Lie Algebra Homology and Cohomology

Shen-Ning Tung

November 26, 2013

Abstract

In this project we give an application of derived functor. Starting with a Lie algebra $\mathfrak{g}$ over the field k, we pass to the universal enveloping algebra $U(\mathfrak{g})$ and define homology $H_n(\mathfrak{g}, M)$ and cohomology groups $H^n(\mathfrak{g}, M)$ for every (left) $\mathfrak{g}$-module M, by regarding M as a $U(\mathfrak{g})$-module. In the last section, we prove Whitehead's theorem: If $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra over field of characteristic 0 and M is a nontrivial irreducible module then $H_i(\mathfrak{g}, M) = H^i(\mathfrak{g}, M) = 0$, and then deduce Weyl's theorem and Whitehead's lemmas as corollaries.

1 Chain complexes

Definition 1 *A chain complex $C_\bullet$ of R-modules is a family $\{C_n\}_{n\in\mathbb{Z}}$ of R-modules, together with R-modules maps $d = d_n : C_n \to C_{n-1}$ such that each composite $d \circ d : C_n \to C_{n-2}$ is zero. The maps d_n are called differentials of $C_\bullet$. The n-th homology groups of $C_\bullet$ is the quotient $H_n(C_\bullet) = Ker(d_n)/Im(d_{n+1})$.*

Definition 2 *A chain complex map $f : C_\bullet \to D_\bullet$ is a family of R-module homomorphisms $f_n : C_n \to D_n$ commuting with d in the sense that $f_{n-1} \circ d_n = d'_{n-1} \circ f_n$ That is, such that the following diagram commutes*

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & C_{n+1} \arrow[r, "d"] \arrow[d, "f"] & C_n \arrow[r, "d"] \arrow[d, "f"] & C_{n-1} \arrow[r, "d"] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r, "d'"] & D_{n+1} \arrow[r, "d'"] & D_n \arrow[r, "d'"] & D_{n-1} \arrow[r, "d'"] & \cdots
\end{tikzcd}$$

By the diagram above, we have $Ker(d_n) \subseteq Ker(d'_n)$ and $Im(d_{n+1}) \subseteq Im(d'_{n+1})$. Hence f induces a map $H_n(f) : H_n(C_\bullet) \to H_n(D_\bullet)$.

Similarly, we can define cochain complex and cohomology by dualizing.

Proposition 1 *Let* $0 \longrightarrow A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \longrightarrow 0$ *be an exact sequence of chain complexes. Then there are natural maps* $\partial_n : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$, *called connecting homomorphisms, such that*

$$\cdots \xrightarrow{g} H_{n+1}(C_\bullet) \xrightarrow{\partial} H_n(A_\bullet) \xrightarrow{f} H_n(B_\bullet) \xrightarrow{g} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \xrightarrow{f} \cdots$$

is an exact sequence.

Similarly, if $0 \longrightarrow \dot{A} \xrightarrow{f} \dot{B} \xrightarrow{g} \dot{C} \longrightarrow 0$ *be an exact sequence of cochain complexes, there are natural maps* $\partial^n : H^n(\dot{C}) \to H^{n+1}(\dot{A})$, *called connecting homomorphisms, and a long exact sequence*

$$\cdots \xrightarrow{g} H^{n-1}(\dot{C}) \xrightarrow{\partial} H^n(\dot{A}) \xrightarrow{f} H^n(\dot{B}) \xrightarrow{g} H^n(\dot{C}) \xrightarrow{\partial} H^{n+1}(\dot{A}) \xrightarrow{f} \cdots .$$

Definition of connecting homomorphism

$$\begin{tikzcd}
 & & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d'_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "d''_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d, "d_n"] & B_n \arrow[r, "g_n"] \arrow[d, "d'_n"] & C_n \arrow[r] \arrow[d, "d''_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "d_{n-1}"] & B_n \arrow[r, "g_{n-1}"] \arrow[d, "d'_{n-1}"] & C_n \arrow[r] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r, "f_{n-2}"] & B_{n-2} & &
\end{tikzcd}$$

Let $\bar{c} \in H_n(C_\bullet)$, and let $c \in Ker(d''_n)$ represent $\bar{c}$. Choose $b \in B_n$ such that $g_n(b) = c$. Let $b' = d'_n(b)$. Then $g_{n-1}(b') = 0$. Therefore, there exists $a \in A_{n-1}$ such that $b' = f_{n-1}(a)$. Since $(d'_{n-1}f_{n-1})(a) = (d'_{n-1}d'_n)(b) = 0$, it follows that $(f_{n-2}d_{n-1})(a) = 0$ i.e. $a \in Ker(d_{n-1})$. Let $\bar{a}$ be the canonical image of a in $H_{n-1}(A_\bullet)$. It is easily seen that $\bar{a}$ does not depend on the choice of c and b. We define $\partial_n : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ by $\partial_n(\bar{c}) = \bar{a}$. Clearly, ∂_n is an R-homomorphism.

Similarly, we can define the connecting homomorphism for cochain complex.

2 δ-funtors and Derived functors

Definition 3 *A (covariant) homological (resp. cohomological)* δ*-functor between abelian category* $\mathcal{A}$ *and* $\mathcal{B}$ *is a collection of additive functors* $T_n : \mathcal{A} \to \mathcal{B}$ *(resp.* $T^n : \mathcal{A} \to \mathcal{B}$*) for* $n \geq 0$*, together with morphisms*

$$\delta_n : T(C) \to T(A)$$
$$(resp. \quad \delta^n : T(C) \to T(A))$$

defined for each short exact sequence $0 \to A \to B \to C \to 0$ *in* $\mathcal{A}$. *With two condition imposed:*

1. *For each short exact sequence as above, there is a long exact sequence*

$$\ldots \longrightarrow T_{n+1}(C) \xrightarrow{\delta} T_n(A) \longrightarrow T_n(B) \longrightarrow T_n(C) \xrightarrow{\delta} T_{n-1}(C) \longrightarrow \ldots$$

$$(\ldots \longrightarrow T^{n-1}(C) \xrightarrow{\delta} T^n(A) \longrightarrow T^n(B) \longrightarrow T^n(C) \xrightarrow{\delta} T^{n+1}(C) \longrightarrow \ldots)$$

   In particular, T_0 *is right exact and* T^0 *is left exact.*

2. *For each morphism of short exact sequence from* $0 \to A' \to B' \to C' \to 0$ *to* $0 \to A \to B \to C \to 0$, *the* δ*'s give a commutative diagram*

$$\begin{tikzcd} T_n(C') \arrow[r, "\delta"] \arrow[d] & T_{n-1}(A') \arrow[d] \\ T_n(C) \arrow[r, "\delta"] & T_{n-1}(A) \end{tikzcd} \qquad \textit{resp.} \qquad \begin{tikzcd} T^n(C') \arrow[r, "\delta"] \arrow[d] & T^{n+1}(A') \arrow[d] \\ T^n(C) \arrow[r, "\delta"] & T^{n+1}(A) \end{tikzcd}$$

Definition 4 *A morphism* $S \to T$ *of* δ*-functors is a system of natural transformations* $S_n \to T_n$*(resp.* $S^n \to T^n$*) that commute with* δ.

A homological δ*-functor* T *is universal if, given any other* δ*-functor* S *and a natural transformation* $f_0 : S_0 \to T_0$, *there exists a unique morphism* $\{f_n : S_n \to T_n\}$ *of* δ*-functors that extends* f_0.

A cohomological δ*-functor* T *is universal if, given* S *and* $f^0 : S^0 \to T^0$, *there exists a unique morphism* $\{f^n : S^n \to T^n\}$ *of* δ*-functors extending* f^0.

Definition 5 *We say an abelian category* $\mathcal{A}$ *has enough projectives if for every object* A *of* $\mathcal{A}$ *there is a surjection* $P \to A$ *with* P *projective. Similarly, we say a category* $\mathcal{A}$ *has enough injectives if for every object* A *of* $\mathcal{A}$ *there is a injection* $A \to I$ *with* I *injective.*

Definition 6 *Let* M *be an object of* $\mathcal{A}$. *A projective resolution of* M *is a pair* $(P_\bullet, \varepsilon)$, *where* P *is a left complex with each* P_i *projective together with a map* $\varepsilon : P_0 \to M$ *so that the augmented complex*

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is exact.

Fact The category $\mathbf{mod} - R$ has enough projectives.

Definition 7 *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor between two abelian categories. If $\mathcal{A}$ has enough projectives, we can construct the left derived functors $L_iF(i \geq 1)$ of F as follows. If A is an object of $\mathcal{A}$, choose (once and for all) a projective resolution $P \to A$ and define*

$$L_iF(A) = H_i(F(P))$$

We always have $L_0F(A) \cong F(A)$.

Facts

1. The object $L_iF(A)$ of $\mathcal{B}$ are well defined up to natural isomorphism. That is, if $Q \to A$ is a second projective resolusion, then there is a canonical isomorphism:

$$L_iF(A) = H_i(F(P)) \longrightarrow H_iF(Q)\ .$$

2. If A is projective, then $L_iF(A) = 0$ for $i \neq 0$.

3. If $f : A' \to A$ is any map in $\mathcal{A}$, there is a natural map $L_iF(f) : L_iF(A') \to L_iF(A)$ for each i.

4. The derived functors L_*F form a homological δ-functor.

5. Assume that $\mathcal{A}$ has enough projectives. Then for any right exact functor $F : \mathcal{A} \to \mathcal{B}$, the derived functors L_nF form a universal δ-functor.

Definition 8 *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between two abelian categories. If $\mathcal{A}$ has enough injectives, we can construct the right derived functors $R^iF(i \geq 0)$ of F as follows. If A is an object of $\mathcal{A}$, choose an injective resolution $A \to I$ and define*

$$R^iF(A) = H^i(F(I))$$

Note that since $0 \to F(A) \to F(I^0) \to F(I^1)$ is exact, we always have $R^0F(A) \cong F(A)$.

Since F also defines a right exact functor $F^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$, and $\mathcal{A}^{op}$ has enough projectives, we can construct the left derived functors L_iF^{op} as well. Since I becomes a projective resolution of A in $\mathcal{A}^{op}$, we see that

$$R^iF(A) = (L_iF^{op})^{op}(A)$$

Therefore all the results about right exact functors apply to left exact functors.

3 Ext Functors and Tor Functors

Definition 9 *(Tor functors) Let B be a left R-module, so that $T(A) = A \otimes_R B$ is a right exact functor from* $\mathbf{mod} - R$ *to* $\mathbf{Ab}$*. We define the abelian groups*

$$Tor_n^R(A, B) = (L_n T)(A)$$

In particular, $Tor_0^R(A, B) \cong A \otimes_R B$.

Definition 10 *(Ext functors) For each R-module A, the functor $F(B) = Hom_R(A, B)$ is left exact. Its right derived functors are called the Ext groups:*

$$Ext_R^i(A, B) = R^i Hom_R(A, -)(B)$$

In particular, $Ext_R^0(A, B)$ is $Hom(A, B)$.

Recall that these groups are computed by finding a projective(resp. injective) resolution $P_\bullet \to A$(resp. $\dot{I} \leftarrow B$) and taking the homology(resp. cohomology) of $P_\bullet \otimes_R B$(resp. $Hom(A, \dot{I})$).

Remark.

1. Tor and Ext satisfy all the facts above. In particular, they are independent of choice of resolutions and are universal δ-funtors.

2. $L_n(A \otimes_R)(B) \cong L_n(\otimes_R B)(A) = Tor_n^R(A, B)$ for all n. This means that we can alsp compute $Tor_n^R(A, B)$ by finding projective resolution of B and taking the homology.

3. $R^n(Hom_R(A, -)) \cong R^n(Hom_R(-, B)) = Ext_R^n(A, B)$ for all n. This means that we can also compute Ext_R^n by finding projective resolution of A and taking the cohomology.

4 Lie algebra Homology and Cohomology

Let $\mathfrak{g}$ a Lie Algebra over field k and M a left $\mathfrak{g}$-module, we have

1. The invariant submodule $M^{\mathfrak{g}}$ of a $\mathfrak{g}$-module M,

$$M^{\mathfrak{g}} = \{m \in M \mid xm = 0 \quad \forall x \in \mathfrak{g}\}$$

   Considering k as a trivial $\mathfrak{g}$-module, we have $M^{\mathfrak{g}} \cong Hom_{\mathfrak{g}}(k, M)$.

2. The coinvariant $M_{\mathfrak{g}}$ of a $\mathfrak{g}$-module M, $M_{\mathfrak{g}} = M/\mathfrak{g}M$.

Definition 11 *Let M be a $\mathfrak{g}$-module. we write $H_*(\mathfrak{g}, M)$ or $H_*^{Lie}(\mathfrak{g}, M)$ for the left derived funtors $L_*(-_{\mathfrak{g}})(M)$ of $-_{\mathfrak{g}}$ and call them the homology groups of $\mathfrak{g}$ with coefficients in M. By definition, $H_0(\mathfrak{g}, M) = M_{\mathfrak{g}}$.*
Similarly, we write $H^*(\mathfrak{g}, M)$ or $H_{Lie}^*(\mathfrak{g}, M)$ for the right derived funtors $R^*(-^{\mathfrak{g}})(M)$ of $-^{\mathfrak{g}}$ and call them the cohomology groups of $\mathfrak{g}$ with coefficients in M. By definition, $H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}$.

Example 1 *Let $\mathfrak{g}$ be a free k-module on basis $\{e_1, ...e_n\}$, made into an abelian Lie algebra with zero bracket. Thus $\mathfrak{g}$-module is just k-module with n commuting endomorphisms $\{e_1, ..., e_n\}$. It follows that $\mathfrak{g}$-module is just R-module where $R = k[e_1, ...e_n]$. If k is trivial $\mathfrak{g}$-module viewed as R-module with e_i acting as 0, then $M_{\mathfrak{g}} = k \otimes_R M$ and $M^{\mathfrak{g}} = Hom_R(k, M)$. Therefore we have*

$$H_*^{Lie}(\mathfrak{g}, M) = Tor_*^R(k, M) \quad and \quad H_{Lie}^*(\mathfrak{g}, M) = Ext_R^*(k, M).$$

Example 2 *If $\mathfrak{f}$ is free Lie algebra on set X, then an $\mathfrak{f}$-module is a k-module M with an arbitrary set $\{e_x : x \in X\}$ of endormorphisms. Hence, if view k as trivial $\mathfrak{f}$-module, then $M_{\mathfrak{f}} = k \otimes_R M$ and $M^{\mathfrak{f}} = Hom_R(k, M)$, where R is the free associative k-algebra $k\{X\}$. Therefore,*

$$H_*^{Lie}(\mathfrak{f}, M) = Tor_*^R(k, M) \quad and \quad H_{Lie}^*(\mathfrak{f}, M) = Ext_R^*(k, M).$$

Proposition 2 *The ideal $J = Xk\{X\}$ of a free ring $k\{X\}$ is free as right $k\{X\}$-module with basis the set X. Hence*

$$0 \longrightarrow J \longrightarrow k\{X\} \longrightarrow k \longrightarrow 0$$

is a free resolution of k as a right $k\{X\}$-module.

Proof. As a free k-module, $k\{X\}$ has basis words W of the set X, and J is a free k-module on basis $W - \{1\}$. Every elements of $W - \{1\}$ is of the form wx with $w \in W$ and $x \in X$, so $\{xw : x \in X, w \in W\}$ is also basis for J as k-module. For each $x \in X$ the k-span $xk\{X\}$ is isomorphic to $k\{X\}$, and J is a direct sum of the $xk\{X\}$, both as k-modules and as right $k\{X\}$-modules. That is, J is a free right $k\{X\}$-module with basis X. $\diamond$

Corollary 1 *If $\mathfrak{f}$ is the free Lie algebra on X, then $H_n^{Lie}(\mathfrak{f}, M) = H_{Lie}^n(\mathfrak{f}, M) = 0$ and for all $n \geq 2$ and all $\mathfrak{f}$-modules M. Moreover, $H_0^{Lie}(\mathfrak{f}, M) = H_{Lie}^0(\mathfrak{f}, M) = k$ while $H_1^{Lie}(\mathfrak{f}, k) = H_{Lie}^1(\mathfrak{f}, k) = \prod_{x \in X} k$.*

Proof. Using the free resolution of k in Proposition 2, $H_*^{Lie}(\mathfrak{f}, M)$ is the homology of the complex $0 \to J \otimes_{\mathfrak{f}} M \to \mathfrak{f} \otimes_{\mathfrak{f}} M \to 0$ and $H_{Lie}^*(\mathfrak{f}, M)$ is the homology of the complex $0 \to Hom_{\mathfrak{f}}(\mathfrak{f}, M) \to Hom_{\mathfrak{f}}(J, M) \to 0$, hence for $n \geqslant 2$ the homology groups and cohomology groups are zero. For $M = k$, the differentials are 0. Furthermore, $J \otimes_{\mathfrak{f}} k \cong \prod_{x \in X} k \cong Hom_{\mathfrak{f}}(J, k)$ and $\mathfrak{f} \otimes_{\mathfrak{f}} k \cong k \cong Hom_{\mathfrak{f}}(\mathfrak{f}, k)$. we get the result for $i = 0, 1$. $\diamond$

5 Universal Enveloping Algebra

Definition 12 *If M is any k-module, the tensor algebra is defined by $T(M) = \oplus_{n=0}^{\infty} M^{\otimes n}$, where $M^{\otimes n}$ denotes $M \otimes ... \otimes M$, the tensor product of n copies of M over k. Writing $i : M \to T(M)$ for the inclusion, $T(M)$ is generated by $i(M)$ as k-algebra.*

Definition 13 *If $\mathfrak{g}$ is a Lie algebra over k, the universal enveloping algebra $U(\mathfrak{g})$ is the quotient of $T(\mathfrak{g})$ by the two sided ideal generated by $\{i([x,y]) - i(x)i(y) - i(y)i(x) : x, y \in \mathfrak{g}\}$.*

For every associated algebra A, there is a natural isomorphism

$$Hom_{Lie}(\mathfrak{g}, Lie(A)) \cong Hom_{k-alg}(U(\mathfrak{g}), A)$$

where $Lie(A)$ is the Lie algebra with underlying set A with Lie bracket defined by $[a, b] = ab - ba$.

Given a $\mathfrak{g}$-module M and a monomial $x_1 \cdots x_n$ in $U(\mathfrak{g})$ ($x_i \in \mathfrak{g}$), define

$$(x_1 \cdots x_n)m = x_1(x_2(\cdots(x_n m)\cdots)), \quad m \in M$$

makes M into a $U(\mathfrak{g})$-module. Conversely, if M is a $U(\mathfrak{g})$-module and $x \in \mathfrak{g}$, define $xm = i(x)m$ ($m \in M$) makes M into a $\mathfrak{g}$-module.

Theorem 1 *If $\mathfrak{g}$ is a Lie algebra, every left $\mathfrak{g}$-module is naturally a left $U(\mathfrak{g})$-module. The category $\mathfrak{g}$-mod is naturally isomorphic to the category $U\mathfrak{g}$-mod of left $U(\mathfrak{g})$-modules.*

Proof. Let M be a k-module and write $E = End_k(M)$ for the k-algebra of all k-module endormorphisms of M. Take $A = E$ in the above identification, we have

$$Hom_{Lie}(\mathfrak{g}, Lie(E)) \cong Hom_{k-alg}(U(\mathfrak{g}), E)$$

A $\mathfrak{g}$-module is a k-module M together with a Lie algebra map $\mathfrak{g} \to Lie(E)$. On the other hand, a $U(\mathfrak{g})$-module M a k-module together with an associative algebra map $U(\mathfrak{g}) \to End_k(M)$, so the theorem follows. $\diamond$

Example 3 *There is a k-algebra homomorphism $\varepsilon : U(\mathfrak{g}) \to k$, sending $i(\mathfrak{g})$ to zero, called the augmentation. We define the augmentation ideal $\mathfrak{J}$ to be the kernal of ε, which is an ideal of $U(\mathfrak{g})$ generated by $i(\mathfrak{g})$ as left module. Therefore $\mathfrak{J}$ is a $U(\mathfrak{g})$-module and $k \cong U(\mathfrak{g})/\mathfrak{J} = U(\mathfrak{g})_{\mathfrak{g}}$.*

Corollary 2 *Let M be a $\mathfrak{g}$-module. Then*

$$H_*^{Lie}(\mathfrak{g}, M) \cong Tor_*^{U_\mathfrak{g}}(k, M) \quad \textit{and} \quad H_{Lie}^*(\mathfrak{g}, M) \cong Ext_{U_\mathfrak{g}}^*(k, M).$$

Proof. To show any two derived funtors are isomorphic, we only need to show the underlying functors are isomorphic. Observe that

$$k \otimes_{U(\mathfrak{g})} M \cong U(\mathfrak{g})/\mathfrak{J} \otimes_{U(\mathfrak{g})} M \cong M/\mathfrak{J}M \cong M/\mathfrak{g}M = M_\mathfrak{g}$$
$$Hom_{U(\mathfrak{g})}(k, M) = Hom_\mathfrak{g}(k, M) = M^\mathfrak{g} \qquad \diamond$$

Example 4 *The map $i : \mathfrak{g} \to U(\mathfrak{g})$ maps $[\mathfrak{g}, \mathfrak{g}]$ to $\mathfrak{J}^2$, hence induced an isomorphism $i : \mathfrak{g}^{ab} \to \mathfrak{J}/\mathfrak{J}^2$, where $\mathfrak{g}^{ab} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.*

Example 5 *Let M and N be left $\mathfrak{g}$-modules. Then $Hom_k(M, N)$ is a $\mathfrak{g}$-module by $xf(m) = xf(m) - f(xm)$, $x \in \mathfrak{g} \quad m \in M$. This gives us a natural isomorphism $Hom_\mathfrak{g}(M, N) \cong Hom_k(M, N)^\mathfrak{g}$. Extend this we get a natural isomorphism of δ funtors:*

$$Ext_{U(\mathfrak{g})}^*(M, N) \cong Hom_{Lie}^*(\mathfrak{g}, Hom_k(M, M)).$$

6 H^1 and H_1

We have the exact sequence of $\mathfrak{g}$-module

$$0 \to \mathfrak{J} \to U_(\mathfrak{g}) \to k \to 0.$$

If M is a $\mathfrak{g}$-module, applying $Tor_*^{U_(\mathfrak{g})}(-, M)$ yields

$$H_n(\mathfrak{g}, M) \cong Tor_n^{U(\mathfrak{g})}(k, M) \cong Tor_{n-1}^{U(\mathfrak{g})}(\mathfrak{J}, M) \quad for \quad n \geq 2$$

and the exact sequence

$$0 \to H_1(\mathfrak{g}, M) \to \mathfrak{J} \otimes_{U(\mathfrak{g})} M \to M \to M_\mathfrak{g} \to 0. \quad (*)$$

Theorem 2 *For any Lie algebra $\mathfrak{g}$, $H_1(\mathfrak{g}, k) \cong \mathfrak{g}^{ab}$.*

Proof. Putting $M = k$ in $(*)$ yields the exact sequence

$$0 \longrightarrow H_1(\mathfrak{g}, k) \longrightarrow \mathfrak{J} \otimes_{U(\mathfrak{g})} k \longrightarrow k \xrightarrow{\sim} k_\mathfrak{g} \longrightarrow 0$$

where

$$\mathfrak{J} \otimes_{U(\mathfrak{g})} k \cong \mathfrak{J} \otimes_{U(\mathfrak{g})} (U(\mathfrak{g})/\mathfrak{J}) \cong \mathfrak{J}/\mathfrak{J}^2 \cong \mathfrak{g}^{ab} \qquad \diamond$$

Corollary 3 *If M is any trivial $\mathfrak{g}$-module, $H_1(\mathfrak{g}, M) \cong \mathfrak{g}^{ab} \otimes_k M$.*

Proof. Since $M = M_{\mathfrak{g}}$, $(*)$ yields $H_1(\mathfrak{g}, M) \cong \mathfrak{J} \otimes_{U(\mathfrak{g})} M \cong (\mathfrak{J} \otimes_{U(\mathfrak{g})} k) \otimes_k M \cong (\mathfrak{J} \otimes_{U(\mathfrak{g})} U(\mathfrak{g})/\mathfrak{J}) \otimes_k M \cong \mathfrak{J}/\mathfrak{J}^2 \otimes_k M \cong \mathfrak{g}^{ab} \otimes_k M$. $\diamond$

Definition 14 *If M is a $\mathfrak{g}$-module, a derivation from $\mathfrak{g}$ into M is a k-linear map $D : \mathfrak{g} \to M$ such that $D([x, y]) = x(Dy) - y(Dx)$. The set of all such derivations is denoted $Der(\mathfrak{g}, M)$; it is a k-submodule of $Hom_k(\mathfrak{g}, M)$. Note that if $\mathfrak{g} = M$, then $Der(\mathfrak{g}, \mathfrak{g})$ is the derivation algebra $Der(\mathfrak{g})$. If M is a trivial $\mathfrak{g}$-module, then $Der(\mathfrak{g}, M) = Hom_k(\mathfrak{g}^{ab}, M)$.*

Example 6 *If $m \in M$, define $D_m(x) = xm$ $x \in \mathfrak{g}$ is a derivation, the D_m is called an inner derivations of $\mathfrak{g}$ into M, they form a k-subalgebra $Der_{Inn}(\mathfrak{g}, M)$ of $Der(\mathfrak{g}, M)$.*

If $\varphi : \mathfrak{J} \to M$ is a $\mathfrak{g}$-homomorphism, let $D_\varphi : \mathfrak{g} \to M$ be defined by $D_\varphi(x) = \varphi(i(x))$. This is a derivation since

$$D_\varphi([x, y]) = \varphi(i(x)i(y) - i(y)i(x)) = x\varphi(i(y)) - y\varphi(i(x)).$$

Lemma 1 *The map $\varphi \mapsto D_\varphi$ is a natural isomorphism of k-modules:*

$$Hom_{\mathfrak{g}}(\mathfrak{J}, M) \cong Der(\mathfrak{g}, M).$$

Proof. The map $\varphi \mapsto D_\varphi$ defines a natural homomorphism, so it suffices to show that it is an isomorphism. Note that $U(\mathfrak{g}) \otimes_k \mathfrak{g} \to U(\mathfrak{g})\mathfrak{g} = \mathfrak{J}$ is onto with kernal generated by the terms $(u \otimes [xy] - ux \otimes y + uy \otimes x)$ with $u \in U(\mathfrak{g})$ and $x, y \in \mathfrak{g}$.
Given a derivation $D : \mathfrak{g} \to M$, consider the map

$$f : U(\mathfrak{g}) \otimes_k \mathfrak{g} \to M \quad f(u \otimes x) = u(D(x))$$

Since D is a derivation, $f(u \otimes [xy] - ux \otimes y + uy \otimes x) = 0$ for all u, x and y. Therefore f induced a map $\varphi : \mathfrak{g} \to M$, which is a left $\mathfrak{g}$-module map. Since $D_\varphi(x) = \varphi(i(x)) = f(1 \otimes x) = Dx$ lifts D to a element in $Hom_{\mathfrak{g}}(\mathfrak{J}, M)$. On the other hand, given $D = D_h$ for some $h \in Hom_{\mathfrak{g}}(\mathfrak{J}, M)$, we have $\varphi(ux) = u(Dx) = uh(x) = h(ux)$ for all $u \in U(\mathfrak{g}), x \in \mathfrak{g}$. Hence $\varphi = h$ as map from $\mathfrak{J} = U(\mathfrak{g})\mathfrak{g}$ to M. $\diamond$

Theorem 3 $H^1(\mathfrak{g}, M) \cong Der(\mathfrak{g}, M)/Der_{Inn}(\mathfrak{g}, M)$.

Proof. If $\varphi : \mathfrak{J} \to M$ extends to a $\mathfrak{g}$-map $U(\mathfrak{g}) \to M$ sending 1 to $m \in M$, then

$$D_\varphi(x) = \varphi(x \cdot 1) = xm = D_m(x)$$

Hence D_φ is an inner derivation. This shows that the image of

$$M \to Hom_{\mathfrak{g}}(\mathfrak{J}, M) = Der(\mathfrak{g}, M)$$

is a submodule of inner derivations, as desired. $\diamond$

Corollary 4 *If M is trivial $\mathfrak{g}$-module*

$$H^1(\mathfrak{g}, M) \cong Der(\mathfrak{g}, M) \cong Hom_{Lie}(\mathfrak{g}, M) \cong Hom_k(\mathfrak{g}^{ab}, M).$$

7 The Chevalley-Eilenberg complex

Let $\mathfrak{g}$ be a Lie algebra over k. Denote $\wedge^p\mathfrak{g}$ the p-th exterior product of $\mathfrak{g}$, which is generated by monomials $x_1 \wedge ... \wedge x_n$ with $x_i \in \mathfrak{g}$. Define chain complex $V_p(\mathfrak{g}) = U(\mathfrak{g}) \otimes_k \wedge^p\mathfrak{g}$, which is free as a left $U_\mathfrak{g}$-module. Since $\wedge^0(\mathfrak{g}) = k$ and $\wedge^1 = \mathfrak{g}$, so $V_0 = U(\mathfrak{g})$ and $V_1 = U(\mathfrak{g}) \otimes_k \mathfrak{g}$. We define $\varepsilon : V_0(\mathfrak{g}) = U(\mathfrak{g}) \to k$ $i(g) \mapsto 0$ to be the augmentation and $d_1 : V_1(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathfrak{g} \to V_0(\mathfrak{g}) = U(\mathfrak{g})$ to be the product map $d(u \otimes x) = ux$ whose image is the ideal $\mathfrak{J}$. We have the exact sequence

$$V_1(\mathfrak{g}) \xrightarrow{d} V_0(\mathfrak{g}) \xrightarrow{\varepsilon} k \longrightarrow 0$$

Definition 15 *For $p \geq 2$, let $d : V_p(\mathfrak{g}) \to V_{p-1}(\mathfrak{g})$ be given by $d(u \otimes x_1 \wedge ... \wedge x_p) = \theta_1 + \theta_2$, where $u \in U(\mathfrak{g})$ $x_i \in \mathfrak{g}$ and*

$$\theta_1 = \sum_{i=1}^{p} (-1)^{i+1} u x_i \otimes x_1 \wedge ... \wedge \hat{x}_i \wedge ... \wedge x_p$$

$$\theta_2 = \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge ... \wedge \hat{x}_i \wedge ... \wedge \hat{x}_j \wedge ... \wedge x_p$$

$V_*(\mathfrak{g})$ with this differential is called Chevalley-Eisenberg complex.

If $p = 2$, then $d(u \otimes x \wedge y) = ux \otimes y - uy \otimes x - u \otimes [xy]$.

Theorem 4 *$V_*(\mathfrak{g}) \to^\varepsilon k$ is a projective resolution of the $\mathfrak{g}$-module k.*

Corollary 5 *(Chevalley-Eilenberg) If M is a right $\mathfrak{g}$-module, then the homology modules $H_*(\mathfrak{g}, M)$ are the homology of the chain complex*

$$M \otimes_{U_\mathfrak{g}} V_*(\mathfrak{g}) \cong M \otimes_{U_\mathfrak{g}} U_\mathfrak{g} \otimes_k \wedge^*\mathfrak{g} \cong M \otimes_k \wedge^*\mathfrak{g}.$$

If M is a left $\mathfrak{g}$-module, then the cohomology modules $H^*(\mathfrak{g}, M)$ are the cohomolofy of the cochain complex

$$Hom_\mathfrak{g}(V_*(\mathfrak{g}), M) = Hom_\mathfrak{g}(U_\mathfrak{g} \otimes \wedge^*\mathfrak{g}, M) \cong Hom_k(\wedge^*\mathfrak{g}, M).$$

In this complex, an n-cochain $f : \wedge^n\mathfrak{g} \to M$ is just an alternating k-multilinear function $f(x_1, ..., x_n)$ of n variables in $\mathfrak{g}$ taking values in M. The coboundary of such an n-cochain is the n+1 cochain

$$df(x_1, ..., x_{n+1}) = \sum (-1)^n x_i f(x_1, ..., \hat{x}_i, ..., x_{n+1})$$
$$+ \sum (-1)^{i+j} f([x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+1}).$$

We thus constructed cochain complex $(C^*(\mathfrak{g}, M), d)$ where $C^*(\mathfrak{g}, M) = Hom_k(\mathfrak{g}, M)$. Denote by

$$Z^p(\mathfrak{g}, M) = \{f \in C^p(\mathfrak{g}, M) | dc = 0\}$$

the space of p-cocycles, and by

$$B^p(\mathfrak{g}, M) = \{f \in C^p(\mathfrak{g}, M) | \exists f' \in C^{p-1}(\mathfrak{g}, M) \, such \, that \, c = dc'\}$$

the space of p-coboundaries. Then $H^p_{Lie}(\mathfrak{g}, M) = Z^p(\mathfrak{g}, M)/B^p(\mathfrak{g}, M)$. In particular, we have

$$Z^1(\mathfrak{g}, M) = \{c \in C^1(\mathfrak{g}, M) | c([x, y]) = x \cdot c(y) - y \cdot c(x)\} = Der(\mathfrak{g}, M)$$

$$B^1(\mathfrak{g}, M) = \{c \in C^1(\mathfrak{g}, M) | c(x) = x \cdot m \quad m \in M\} = Der_{Inn}(\mathfrak{g}, M)$$

Reinterpret $H^0(\mathfrak{g}, M)$ and $H^1(\mathfrak{g}, M)$

$$H^0(\mathfrak{g}, M) = Z^0(\mathfrak{g}, M) = \{m \in M | dm = 0\} = \{m \in M | x \cdot m = 0 \quad \forall x \in \mathfrak{g}\}$$

$$H^1(\mathfrak{g}, M) = Z^1(\mathfrak{g}, M)/B^1(\mathfrak{g}, M) = Der(\mathfrak{g}, M)/Der_{Inn}(\mathfrak{g}, M).$$

Example 7 *Consider $\mathfrak{sl}(2, \mathbb{C}) =< x, y, h >$, where*

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with bracket relation $[x, y] = h$, $[h, x] = 2x$, $[h, y] = -2y$. This shows easily that $H^1(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})) = 0$, i.e. all derivations are inner, and that $H^1(\mathfrak{sl}(2, \mathbb{C}), k) = 0$.

Application If $\mathfrak{g}$ is a n-dimensional over k, then $H^i(\mathfrak{g}, M) = H_i(\mathfrak{g}, M) = 0 \quad \forall i > n$. $(\wedge^i \mathfrak{g} = 0)$

8 H^2 and Extensions

Definition 16 *An extension of Lie algebra of $\mathfrak{g}$ by M is a short exact sequence $0 \longrightarrow M \xrightarrow{i} \mathfrak{e}_1 \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$ in which M is an abelian Lie algebra.*

Indeed M is a $\mathfrak{g}$-module via $x \cdot m = [\tilde{g}, m]$ in e where $\tilde{g}$ lift of g in $\mathfrak{e}$, $x \in \mathfrak{g}$ $m \in M$.

Give $\mathfrak{g}$-module M, say two extensions are equivalent if $\exists\varphi : \mathfrak{e}_1 \to \mathfrak{e}_2$ isomorphism such that the diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & \mathfrak{e}_1 \arrow[r, "\pi"] \arrow[d, "\varphi"] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "i'"] & \mathfrak{e}_2 \arrow[r, "\pi'"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

Denote $Ext(\mathfrak{g}, M)$ the set of equivalent classes of extensions.

Classification Theorem
Let M be a $\mathfrak{g}$-module. The set $Ext(\mathfrak{g}, M)$ of equivalent classes extensions of $\mathfrak{g}$ by M is in 1-1 correspondence with $H^2_{Lie}(\mathfrak{g}, M)$.

Sketch of proof. In one direction, associate to a given 2-cocycle c the extension $\mathfrak{e}_c = M \oplus \mathfrak{g}$(as vector space) with the bracket

$$[(a, x), (b, y)] = (x \cdot b - y \cdot a + c(x, y), [x, y]).$$

In other direction, choose a linear section $s([x, y]) - [s(x), s(y)]$ of π. The default of s being a Lie algebra morphism gives a cocycle:

$$c(x, y) = s([x, y]) - [s(x), s(y)] \quad \forall\, x, y \in \mathfrak{g}$$

As $Ker(\pi) = Im(i)$, c takes values in $i(M) \cong M$ $\diamond$

9 H^3 and Crossed Modules

Definition 17 *A crossed module of Lie algebras is a homomorphism of Lie algebra $\mu : \mathfrak{m} \to \mathfrak{n}$ together with an action of $\mathfrak{n}$ on $\mathfrak{m}$ by derivations, denoted by $m \longmapsto n \cdot m$, such that for all $m, m' \in \mathfrak{m}$ and for all $n \in \mathfrak{n}$*

(a) $\mu(n \cdot m) = [n, \mu(m)]$

(b) $\mu \cdot m' = [m, m']$

The requirements (a) and (b) imply that the cokernel of μ, denote by $\mathfrak{g}$, is a Lie algebra, that the kernel of μ, denote by M, is a central ideal, and that $\mathfrak{g}$ acts on M. Therefore a crossed module corresponds to a 4 term exact sequence of Lie algebras

$$0 \longrightarrow M \longrightarrow \mathfrak{m} \xrightarrow{\mu} \mathfrak{n} \longrightarrow \mathfrak{g} \longrightarrow 0.$$

Fact. $H^3(\mathfrak{g}, M)$ is isomorphic to the set of equivalent classes of crossed modules with cokernal $\mathfrak{g}$ and kernal M.

10 Semisimple Lie Algebras

Let $\mathfrak{g}$ be finite dimensional Lie algebra over k where char k=0.

Proposition 3 *If $\mathfrak{g}$ is finite dimensional and semisimple, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Consequently, $H_1(\mathfrak{g}, k) = H^1(\mathfrak{g}, k) = 0$.*

Proof. Since $\mathfrak{g} = \mathfrak{g}_1 \times ... \times \mathfrak{g}_k$ where $\mathfrak{g}_i$ simple Lie algebra and $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$ (If $[\mathfrak{g}_i, \mathfrak{g}_i] = 0$, then $\mathfrak{g}_i \subseteq Rad(\mathfrak{g}) = 0$), hence $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. This gives us $\mathfrak{g}^{ab} = 0$. On the other hand, we saw in Corollary 2 and Corollary 4 that $H_1(\mathfrak{g}, k) \cong \mathfrak{g}^{ab}$ and $H^1(\mathfrak{g}, k) \cong Hom_k(\mathfrak{g}^{ab}, k)$. $\diamond$

The Casimir operator
Let $\mathfrak{g}$ be a semisimple Lie algebra and let M a m-dim $\mathfrak{g}$-module, if $\mathfrak{h}$ be the image of the structure map

$$\rho : \mathfrak{g} \to Lie(End_k(M)) \cong \mathfrak{gl}_m(k)$$

then $\rho \cong \mathfrak{h} \times ker(\rho) \quad \mathfrak{h} \subset \mathfrak{gl}_m(k)$. By Cartan's criterion, tr is non-degenerate on $\mathfrak{h}$. Choose $e_1, ..., e_r$ basis of $\mathfrak{h}$, then there is a dual basis $e^1, ..., e^r$ dual basis of $\mathfrak{h}$ with respect to tr. Then the element $c_M := \sum e_i e^i \in U_{\mathfrak{g}}$ is called the Casimir operator for M, which is independent of choice of basis.

Recall c_M has the following properties:

1. If $x \in \mathfrak{g}$ and $[e_i, x] = c_{ij} e_j$, then $[x, e^j] = \sum c_{ij} e^j$.
2. $c_M \in center(U(\mathfrak{g}))$ and $c_M \in \mathfrak{J}$.
3. Image of c_M in $End_k(M)$ is $\frac{r}{m} id_M$, where r=$dim(\mathfrak{h})$.

Theorem 5 *Let $\mathfrak{g}$ be a semisimple over field of characteristic 0. If M is a simple $\mathfrak{g}$-module $M \neq k$, then*

$$H^i_{Lie}(\mathfrak{g}, M) = H^{Lie}_i(\mathfrak{g}, M) = 0 \quad \forall i.$$

Proof. Let C be the center of $U(\mathfrak{g})$. We saw in Corollary 2 that $H_*(\mathfrak{g}, M) = Tor_*^{U(\mathfrak{g})}(k, M)$ and $H^*(\mathfrak{g}, M) = Ext^*_{U(\mathfrak{g})}(k, M)$ are naturally C-modules. Multiplication by $c \in C$ is induced by $c : k \to k$ as well as $c : M \to M$. Since the Casmier element c_M acts by 0 on k (as $c_M \in \mathfrak{J}$) and by the invertible scalar r/m on M, we must have $0 = r/m$ on $H_*(\mathfrak{g}, M)$ and $H^*(\mathfrak{g}, M)$. This can only happen if they are zero. $\diamond$

Corollary 6 *(Whitehead's first lemma) Let $\mathfrak{g}$ be a semisimple Lie algebra over field of characteristic 0. If M is any finite dimensional $\mathfrak{g}$-module, then $H^1_{Lie}(\mathfrak{g}, M) = 0$. That is, $Der(\mathfrak{g}, M) = Der_{Inn}(\mathfrak{g}, M)$.*

Proof. Induction on $dim(M)$. If M is simple, then either $M = k$ and $H^1(\mathfrak{g}, k) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = 0$ or else $M \neq k$ and $H^*(\mathfrak{g}, M) = 0$ by the theorem. Otherwise, M contains a proper submodule L. By induction, $H^1(\mathfrak{g}, L) = H^1(\mathfrak{g}, M/L) = 0$. Since $0 \to L \to M \to M/L \to 0$ induces exact sequence of cohomology,

$$...H^1(\mathfrak{g}, L) \to H^1(\mathfrak{g}, M) \to H^1(\mathfrak{g}, L/M)...$$

We have $H^1(\mathfrak{g}, M) = 0$. By Theorem 3, $H^1(\mathfrak{g}, M) \cong Der(\mathfrak{g}, M)/Der_{inn}(\mathfrak{g}, M)$. $\diamond$

Theorem 6 *(Weyl's Theorem) Let $\mathfrak{g}$ be a semisimple Lie algebra over field of characteristic 0. Then every finite dimensional $\mathfrak{g}$-module M is completely reducible i.e. direct sum of $\mathfrak{g}$-module.*

Proof. Suppose M is not a direct sum of simple modules. Since $dim(M)$ is finite, M contains a submodule M_1 minimal with respect to this property. Clearly M_1 is not simple, so it contains a proper $\mathfrak{g}$-submodule M_0. By induction, both M_0 and $M_2 = M_1/M_0$ are direct sums of simple $\mathfrak{g}$-modules but M_1 is not, so the extension of M_2 by M_0 is not split. Hence it is represented by a nonzero element of

$$Ext^1_{U(\mathfrak{g})}(M_2, M_0) \cong H^1_{Lie}(\mathfrak{g}, Hom_k(M_2, M_0))$$

by Example 5 and this contradicts Whitehead's first lemma. $\diamond$

Corollary 7 *(Whitehead's Second lemma) Let $\mathfrak{g}$ be a semisimple Lie algebra over field of characteristic 0. If M is any finite dimensional $\mathfrak{g}$-module, then $H^2_{Lie}(\mathfrak{g}, M) = 0$.*

Proof. Since H^* commutes with direct sums, and M is a direct sum of simple $\mathfrak{g}$-modules by Weyl's theorem, we may assume that M is simple. If $M \neq k$, then $H^2_{Lie}(\mathfrak{g}, M) = 0$ by Theorem 5, so it suffices to show that $H^2(\mathfrak{g}, k) = 0$. That is, every Lie algebra extension

$$0 \longrightarrow k \longrightarrow \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

splits. We claim that $\mathfrak{e}$ can be made into a $\mathfrak{g}$-module in such a way that π is a $\mathfrak{g}$-map. To see this, let $\tilde{x}$ be any lift of $x \in \mathfrak{g}$ to $\mathfrak{e}$ and define $x \cdot e$ to be $[\tilde{x}, e]$ for $e \in \mathfrak{e}$. This is independent of choice of $\tilde{x}$ because k is in the center of $\mathfrak{e}$. This defines a $\mathfrak{g}$-module, and by construction $\pi(x \cdot e) = [x, \pi(e)]$.

By Weyl's theorem, $\mathfrak{e}$ and $\mathfrak{g}$ splits as $\mathfrak{g}$-modules, and there is a $\mathfrak{g}$-module homomorphism $\tau : \mathfrak{g} \to \mathfrak{e}$ splitting π such that $\mathfrak{e} \cong k \times \mathfrak{g}$ as a $\mathfrak{g}$-module. If we choose $\tilde{x} = \tau(x)$, then it is clear that τ is a Lie algebra homomorphism and that $\mathfrak{e} \cong k \times \mathfrak{g}$ as a Lie algebra. This proves that $H^2(\mathfrak{g}, k) = 0$. $\diamond$

Theorem 7 *(Levi-Malcev) If $\mathfrak{g}$ is a finite dimensional Lie algebra over a field of characteristic zero, then there is a semisimple Lie subalgebra calL of $\mathfrak{g}$(called a Levi factor of $\mathfrak{g}$) such that $\mathfrak{g}$ is isomorphic to the semidirect product*

$$\mathfrak{g} \cong (rad\mathfrak{g}) \rtimes \mathcal{L}.$$

Proof. Since $\mathfrak{g}/rad(\mathfrak{g})$ is semisimple, it suffices to show that the following Lie algebra extension splits.

$$0 \to rad(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/rad(\mathfrak{g}) \to 0$$

If $rad(\mathfrak{g})$ is abelian, then this extension is classified by $H^2(\mathfrak{g}/rad\mathfrak{g}, rad\mathfrak{g})$, which vanishes by Whitehead's second lemma, so every extension splits.

If $rad(\mathfrak{g})$ is not abelian, we proceed by induction on the derived length of $rad(\mathfrak{g})$. Let $\mathfrak{r}$ denote the ideal $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$. Since $rad(\mathfrak{g}/\mathfrak{r}) = rad(\mathfrak{g})/\mathfrak{r}$ is abelian, the extension

$$0 \to (rad\mathfrak{g})/\mathfrak{r} \to \mathfrak{g}/\mathfrak{r} \to \mathfrak{g}/(rad\mathfrak{g}) \to 0$$

splits. Hence there is an ideal $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{r}$ such that $\mathfrak{g}/\mathfrak{r} \cong (rad\mathfrak{g})/\mathfrak{r} \times \mathfrak{h}/\mathfrak{r}$ and $\mathfrak{h}/\mathfrak{r} \cong \mathfrak{g}/(rad\mathfrak{g})$. Now $rad(\mathfrak{h}) = rad(\mathfrak{g}) \cap \mathfrak{h} = \mathfrak{r}$, and $\mathfrak{r}$ has a smaller derived length than $rad\mathfrak{g}$. By induction there is a Lie subalgebra $\mathcal{L}$ of $\mathfrak{h}$ such that $\mathfrak{h} \cong \mathfrak{r} \rtimes \mathcal{L}$ and $\mathcal{L} \cong \mathfrak{h}/\mathfrak{r} \cong \mathfrak{g}/(rad\mathfrak{g})$. Then $\mathcal{L}$ is our desired Levi factor of $\mathfrak{g}$. $\diamond$

References

[1] C. Weibel: An introduction to Homological Algebra

[2] P.J. Hilton and U. Stammbach: A Course in Homological Algebra

[3] H. Cartan and S. Eilenberg: Homological Algebra

[4] F. Wagemann: Introduction to Lie algebra cohomology with a view towards BRST cohomology

[5] S. Raghavan, R. Balwant Singh and R. Sridharan: Homological methods in Commutative Algebra