for every algebraic field extension } E/F,\]

where $\text{Br}(E)$ denotes the Brauer group of $E$. This class of fields was studied in detail by J.-P. Serre in connection with his celebrated Conjecture I; see [Se, §II.3]. Serre referred to fields satisfying (1.1) as “fields of dimension $\leq 1$”. If $F$ is perfect, this condition is equivalent to the cohomological dimension of the absolute Galois group $\text{Gal}(F)$ being $\leq 1$; see [Se, Proposition II.3.1.6]. In particular, this condition is satisfied by all finite fields, all algebraically closed fields and all field extensions of transcendence degree 1 over an algebraically closed field. For proofs of these assertions and further examples, see [Se, §II.3.3].

Our first main result is as follows.

**Theorem 1.2.** Let $F$ be a field satisfying (1.1), $A$ be a finite-dimensional $F$-algebra, $K/F$ be a separable algebraic field extension and $M$ be an $A_K$-module. Then $M$ has a minimal field of definition $F \subset K_0 \subset K$ such that $[K_0 : F] < \infty$.

To illustrate Theorem 1.2, let us consider a simple case, where $\text{char}(F) = 0$, $A := FG$ is the group algebra of a finite group $G$, and $M$ is absolutely irreducible $KG$-module. Denote the character of $G$ associated to $M$ by $\chi : G \to K$. We claim that in this case the minimal field of definition is $F(\chi)$, the field generated over $F$ by the character values $\chi(g)$, as $g$ ranges over $G$. Indeed, it is clear that $F(\chi)$ has to be contained in any field of definition $F \subset K_0 \subset K$ of $M$. Thus to prove the above assertion, we only need to show that $M$ descends to $F(\chi)$. The minimal degree of a finite field extension $L/F(\chi)$, such that $M$ is defined over $L$ (i.e., there exists an $LG$-module with character $\chi$), is the Schur index $s_M$; cf. [Cr1, Definition 41.4]. Thus it suffices to show that $s_M = 1$. By [Cr1, Theorem (70.15)], $s_M$ is the index of the endomorphism algebra $\text{End}_A(M)$ of $M$, which is a central simple algebra over $F(\chi)$. Since $F$ satisfies condition (1.1) and $[F(\chi) : F] < \infty$, the index of every central simple algebra over $F(\chi)$ is 1. In particular, $s_M = 1$, and $M$ descends to $F(\chi)$, as claimed.

**Algebras of finite representation type.** A finite-dimensional $F$-algebra $A$ is said to be of finite representation type if there are only finitely many indecomposable finitely generated $A$-modules (up to isomorphism).

Our next result shows that for algebras of finite representation type Theorem 1.2 remains valid even if the field extension $K/F$ is not assumed to be algebraic.

**Theorem 1.3.** Let $F$ be a field satisfying (1.1), $A$ be a finite-dimensional $F$-algebra of finite representation type, $K/F$ be a field extension, and $M$ be an $A_K$-module. Assume
further that $F$ is perfectly closed in $K$. Then $M$ has a minimal field of definition $F \subset K_0 \subset K$ such that $[K_0 : F] < \infty$.

**Essential dimension.** Given the $A_K$-module $M$, the essential dimension $ed(M)$ of $M$ over $F$ is defined as the minimal value of the transcendence degree $trdeg(K_0/F)$, where the minimum is taken over all fields of definition $F \subset K_0 \subset K$. The integer $ed(M)$ may be viewed as a measure of the complexity of $M$. Note that $ed(M)$ is well-defined, irrespective of whether $M$ has a minimal field of definition or not. We also remark that this number implicitly depends on the base field $F$, which is assumed to be fixed throughout. As a consequence of Theorem 1.3, we will deduce the following.

**Theorem 1.4.** Let $F$ be a field satisfying (1.1), $A$ be finite-dimensional $F$-algebra of finite representation type, $K/F$ be a field extension, and $M$ be an $A_K$-module. Then $ed(M) = 0$.

Both Theorem 1.3 and 1.4 fail if we do not require $F$ to satisfy (1.1); see Section 6.

**The essential dimension of the functor of $A$-modules.** We will also be interested in the essential dimension $ed(Mod_A)$ of the functor $Mod_A$ from the category of field extensions of $F$ to the category of sets, which associates to a field $K$, the set of isomorphism classes of $A_K$-modules. By definition,

$$ed(Mod_A) := \sup ed(M),$$

where the supremum is taken over all field extensions $K/F$ and all finitely generated $A_K$-modules $M$. The value of $ed(Mod_A)$ may be viewed as a measure of the complexity of the representation theory of $A$. For generalities on the notion of essential dimension we refer the reader to [BF, Re1, Re2, Me1, Me2]. Essential dimensions of representations of finite groups and finite-dimensional algebras are studied in [KRP] and [BDH, Section 3].

Note that while $ed(M) < \infty$, for any given $A_K$-module $M$ (see Lemma 2.1), $ed(Mod_A)$ may be infinite. In particular, in the case, where $A = FG$ is the group algebra of a finite group $G$ over a field $F$, it is shown in [KRP, Theorem 14.1] that $ed(Mod_A) = \infty$, provided that $F$ is a field of characteristic $p > 0$ and $G$ has a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Our final main result is the following amplification of [KRP, Theorem 14.1].

**Theorem 1.5.** Let $G$ be a finite group and $F$ be a field of characteristic $p$. Then the following conditions are equivalent:

1. The $p$-Sylow subgroup of $G$ is cyclic,
2. $ed(Mod_{FG}) = 0$,
3. $ed(Mod_{FG}) < \infty$.

Note that by a theorem of D. Higman [Hi], the condition that the $p$-Sylow subgroup of $G$ is cyclic is equivalent to the group algebra $FG$ being of finite representation type.

2. **Preliminaries on fields of definition**

**Lemma 2.1.** Let $A$ be a finite-dimensional $F$-algebra, $K/F$ be a field extension and $M$ be an $A_K$-module. Then $M$ descends to an intermediate subfield $F \subset E \subset K$, where $E/F$ is finitely generated.
Lemma 2.3. Let 

(a) If \( N \) is a field of definition for extensions, \( N \) and \( A \) are isomorphic as \( E \)-algebra, and \( M \) be a field of definition for \( K/F \). Then \( M \) descends to \( E \). □

Next we recall the classical theorem of Noether and Deuring. For a proof, see [CR1, (29.7)] or [BP, Lemma 5.1].

Theorem 2.2. (Noether-Deuring Theorem) Let \( K/E \) be a field extension, \( A \) be a finite-dimensional \( E \)-algebra, and \( M, M' \) be \( A \)-modules. If \( M_K = K \otimes_E M \) and \( M'_K = K \otimes_E M' \) are isomorphic as \( A_K \)-modules, then \( M \) and \( M' \) are isomorphic as \( A \)-modules.

Lemma 2.3. Let \( F \) be a field, \( A \) be a finite-dimensional \( F \)-algebra, \( F < E < K \) be field extensions, \( N \) be \( A_E \)-module, and \( F < E_0 < E \) be an intermediate field. Then

(a) \( N_K \) descends to \( E_0 \) if and only if \( N \) descends to \( E_0 \).

(b) If \( F < E_\text{min} < E \) is a minimal field of definition for \( N_K \), then \( E_\text{min} \) is a minimal field of definition for \( N \).

Proof. (a) If \( N \) descends to \( E_0 \), then clearly so does \( N_K \). Conversely, suppose \( N_K \) descends to \( E_0 \). That is, there exists a \( E_0 \)-module \( M \) such that \( K \otimes_{E_0} M \simeq N_K \) as an \( A_K \)-module. The \( A_E \)-modules \( M_E = E \otimes_{E_0} M \) and \( N \) become isomorphic to \( M_K = N_K \) over \( K \). By Theorem 2.2, \( M_E \simeq N \) as \( A_E \)-modules. Thus \( N \) descends to \( E_0 \), as desired.

(b) Clearly \( E \) is a field of definition for \( N_K \). Hence, by definition of \( E_\text{min} \), \( E_\text{min} \subset E \).

On the other hand, by part (a), \( E_\text{min} \) is a field of definition for \( N \), and part (b) follows. □

We finally come to the main result of this section.

Proposition 2.4. Suppose \( F \) is a field satisfying (1.1), \( A \) is a finite-dimensional \( F \)-algebra, \( K/F \) is a field extension, \( M \) is an indecomposable \( A_K \)-module, and \( F < K_0 < K \) is an intermediate field, such that \( [K_0 : F] < \infty \).

If \( M^n \) is defined over \( K_0 \) for some positive integer \( n \), then so is \( M \).

Proof. Set \( \text{End}_{A_K}(M) \) to be the quotient of \( \text{End}_{A_K}(M) \) by its Jacobson radical. By our assumption \( M^n \simeq K \otimes_K N \) for some \( A_{K_0} \)-module \( N \). By Fitting’s Lemma,

\[
\text{End}_{A_K}(M^n) \simeq M_n(D),
\]

where \( D \) is a finite-dimensional division algebra over some field extension \( K' \) of \( K \), where \( [K' : K] < \infty \). On the other hand,

\[
M_n(D) \simeq \text{End}_{A_K}(M^n) \simeq \text{End}_{A_K}(K \otimes_{K_0} N) \simeq K \otimes_{K_0} \text{End}_{A_{K_0}}(N).
\]

We conclude that \( \text{End}_{A_{K_0}}(N) \) is a simple algebra over \( K_0 \), i.e.,

\[
\text{End}_{A_{K_0}}(N) \simeq M_m(D_0)
\]

over \( K_0 \), for some integer \( m \geq 0 \) and some finite-dimensional central division algebra \( D_0 \) over a field \( K'_0 \) such that \( K'_0/K_0 \) is a field extension of finite degree. Now recall that we are assuming that \( F \) satisfies (1.1) and

\[
F < K_0 < K'_0
\]
are field extensions of finite degree. Hence, every finite-dimensional division algebra over $K_0'$ is commutative. In particular, $D_0 = K_0'$ is a field, and
$$M_n(D) \simeq K \otimes_{K_0} \text{End}^{\text{ss}}_{A_{K_0}}(N) \simeq K \otimes_{K_0} M_m(K_0').$$
Since $M_n(D)$ is a simple algebra, we conclude that $K \otimes_{K_0} K_0'$ is a field. Moreover, the index of $M_m(K \otimes_{K_0} K_0')$ is 1; hence, $D = K'$ is commutative, $K \otimes_{K_0} K_0' = K'$, and $m = n$.

Now (2.6) tells us that $N \simeq M_0^n$ as an $A_{K_0}$-module, for some indecomposable $A_{K_0}$-module $M_0$. Since $K \otimes_{K_0} N \simeq M^n$, by the Krull-Schmidt theorem $K \otimes_{K_0} M_0 \simeq M$. Thus $M$ descends to $K_0$, as claimed. \hfill \Box

3. Proof of Theorem 1.2

We begin with a simple criterion for the existence of a minimal field of definition.

**Lemma 3.1.** Let $A$ be a finite-dimensional $F$-algebra, and $K/F$ be a field extension, and $M$ be an $A_K$-module, satisfying conditions (a) and (b) below. Then $M$ has a minimal field of definition.

(a) Suppose $M$ descends to an intermediate field $F \subset L \subset K$, i.e., $M \simeq K \otimes_L N$ for some $A_L$-module $N$. Then $N$ further descends to a subfield $F \subset E \subset L$, where $[E : F] < \infty$.

(b) Suppose $M$ descends to an intermediate field $F \subset E \subset K$ such that $[E : F] < \infty$. That is, $M \simeq K \otimes_E N$ for some $A_E$-module $N$. Then $N$ has a minimal field of definition $E_{min} \subset E$.

**Proof.** Condition (a) implies that $M$ is defined over some $F \subset E \subset K$ with $[E : F] < \infty$. Let the $A_E$-module $N$ and the field $E_{min} \subset E$ be as in (b).

We claim that $E_{min}$ is independent of the choice of $E$. That is, suppose $F \subset E' \subset K$ is another field of definition of $M$ with $[E' : F] < \infty$, $M := K \otimes_{E'} N'$ for some $A_{E'}$-module $N'$. Let $E'_{min} \subset E'$ be the minimal field of definition of $N'$, so that $N' := E' \otimes_{E'_{min}} N'_{min}$. Then our claim asserts that $E_{min} = E'_{min}$. If we can prove this claim, then clearly $E_{min}$ is the minimal field of definition for $M$. Our proof of the claim will proceed in two steps.

First assume $E \subset E'$. By Lemma 2.3(b), $E'_{min}$ is a minimal field of definition for $N$. By uniqueness of the minimal field of definition for $N$, $E_{min} = E'_{min}$.

Now suppose $F \subset E \subset K$ and $F \subset E' \subset K$ are fields of definition for $M$ such that $[E : F] < \infty$ and $[E' : F] < \infty$. Let $E''$ be the composite of $E$ and $E'$ in $K$ and $E''_{min}$ be the minimal field of definition of $N'_{E''} \simeq N'_{E''}$. (Note that $N_{E''}$ and $N'_{E_{min}}$ become isomorphic over $K$; hence, by Theorem 2.2, they are isomorphic over $E''$.) Then, $[E'' : F] < \infty$, and $E, E' \subset E''$. As we just showed, $E_{min} = E''_{min}$ and $E'_{min} = E''_{min}$. Thus $E_{min} = E'_{min}$, as desired. \hfill \Box

We now proceed with the proof of Theorem 1.2.

**Reduction 3.2.** For the purpose of proving Theorem 1.2, we may assume without loss of generality that

(i) $K$ is a finite extension of $F$.

(ii) $K$ is a Galois extension of $F$. 
Proof. (i) follows from Lemma 3.1. Indeed, we are assuming that Theorem 1.2 holds whenever $K$ is a finite extension of $F$. That is, condition (b) of Lemma 3.1 holds. On the other hand, condition (a) of Lemma 3.1 follows from Lemma 2.1.

(ii) By part (i), we may assume that $K/F$ is finite. Let $L$ be the normal closure of $K$ over $F$. Then $L/F$ is finite Galois. Lemma 2.3(b) now tells us that if $M_L := L \otimes_K M$ has a minimal field of definition then so does $M$. \hfill \Box

We are now ready to finish the proof of Theorem 1.2. In view of Reduction 3.2, it remains to establish the following.

Lemma 3.3. Let $F$ be a field satisfying (1.1), $A$ be a finite-dimensional $F$-algebra, $K/F$ be a finite Galois extension, and $M$ be an $A_K$-module. The Galois group $G := \text{Gal}(K/F)$ acts on the set of isomorphism classes of $A_K$-modules via

$$g: N \rightarrow g^*N := K \otimes gN.$$  

Let $G_M$ be the stabilizer of $M$ under this action. Then the fixed field $K^{G_M}$ of $G_M$ is the minimal field of definition for $M$.

Proof. Suppose $M$ is defined over $K_0$, where $F \subset K_0 \subset K$. Then clearly $g^*M \simeq M$ for every $g \in \text{Gal}(K/K_0)$. Hence, $\text{Gal}(K/K_0) \subset G_M$ and consequently, $K^{G_M} \subset K_0$. This shows that $K^{G_M}$ is contained in every field of definition of $M$.

It remains to show that $M$ descends to $K_0 := K^{G_M}$. Write $M = M_1^d_1 \oplus \cdots \oplus M_r^d_r$, where $M_1, \ldots, M_r$ are distinct indecomposables. The condition that $g^*M \simeq M$ for any $g \in G_M$ is equivalent to the following: if $M_j \simeq g^*M_i$ for some $g \in \text{Gal}(K/K_0)$, then $d_i = d_j$. Grouping $G_M$-conjugate indecomposables together, we see that $M \simeq S_1 \oplus \cdots \oplus S_m$, where each $S_1, \ldots, S_m$ is the $G_M$-orbit sum of one of the indecomposable modules $M_i$. (Here the orbit sums $S_1, \ldots, S_m$ may not be distinct.) It thus suffices to show that each orbit sum is defined over $K_0$.

Consider a typical $G_M$-orbit sum $S := M_1 \oplus \cdots \oplus M_s$, where we renumber the indecomposable factors of $M$ so that $M_1, \ldots, M_s$ are the $G_M$-translates of $M_1$. Let $H$ be the stabilizer of $M_1$ in $G_M$. That is,

$$H := \{h \in G_M \mid h^*M_1 \simeq M_1 \}.$$  

Let $K_1 := K^H$. Then

$$K \otimes_{K_1} (M_1)^{\downarrow_{K_1}} = \bigoplus_{h \in H} h^*M_1 = M_1^{[H]}.$$  

In particular, this tells us that $M_1^{[H]}$ descends to $K_1$. By Proposition 2.4, so does $M_1$. In other words, $M_1 \simeq K \otimes_{K_1} N_1$ for some $K_1$-module $N_1$. We claim that

$$K \otimes_{K_0} (N_1)^{\downarrow_{K_0}} \simeq S.$$  

If we can prove this claim, then $S$ descends to $K_0$, and we are done.

To prove the claim, note that on the one hand,

$$K \otimes_{K_0} (M_1)^{\downarrow_{K_0}} = \prod_{g \in G_M} g^*M_1 = S^{[H]}.$$  

On the other hand, since $M_1 \simeq K \otimes_{K_1} N_1$, we have

$$(M_1)^{\downarrow_{K_0}} \simeq ((M_1)^{\downarrow_{K_1}})^{\downarrow_{K_0}} \simeq (N_1^{[H]})^{\downarrow_{K_0}}.$$
and thus
\[(3.6) \quad K \otimes K_0 (M_1 \downarrow K_0) = (K \otimes K_0 ((N_1) \downarrow K_0)|^H) \simeq (K \otimes K_0 (N_1) \downarrow K_0)|^H.\]

Comparing (3.5) and (3.6), we obtain
\[(3.7) \quad (K \otimes K_0 (N_1) \downarrow K_0)|^H \simeq S|^H.\]

The desired isomorphism (3.4) follows from (3.7) by the Krull-Schmidt theorem. This completes the proof of Lemma 3.3 and thus of Theorem 1.2. \qed

4. Algebras of finite representation type

A finite-dimensional \(F\)-algebra \(A\) is said to be of finite representation type if there are only finitely many indecomposable finitely generated \(A\)-modules (up to isomorphism).

**Theorem 4.1.** Let \(F\) be a field satisfying (1.1), \(A\) be finite-dimensional \(F\)-algebra of finite representation type, and \(K/F\) be a field extension (not necessarily algebraic) such that \(F\) is perfectly closed in \(K\). (That is, for every subextension \(F \subset E \subset K\) with \([E:F] < \infty\), \(E\) is separable over \(F\).) Suppose \(M\) is an indecomposable \(A_K\)-module. Then

(a) \(M\) descends to an intermediate subfield \(F \subset E \subset K\) such that \([E:F] < \infty\).

(b) \(M\) is a direct summand of \(K \otimes F N\) for some indecomposable \(A_F\)-module \(N\).

**Proof.** (a) Consider the \(A\)-module \(M_{\downarrow F}\). Generally speaking this module is not finitely generated over \(A\). Nevertheless, since \(A\) has finite representation type, thanks to a theorem of H. Tachikawa [Ta, Corollary 9.5], \(M_{\downarrow F}\) can be written as a direct sum of finitely generated indecomposable \(A\)-modules. Denote one of these modules by \(N\). That is,
\[(4.2) \quad M_{\downarrow F} \simeq N \oplus N',\]

for some \(A\)-module \(N'\) (not necessarily finitely generated).

Let us now take a closer look at \(N\). By Fitting’s lemma, \(E := \text{End}_A^a(N)\) is a finite-dimensional division algebra over \(F\). Since \(F\) is a field satisfying (1.1), \(E\) is a field extension of \(F\). Now set \(F' := E \cap K\) and \(m = [F':F]\). Since \(F\) is perfectly closed in \(K\), \(F'\) is finite and separable over \(F\). Thus
\[\text{End}_A^a(F' \otimes_F N) \simeq F' \otimes_F \text{End}_A^a(N) \simeq E \times \cdots \times E.\]

This tells us that over \(F'\), \(N\) decomposes into a direct sum of \(m\) indecomposables,
\[(4.3) \quad F' \otimes_F N = N_1 \oplus \cdots \oplus N_m.\]

By the definition of \(F'\), \(K \otimes F' E\) is a field. Hence, each indecomposable \(A_{F'}\)-module \(N_i\) remains indecomposable over \(K\).

Tensoring both sides of (4.2) with \(K\), we obtain an isomorphism of \(A_K\)-modules
\[K \otimes M_{\downarrow F} \simeq (K \otimes F N) \oplus (K \otimes F N')\]
\[= (\bigoplus_{i=1}^m K \otimes F' N_i) \oplus (K \otimes F N')\]
\[= (K \otimes F N_1) \oplus N'',\]
where $N'' := \left( \bigoplus_{i=2}^{m} K \otimes_{F'} N_i \right) \oplus (K \otimes_{F} N')$. Note that

$$K \otimes_{F'} M_i \simeq \bigoplus_B M,$$

where $B$ is a basis of $K$ as an $F'$-vector space. As we mentioned above, $K \otimes_{F'} N_1$ is an indecomposable $A_K$-module. Since $K \otimes_{F'} N_1$ is finitely generated and is contained in $\bigoplus_B M$, it lies in the direct sum of finitely many copies of $M$, say, in $M' := M \oplus \cdots \oplus M$ ($r$ copies). Thus we have maps

$$K \otimes_{F'} N_1 \hookrightarrow M' \hookrightarrow \bigoplus_B M \twoheadrightarrow K \otimes_{F'} N_1$$

whose composite is the identity, and so $K \otimes_{F'} N_1$ is isomorphic to a direct summand of $M'$. By the Krull-Schmidt Theorem, $K \otimes_{F'} N_1 \simeq M$. In particular, $M$ descends to $F'$, as claimed.

(b) By (4.3), $N$ is an indecomposable $A$-module, and $N_1$ is a direct summand of $F' \otimes_{F} N$. Hence, $M \simeq K \otimes_{F'} N_1$ is a direct summand of $K \otimes_{F} N$, as desired. \hfill \Box

**Corollary 4.4.** Let $F$ be a field satisfying (1.1), $A$ be finite-dimensional $F$-algebra of finite representation type, and $K/F$ be a field extension such that $F$ is perfectly closed in $K$. Then $A_K$ is also of finite representation type.

*Proof.* By our assumption $A$ has finitely many indecomposable modules $N^{(1)}, \ldots, N^{(d)}$. By Theorem 4.1(b) every indecomposable $A_K$-module is isomorphic to a direct summand of $K \otimes_{F} N^{(i)}$ for some $i$. By the Krull-Schmidt Theorem, each $K \otimes_{F} N^{(i)}$ has finitely many direct summands (up to isomorphism), and the corollary follows. \hfill \Box

5. PROOF OF THEOREMS 1.3 AND 1.4

We will deduce Theorem 1.3 from Lemma 3.1. $M$ satisfies condition (b) of Lemma 3.1 by Theorem 1.2. It thus remains to show that $M$ satisfies condition (a) of Lemma 3.1. For notational simplicity, we may assume that $K = L$ and $M = N$. That is, we want to show that $M$ descends to some intermediate field $F \subset E \subset K$ with $[E : F] < \infty$. Note that in the case, where $M$ is indecomposable, this is precisely the content of Theorem 4.1(a).

In general, write $M = M_1 \oplus \cdots \oplus M_r$ as a direct product of (not necessarily distinct) indecomposables. By Theorem 4.1(a), each $M_i$ descends to an intermediate field $F \subset K_i \subset K$ such that $[K_i : F] < \infty$. Let $E$ be the compositum of $K_1, \ldots, K_r$ inside $K$. Then $[E : F] < \infty$, and $M$ descends to $E$. This completes the proof of Theorem 1.3. \hfill \Box

We now proceed with the proof of Theorem 1.4. Denote the perfect closure of $F$ in $K$ by $F^{pf}$. By Theorem 1.3, $M$ descends to an intermediate field $F^{pf} \subset K_0 \subset K$ such that $[K_0 : F^{pf}] < \infty$. Hence, $K_0$ is algebraic over $F$, and consequently, $\text{ed}(M) \leq \text{trdeg}_F(K_0) = 0$, as desired. \hfill \Box

6. AN EXAMPLE

In this section we will show by example that both Theorems 1.3 and 1.4 fail if we do not require $F$ to be a field satisfying (1.1). Let $F = \mathbb{Q}$ and $A$ be the quaternion algebra

$$A = \mathbb{Q}\{x, y\}/(x^2 = y^2 = -1, \ xy = -yx).$$
and $K/F$ be any field having two elements $a$ and $b$ satisfying $a^2 + b^2 = -1$. Then $A$ has a two dimensional $A_K$-module $M$ given by

\[(6.1) \quad x \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad y \mapsto \begin{pmatrix} b & -a \\ -a & b \end{pmatrix}.
\]

Note that the multiplicative subgroup of $A$ generated by $x$ and $y$ is isomorphic to the quaternion group $Q_8$. Thus $A$ is naturally a quotient of the group algebra $QQ_8$ of $Q_8$ over $Q$. Since $QQ_8$ is of finite representation type, one readily concludes that so is $A$.

**Lemma 6.2.** The following conditions on an intermediate field $Q \subset E \subset K$ are equivalent:

- (a) $\varphi$ descends to $E$,
- (b) $A$ splits over $E$,
- (c) there exist elements $a_0, b_0 \in E$ such that $a_0^2 + b_0^2 = -1$.

**Proof.** (a) $\implies$ (b). Suppose $M$ descends to an $A_E$-module $N$. Since $A_E := E \otimes_Q A$ is a central simple 4-dimensional algebra over $E$, the homomorphism of algebras given by $A_E \to \text{End}_E(N) \simeq M_2(E)$ is an isomorphism. In other words, $E$ splits $A$.

(b) $\implies$ (a). Conversely, suppose $E$ splits $A$. Then the representation of $A \to \text{End}_K(M)$ factors as follows:

$$A \to E \otimes_Q A \simeq M_2(E) \to M_2(K).$$

This shows that $\varphi$ descends to $E$.

The equivalence of (b) and (c) a special case of Hilbert’s criterion for the splitting of a quaternion algebra; see the equivalence of conditions (1) and (7) in [Lam, Theorem III.2.7] as well as Remark (B) on [Lam, p. 59].

**Proposition 6.3.** Let $a$ and $b$ be independent variables over $F = Q$, $E$ be the field of fractions of $\mathbb{Q}[a,b]/(a^2 + b^2 + 1)$, and $M$ be the 2-dimensional $A_E$-module given by (6.1). Then

- (a) $\text{ed}(M) = 1$,
- (b) $M$ does not have a minimal field of definition.

**Proof.** (a) The assertion of part (a), follows from [KRP, Example 6.1]. For the sake of completeness, we will give an independent proof.

Suppose $M$ descends to an intermediate subfield $Q \subset E_0 \subset E$. Since $\text{trdeg}_Q(E) = 1$, $\text{trdeg}_Q(E_0) = 0$ or 1. Our goal is to show that $\text{trdeg}_Q(E_0) \neq 0$. Assume the contrary, i.e., $E_0$ is algebraic over $Q$.

Note that $E$ is the function field of the conic curve $a^2 + b^2 + c^2 = 0$ in $\mathbb{P}^2$. Since this curve is absolutely irreducible, $Q$ is algebraically closed in $E$. Since $E_0$ is algebraic over $Q$, we conclude that $E_0 = Q$. On the other hand, $M$ does not descend to $Q$ by Lemma 6.2, a contradiction.

(b) Suppose $M$ descends to $E_1 \subset E$. Our goal is to show that $M$ descends to a proper subfield $E_3 \subset E_1$. By Lemma 6.2(c) there exist $a_1$ and $b_1$ in $E_1$ such that $a_1^2 + b_1^2 = -1$. If $Q(a_1, b_1)$ is properly contained in $E_1$, then we are done. Thus we may assume without
loss of generality that $E_1 = \mathbb{Q}(a_1, b_1)$. Set $E_3 := \mathbb{Q}(a_3, b_3)$, where $a_3 := a_1^3 - 3a_1b_1^2$ and $b_3 = 3a_1^2b_1 - b_1^3$. We claim that (i) $A$ splits over $E_3$, and (ii) $E_3 \subsetneq E_1$.

In order to establish (i) and (ii), let us consider the following diagram

$$
\begin{array}{ccc}
E_1(i) & \longrightarrow & E_3(i) \\
\downarrow & & \downarrow \\
E_3 & \longrightarrow & E_1
\end{array}
$$

of field extensions. Here as usual, $i$ denotes a primitive 4th root of 1. It is easy to see that $E_1(i) = \mathbb{Q}(i)(a_1, b_1) = \mathbb{Q}(i)(z)$ is a purely transcendental extension of $\mathbb{Q}(i)$, where $z = a_1 + b_1i$ and $\frac{1}{z} = -a_1 + b_1i$. Similarly $E_3(i) = \mathbb{Q}(i)(z^3)$, where $z^3 = a_3 + b_3i$ and $\frac{1}{z^3} = -a_3 + b_3i$. In particular, this shows $a_3^2 + b_3^2 = -1$, thus proving (i). Moreover, since $z$ is transcendental over $\mathbb{Q}(i)$, we have

$$[E_1(i) : E_3(i)] = [\mathbb{Q}(i)(z) : \mathbb{Q}(i)(z^3)] = 3$$

and thus

$$[E_1 : E_3] = \frac{[E_3(i) : E_3] \cdot [E_1(i) : E_3(i)]}{[E_1(i) : E_1]} = \frac{2 \cdot 3}{2} = 3.$$ 

This proved (ii). $\square$

**Remark 6.4.** Write $z^n = a_n + b_ni$ for suitable $a_n, b_n \in E_1$ and set $E_n = \mathbb{Q}(a_n, b_n)$. We showed above that $[E_1 : E_3] = 3$ and thus $E_3 \subsetneq E_1$. The same argument yields $[E_1 : E_n] = n$ for any positive integer $n$.

**7. Proof of Theorem 1.5**

We shall actually prove a stronger, more natural theorem, about blocks of finite group algebras. Theorem 1.5 will follow from the fact that $p$-Sylow subgroups of a finite group $G$ are cyclic if and only if every block over a field $F$ of characteristic $p$ has cyclic defect; see [Ha] or [CR2, Theorem 62.21].

**Theorem 7.1.** Let $B$ be a block of a finite group algebra $FG$, where $F$ is a field of characteristic $p$. Then the following are equivalent:

1. $B$ has cyclic defect,
2. $\text{ed}(\text{Mod}_B) = 0$,
3. $\text{ed}(\text{Mod}_B) < \infty$.

The implication (1) $\implies$ (2) is a direct consequence of Theorem 1.4. The implication (2) $\implies$ (3) is obvious.

The remainder of this section will be devoted to proving that (3) $\implies$ (1). We shall show that if $B$ has non-cyclic defect, then $\text{ed}(\text{Mod}_B) = \infty$. Let $K$ be an extension field of $F$, let $e$ be the block idempotent of $B$, let $D$ be a defect group of $B$, and let $N = \Phi(D)$,
the Frattini subgroup of $D$. If $D$ is not cyclic, $D/N$ is elementary abelian of rank $r \geq 2$, with basis the images of elements $g_1, \ldots, g_r \in D$. Since $D$ is a defect group of $B$, any $KD$-module $M$ is a summand of $\text{Res}_{G,D}(e. \text{Ind}_{D,G}(M))$.

Now let $n > 0$, and let $K = F(t_{1,1}, \ldots, t_{n,r})$ be a function field in $nr$ indeterminates, and let $M_i$ ($1 \leq i \leq n$) be the two dimensional $KD$-module

$$g_j \mapsto \begin{pmatrix} 1 & t_{i,j} \\ 0 & 1 \end{pmatrix}.$$ 

Then $J^2(KD)$ is in the kernel of $M_i$, so $M_i$ is really a module for $KD/J^2(KD)$, which has a basis $1, (g_1 - 1), \ldots, (g_r - 1)$. The last $r$ elements of this list form a basis for $J(KD)/J^2(KD)$, and we form a vector space $V$ with basis $(g_1 - 1), \ldots, (g_r - 1)$. The kernel of $M_i$ as a module for $KD/J^2(KD)$ is the codimension one subspace $H_i$ of

$$J(KD)/J^2(KD) \cong V$$

given by

$$(7.2) \quad H_i := \{ \lambda_j (g_j - 1) \mid \sum_j t_{i,j} \lambda_j = 0 \}.$$ 

By the Mackey decomposition theorem, the module $M'_i = \text{Res}_{G,D}(e. \text{Ind}_{D,G}(M_i))$ is a direct sum of at least one copy of $M_i$, some conjugates of $M_i$ by elements of $N_G(D)$, and some modules of the form $\text{Ind}_{D \times D,D,D}(\text{Res}_{D,D,D} g) M$. It follows that the Jordan canonical form of elements of $V$ on $M'_i$ is constant, except on a set $S_i$, which is a finite union of hyperplanes $N_G(D)$-conjugates of $H_i$ and linear subspaces of smaller dimension.

Now let $M := \bigoplus_i M_i$. Our goal is to show that

$$\text{ed}(e. \text{Ind}_{D,G}(M)) \geq n(r - 1).$$

This will imply that $\text{ed}(\text{Mod}_B(n)) \geq n(r - 1)$ for every $n > 0$ and thus $\text{ed}(\text{Mod}_B) = \infty$, as desired.

Note that $e. \text{Ind}_{D,G}(M)$ is a module whose restriction to $D$ is $\bigoplus_i M'_i$. If $e. \text{Ind}_{D,G}(M)$ descends to an intermediate subfield $F \subset K_0 \subset K$, then so does the set $\bigcup_i S_i \subset V$ and its natural image in $\mathbb{P}(V) = \mathbb{P}^{r-1}$, which we will denote by $S$. To complete the proof of Theorem 7.1, it remains to show that if $S$ descends to $K_0$, then

$$(7.3) \quad \text{trdeg}_F(K_0) \geq n(r - 1).$$

**Lemma 7.4.** Let $S \subset \mathbb{P}^{r-1}$ be a projective variety defined over a field $K$. Assume that a hyperplane $H$ given by $a_1x_1 + a_2x_2 + \cdots + a_rx_r = 0$ is an irreducible component of $S$ for some $a_1, \ldots, a_r \in K$ (not all zero). Suppose $S$ descends to a subfield $K_0 \subset K$. Then each ratio $a_j/a_i$ is algebraic over $K_0$, as long as $a_i \neq 0$.

To deduce the inequality (7.3) from Lemma 7.4, recall that in our case $S$ is the union of the hyperplanes $H_1, \ldots, H_n$, a finite number of other hyperplanes (translates of $H_1, \ldots, H_n$ by elements of $N_G(D)$) and lower-dimensional linear subspaces of $\mathbb{P}(V) = \mathbb{P}^{r-1}$. In the basis $(g_1 - 1), \ldots, (g_r - 1)$ of $V$, $H_i$ is given by $t_{i,1}x_1 + t_{i,2}x_2 + \cdots + t_{i,r}x_r = 0$; see (7.2). Thus by Lemma 7.4 the elements $t_{i,j}/t_{i,1}$ are algebraic over $K_0$ for every $i = 1, \ldots, n$ and every $j = 2, \ldots, r$. In other words, if $K_1$ is the algebraic closure of $K_0$ in $K$, then each $t_{i,j}/t_{i,1} \in K_1$, and thus $\text{trdeg}_F(K_0) = \text{trdeg}_F(K_1) \geq n(r - 1)$, as desired.
Proof of Lemma 7.4. We may assume without loss of generality that $K_0$ is algebraically closed. To reduce to this case, we replace $K_0$ by its algebraic closure $\overline{K_0}$ and $K$ by a compositum of $K$ and $\overline{K_0}$. If we know that each $a_{i,j}$ is algebraic over $\overline{K_0}$ (or equivalently, is contained in $\overline{K_0}$), then $a_{i,j}$ is algebraic over $K_0$.

Now assume that $K_0$ is algebraically closed. Since $S$ is defined over $K_0$, every irreducible component of $S$ is defined over $K_0$. In particular, $H$ is defined over $K_0$. That is, the point $(a_1 : \cdots : a_r)$ of the dual projective space $\mathbb{P}^{r-1}$ is defined over $K_0$. Equivalently, $a_i/a_j \in K_0$ whenever $a_i \neq 0$. This completes the proof of the claim and thus of Lemma 7.4 and Theorem 7.1.

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