Lie Algebra Homology and Cohomology

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Abstract

In this project we give an application of derived functor. Starting with a Lie algebra \( g \) over the field \( k \), we pass to the universal enveloping algebra \( U(g) \) and define homology \( H_n(g, M) \) and cohomology groups \( H^*(g, M) \) for every (left) \( g \)-module \( M \), by regarding \( M \) as a \( U(g) \)-module. In the last section, we prove Whitehead’s theorem: If \( g \) is a finite dimensional semisimple Lie algebra over field of characteristic 0 and \( M \) is a nontrivial irreducible module then \( H_i(g, M) = H^i(g, M) = 0 \), and then deduce Weyl’s theorem and Whitehead’s lemmas as corollaries.

1 Chain complexes

Definition 1 A chain complex \( C \) of \( R \)-modules is a family \( \{ C_n \}_{n \in \mathbb{Z}} \) of \( R \)-modules, together with \( R \)-modules maps \( d = d_n : C_n \to C_{n-1} \) such that each composite \( d \circ d : C_n \to C_{n-2} \) is zero. The maps \( d_n \) are called differentials of \( C \). The \( n \)-th homology groups of \( C \) is the quotient \( H_n(C) = \ker(d_n)/\text{im}(d_{n+1}) \).

Definition 2 A chain complex map \( f : C \to D \) is a family of \( R \)-module homomorphisms \( f_n : C_n \to D_n \) commuting with \( d \) in the sense that \( f_{n-1} \circ d_n = d'_{n-1} \circ f_n \). That is, such that the following diagram commutes

\[
\begin{array}{ccccccc}
\cdots & \xrightarrow{d} & C_{n+1} & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} & \xrightarrow{d} & \cdots \\
\downarrow{f} & & \downarrow{f} & & \downarrow{f} & & \\
\cdots & \xrightarrow{d'} & D_{n+1} & \xrightarrow{d'} & D_n & \xrightarrow{d'} & D_{n-1} & \xrightarrow{d'} & \cdots
\end{array}
\]

By the diagram above, we have \( \ker(d_n) \subseteq \ker(d'_n) \) and \( \text{im}(d_{n+1}) \subseteq \text{im}(d'_{n+1}) \). Hence \( f \) induces a map \( H_n(f) : H_n(C) \to H_n(D) \).

Similarly, we can define cochain complex and cohomology by dualizing.
**Proposition 1** Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be an exact sequence of chain complexes. Then there are natural maps $\partial_n : H_n(C) \rightarrow H_{n-1}(A)$, called connecting homomorphisms, such that

$$\cdots \xrightarrow{g} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f} \cdots$$

is an exact sequence.

Similarly, if $0 \rightarrow \hat{A} \xrightarrow{f} \hat{B} \xrightarrow{g} \hat{C} \rightarrow 0$ be an exact sequence of cochain complexes, there are natural maps $\partial^n : \hat{H}^n(\hat{C}) \rightarrow \hat{H}^{n+1}(\hat{A})$, called connecting homomorphisms, and a long exact sequence

$$\cdots \xrightarrow{g} \hat{H}^{n-1}(\hat{C}) \xrightarrow{\partial} \hat{H}^n(\hat{A}) \xrightarrow{f} \hat{H}^n(\hat{B}) \xrightarrow{g} \hat{H}^n(\hat{C}) \xrightarrow{\partial} \hat{H}^{n+1}(\hat{A}) \xrightarrow{f} \cdots .$$

**Definition of connecting homomorphism**

\[
\begin{array}{ccccccccc}
B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} & \rightarrow & 0 \\
\downarrow d'_{n+1} & & \downarrow d''_{n+1} & & \\
0 & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \rightarrow & 0 \\
\downarrow d_n & & \downarrow d'_n & & \downarrow d''_n & & \\
0 & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_n & \rightarrow & 0 \\
\downarrow d_{n-1} & & \downarrow d'_{n-1} & & & & \\
0 & \xrightarrow{f_{n-2}} & B_{n-2} & & & & \\
\end{array}
\]

Let $\tilde{c} \in H_n(C)$, and let $c \in \text{Ker}(d''_n)$ represent $\tilde{c}$. Choose $b \in B_n$ such that $g_n(b) = c$. Let $b' = d'_n(b)$. Then $g_{n-1}(b') = 0$. Therefore, there exists $a \in A_{n-1}$ such that $b' = f_{n-1}(a)$. Since $(d''_{n-1}f_{n-1})(a) = (d'_{n-1}d'_{n})(b) = 0$, it follows that $(f_{n-2}d_{n-1})(a) = 0$ i.e. $a \in \text{Ker}(d_{n-1})$. Let $\tilde{a}$ be the canonical image of $a$ in $H_{n-1}(A)$. It is easily seen that $\tilde{a}$ does not depend on the choice of $c$ and $b$. We define $\partial_n : H_n(C) \rightarrow H_{n-1}(A)$ by $\partial_n(\tilde{c}) = \tilde{a}$. Clearly, $\partial_n$ is an $R$-homomorphism.

Similarly, we can define the connecting homomorphism for cochain complex.

## 2 \(\delta\)-functors and Derived functors

**Definition 3** A (covariant) homological (resp. cohomological) $\delta$-functor between abelian category $A$ and $B$ is a collection of additive functors $T_n : A \rightarrow B$ (resp. $T^n : A \rightarrow B$) for $n \geq 0$, together with morphisms

$$\delta_n : T(C) \rightarrow T(A)$$

(resp. $\delta^n : T(C) \rightarrow T(A)$)
defined for each short exact sequence \( 0 \to A \to B \to C \to 0 \) in \( A \). With two condition imposed:

1. For each short exact sequence as above, there is a long exact sequence

\[
\cdots \to T_{n+1}(C) \overset{\delta}{\to} T_n(A) \to T_n(B) \to T_n(C) \overset{\delta}{\to} T_{n-1}(C) \to \cdots
\]

\[
\cdots \to T^{n-1}(C) \overset{\delta}{\to} T^n(A) \to T^n(B) \to T^n(C) \overset{\delta}{\to} T^{n+1}(C) \to \cdots
\]

In particular, \( T_0 \) is right exact and \( T^0 \) is left exact.

2. For each morphism of short exact sequence from \( 0 \to A' \to B' \to C' \to 0 \) to \( 0 \to A \to B \to C \to 0 \), the \( \delta \)'s give a commutative diagram

\[
\begin{array}{ccc}
T_n(C') & \overset{\delta}{\to} & T_{n-1}(A') \\
\downarrow & & \downarrow \\
T_n(C) & \overset{\delta}{\to} & T_{n-1}(A)
\end{array}
\]

\[
\begin{array}{ccc}
T^n(C') & \overset{\delta}{\to} & T^{n+1}(A') \\
\downarrow & & \downarrow \\
T^n(C) & \overset{\delta}{\to} & T^{n+1}(A)
\end{array}
\]

**Definition 4** A morphism \( S \to T \) of \( \delta \)-functors is a system of natural transformations \( S_n \to T_n \) (resp. \( S^n \to T^n \)) that commute with \( \delta \).

A homological \( \delta \)-functor \( T \) is universal if, given any other \( \delta \)-functor \( S \) and a natural transformation \( f_0 : S_0 \to T_0 \), there exists a unique morphism \( \{ f_n : S_n \to T_n \} \) of \( \delta \)-functors that extends \( f_0 \).

A cohomological \( \delta \)-functor \( T \) is universal if, given \( S \) and \( f^0 : S^0 \to T^0 \), there exists a unique morphism \( \{ f^n : S^n \to T^n \} \) of \( \delta \)-functors extending \( f^0 \).

**Definition 5** We say an abelian category \( A \) has enough projectives if for every object \( A \) of \( A \) there is a surjection \( P \to A \) with \( P \) projective. Similarly, we say a category \( A \) has enough injectives if for every object \( A \) of \( A \) there is a injection \( A \to I \) with \( I \) injective.

**Definition 6** Let \( M \) be an object of \( A \). A projective resolution of \( M \) is a pair \( (P_*, \varepsilon) \), where \( P \) is a left complex with each \( P_i \) projective together with a map \( \varepsilon : P_0 \to M \) so that the augmented complex

\[
\cdots \to P_n \overset{d_n}{\to} P_{n-1} \to \cdots \to P_0 \overset{\varepsilon}{\to} M \to 0
\]

is exact.

**Fact** The category \( \text{mod} - R \) has enough projectives.
Definition 7 Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor between two abelian categories. If $\mathcal{A}$ has enough projectives, we can construct the left derived functors $L_i F(i \geq 1)$ of $F$ as follows. If $A$ is an object of $\mathcal{A}$, choose (once and for all) a projective resolution $P \to A$ and define

$$L_i F(A) = H_i(F(P))$$

We always have $L_0 F(A) \cong F(A)$.

Facts

1. The object $L_i F(A)$ of $\mathcal{B}$ are well defined up to natural isomorphism. That is, if $Q \to A$ is a second projective resolution, then there is a canonical isomorphism:

$$L_i F(A) = H_i(F(P)) \to H_i(F(Q)).$$

2. If $A$ is projective, then $L_i F(A) = 0$ for $i \neq 0$.

3. If $f : A' \to A$ is any map in $\mathcal{A}$, there is a natural map $L_i F(f) : L_i F(A') \to L_i F(A)$ for each $i$.

4. The derived functors $L_\ast F$ form a homological $\delta$-functor.

5. Assume that $\mathcal{A}$ has enough projectives. Then for any right exact functor $F : \mathcal{A} \to \mathcal{B}$, the derived functors $L_n F$ form a universal $\delta$-functor.

Definition 8 Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between two abelian categories. If $\mathcal{A}$ has enough injectives, we can construct the right derived functors $R^i F(i \geq 0)$ of $F$ as follows. If $A$ is an object of $\mathcal{A}$, choose an injective resolution $A \to I$ and define

$$R^i F(A) = H^i(F(I))$$

Note that since $0 \to F(A) \to F(I^0) \to F(I^1)$ is exact, we always have $R^0 F(A) \cong F(A)$.

Since $F$ also defines a right exact functor $F^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$, and $\mathcal{A}^{op}$ has enough projectives, we can construct the left derived functors $L_i F^{op}$ as well. Since $I$ becomes a projective resolution of $A$ in $\mathcal{A}^{op}$, we see that

$$R^i F(A) = (L_i F^{op})^{op}(A)$$

Therefore all the results about right exact functors apply to left exact functors.
3 Ext Functors and Tor Functors

Definition 9 (Tor functors) Let $B$ be a left $R$-module, so that $T(A) = A \otimes_R B$ is a right exact functor from $\text{mod} - R$ to $\text{Ab}$. We define the abelian groups

$$\text{Tor}_n^R(A, B) = (L_nT)(A)$$

In particular, $\text{Tor}_0^R(A, B) \cong A \otimes_R B$.

Definition 10 (Ext functors) For each $R$-module $A$, the functor $F(B) = \text{Hom}_R(A, B)$ is left exact. Its right derived functors are called the Ext groups:

$$\text{Ext}_n^R(A, B) = R^n\text{Hom}_R(A, -)(B)$$

In particular, $\text{Ext}_0^R(A, B)$ is $\text{Hom}(A, B)$.

Remark.

1. Tor and Ext satisfy all the facts above. In particular, they are independent of choice of resolutions and are universal $\delta$-functors.

2. $L_n(A \otimes_R)(B) \cong L_n(\otimes_R B)(A) = \text{Tor}_n^R(A, B)$ for all $n$. This means that we can also compute $\text{Tor}_n^R(A, B)$ by finding projective resolution of $B$ and taking the homology.

3. $R^n(\text{Hom}_R(A, -)) \cong R^n(\text{Hom}_R(-, B)) = \text{Ext}_n^R(A, B)$ for all $n$. This means that we can also compute $\text{Ext}_n^R$ by finding projective resolution of $A$ and taking the cohomology.

4 Lie algebra Homology and Cohomology

Let $\mathfrak{g}$ a Lie Algebra over field $k$ and $M$ a left $\mathfrak{g}$-module, we have

1. The invariant submodule $M^\mathfrak{g}$ of a $\mathfrak{g}$-module $M$,

$$M^\mathfrak{g} = \{m \in M \mid xm = 0 \ \forall x \in \mathfrak{g}\}$$

Considering $k$ as a trivial $\mathfrak{g}$-module, we have $M^\mathfrak{g} \cong \text{Hom}_\mathfrak{g}(k, M)$.

2. The coinvariant $M_\mathfrak{g}$ of a $\mathfrak{g}$-module $M$, $M_\mathfrak{g} = M/\mathfrak{g}M$. 

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Definition 11 Let $M$ be a $\mathfrak{g}$-module. we write $H_*(\mathfrak{g}, M)$ or $H_*^{\text{Lie}}(\mathfrak{g}, M)$ for the left derived functors $L_*(-\mathfrak{g})(M)$ of $-\mathfrak{g}$ and call them the homology groups of $\mathfrak{g}$ with coefficients in $M$. By definition, $H_0(\mathfrak{g}, M) = M_{\mathfrak{g}}$.

Similarly, we write $H^*(\mathfrak{g}, M)$ or $H_*^R(\mathfrak{g}, M)$ for the right derived functors $R_*^*(-\mathfrak{g})(M)$ of $-\mathfrak{g}$ and call them the cohomology groups of $\mathfrak{g}$ with coefficients in $M$. By definition, $H^0(\mathfrak{g}, M) = M^\mathfrak{g}$.

Example 1 Let $\mathfrak{g}$ be a free $k$-module on basis $\{e_1, \ldots, e_n\}$, made into an abelian Lie algebra with zero bracket. Thus $\mathfrak{g}$-module is just a $k$-module with $n$ commuting endomorphisms $\{e_1, \ldots, e_n\}$. It follows that $\mathfrak{g}$-module is just an $R$-module where $R = k[e_1, \ldots, e_n]$. If $k$ is trivial $\mathfrak{g}$-module viewed as $R$-module with $e_i$ acting as 0, then $M_{\mathfrak{g}} = k \otimes_R M$ and $M^\mathfrak{g} = \text{Hom}_R(k, M)$. Therefore we have

$$H_*^{\text{Lie}}(\mathfrak{g}, M) = \text{Tor}_*^R(k, M) \quad \text{and} \quad H_*^R(\mathfrak{g}, M) = \text{Ext}_R^*(k, M).$$

Example 2 If $\mathfrak{f}$ is free Lie algebra on set $X$, then an $\mathfrak{f}$-module is a $k$-module $M$ with an arbitrary set $\{e_x : x \in X\}$ of endormorphisms. Hence, if view $k$ as trivial $\mathfrak{f}$-module, then $M_1 = k \otimes_R M$ and $M^1 = \text{Hom}_R(k, M)$, where $R$ is the free associative $k$-algebra $k\{X\}$. Therefore,

$$H_*^{\text{Lie}}(\mathfrak{f}, M) = \text{Tor}_*^R(k, M) \quad \text{and} \quad H_*^R(\mathfrak{f}, M) = \text{Ext}_R^*(k, M).$$

Proposition 2 The ideal $J = Xk\{X\}$ of a free ring $k\{X\}$ is free as right $k\{X\}$-module with basis the set $X$. Hence

$$0 \rightarrow J \rightarrow k\{X\} \rightarrow k \rightarrow 0$$

is a free resolution of $k$ as a right $k\{X\}$-module.

Proof. As a free $k$-module, $k\{X\}$ has basis words $W$ of the set $X$, and $J$ is a free $k$-module on basis $W - \{1\}$. Every elements of $W - \{1\}$ is of the form $wx$ with $w \in W$ and $x \in X$, so $\{xw : x \in X, w \in W\}$ is also basis for $J$ as $k$-module. For each $x \in X$ the $k$-span $xk\{X\}$ is isomorphic to $k\{X\}$, and $J$ is a direct sum of the $xk\{X\}$, both as $k$-modules and as right $k\{X\}$-modules. That is, $J$ is a free right $k\{X\}$-module with basis $X$. \hfill \Box

Corollary 1 If $\mathfrak{f}$ is the free Lie algebra on $X$, then $H_*^{\text{Lie}}(\mathfrak{f}, M) = H_*^{\text{Lie}}(\mathfrak{f}, M)$, and for all $n \geq 2$ and all $\mathfrak{f}$-modules $M$. Moreover, $H_0^{\text{Lie}}(\mathfrak{f}, M) = H_0^{\text{Lie}}(\mathfrak{f}, M) = k$ while $H_1^{\text{Lie}}(\mathfrak{f}, k) = H_1^{\text{Lie}}(\mathfrak{f}, k) = \prod_{x \in X} k$.

Proof. Using the free resolution of $k$ in Proposition 2, $H_*^{\text{Lie}}(\mathfrak{f}, M)$ is the homology of the complex $0 \rightarrow J \otimes \mathfrak{f} \rightarrow \mathfrak{f} \otimes M \rightarrow 0$ and $H_*^{\text{Lie}}(\mathfrak{f}, M)$ is the homology of the complex $0 \rightarrow \text{Hom}_\mathfrak{f}(\mathfrak{f}, M) \rightarrow \text{Hom}_\mathfrak{f}(J, M) \rightarrow 0$, hence for $n \geq 2$ the homology groups and cohomology groups are zero. For $M = k$, the differentials are 0. Furthermore, $J \otimes k \cong \prod_{x \in X} k \cong \text{Hom}_\mathfrak{f}(J, k)$ and $\mathfrak{f} \otimes k \cong k \cong \text{Hom}_\mathfrak{f}(\mathfrak{f}, k)$. we get the result for $i = 0, 1$. \hfill \Box
5 Universal Enveloping Algebra

**Definition 12** If $M$ is any $k$-module, the tensor algebra is defined by $T(M) = \bigoplus_{n=0}^{\infty} M^\otimes_n$, where $M^\otimes_n$ denotes $M \otimes \ldots \otimes M$, the tensor product of $n$ copies of $M$ over $k$. Writing $i : M \to T(M)$ for the inclusion, $T(M)$ is generated by $i(M)$ as $k$-algebra.

**Definition 13** If $\mathfrak{g}$ is a Lie algebra over $k$, the universal enveloping algebra $U(\mathfrak{g})$ is the quotient of $T(\mathfrak{g})$ by the two sided ideal generated by $\{i([x,y]) - i(x)i(y) - i(y)i(x) : x, y \in \mathfrak{g}\}$.

For every associated algebra $A$, there is a natural isomorphism

$$\text{Hom}_{\text{Lie}}(\mathfrak{g}, \text{Lie}(A)) \cong \text{Hom}_{k-\text{alg}}(U(\mathfrak{g}), A)$$

where $\text{Lie}(A)$ is the Lie algebra with underlying set $A$ with Lie bracket defined by $[a, b] = ab - ba$.

Given a $\mathfrak{g}$-module $M$ and a monomial $x_1 \cdots x_n$ in $U(\mathfrak{g})$ ($x_i \in \mathfrak{g}$), define

$$(x_1 \cdots x_n)m = x_1(x_2(\cdots (x_nm) \cdots)), \quad m \in M$$

makes $M$ into a $U(\mathfrak{g})$-module. Conversely, if $M$ is a $U(\mathfrak{g})$-module and $x \in \mathfrak{g}$, define $xm = i(x)m$ ($m \in M$) makes $M$ into a $\mathfrak{g}$-module.

**Theorem 1** If $\mathfrak{g}$ is a Lie algebra, every left $\mathfrak{g}$-module is naturally a left $U(\mathfrak{g})$-module. The category $\mathfrak{g}$-mod is naturally isomorphic to the category $U\mathfrak{g}$-mod of left $U(\mathfrak{g})$-modules.

Proof. Let $M$ be a $k$-module and write $E = \text{End}_k(M)$ for the $k$-algebra of all $k$-module endomorphisms of $M$. Take $A = E$ in the above identification, we have

$$\text{Hom}_{\text{Lie}}(\mathfrak{g}, \text{Lie}(E)) \cong \text{Hom}_{k-\text{alg}}(U(\mathfrak{g}), E)$$

A $\mathfrak{g}$-module is a $k$-module $M$ together with a Lie algebra map $\mathfrak{g} \to \text{Lie}(E)$. On the other hand, a $U(\mathfrak{g})$-module $M$ a $k$-module together with an associative algebra map $U(\mathfrak{g}) \to \text{End}_k(M)$, so the theorem follows. $\Diamond$

**Example 3** There is a $k$-algebra homomorphism $\varepsilon : U(\mathfrak{g}) \to k$, sending $i(\mathfrak{g})$ to zero, called the augmentation. We define the augmentation ideal $\mathfrak{J}$ to be the kernel of $\varepsilon$, which is an ideal of $U(\mathfrak{g})$ generated by $i(\mathfrak{g})$ as left module. Therefore $\mathfrak{J}$ is a $U(\mathfrak{g})$-module and $k \cong U(\mathfrak{g})/\mathfrak{J} = U(\mathfrak{g})_\mathfrak{g}$.  

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Corollary 2 Let \( M \) be a \( \mathfrak{g} \)-module. Then
\[
H^*_\text{Lie}(\mathfrak{g}, M) \cong \text{Tor}^*_U(k, M) \quad \text{and} \quad H^*_\text{Lie}(\mathfrak{g}, M) \cong \text{Ext}^*_U(k, M).
\]

Proof. To show any two derived functors are isomorphic, we only need to show the underlying functors are isomorphic. Observe that
\[
k \otimes_{U(\mathfrak{g})} M \cong U(\mathfrak{g})/\mathfrak{J} \otimes_{U(\mathfrak{g})} M \cong M/\mathfrak{J} M \cong M/\mathfrak{g} M = M_{\mathfrak{g}}
\]
\[
\text{Hom}_{U(\mathfrak{g})}(k, M) = \text{Hom}_{\mathfrak{g}}(k, M) = M_{\mathfrak{g}}. \quad \Box
\]

Example 4 The map \( i : \mathfrak{g} \to U(\mathfrak{g}) \) maps \([\mathfrak{g}, \mathfrak{g}]\) to \( \mathfrak{J}^2 \), hence induced an isomorphism \( i : \mathfrak{g}_{ab} \to \mathfrak{J}/\mathfrak{J}^2 \), where \( \mathfrak{g}_{ab} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \).

Example 5 Let \( M \) and \( N \) be left \( \mathfrak{g} \)-modules. Then \( \text{Hom}_k(M, N) \) is a \( \mathfrak{g} \)-module by \( xf(m) = xf(m) - f(xm), \ x \in \mathfrak{g} \quad m \in M \). This gives us a natural isomorphism \( \text{Hom}_k(M, N) \cong \text{Hom}_k(M, M)^{\mathfrak{g}} \). Extend this we get a natural isomorphism of \( \delta \) functors:
\[
\text{Ext}^*_U(M, N) \cong \text{Hom}^*_\text{Lie}(\mathfrak{g}, \text{Hom}_k(M, M)).
\]

\section{H^1 and H_1}

We have the exact sequence of \( \mathfrak{g} \)-module
\[
0 \to \mathfrak{J} \to U(\mathfrak{g}) \to k \to 0.
\]

If \( M \) is a \( \mathfrak{g} \)-module, applying \( \text{Tor}^*_U(-, M) \) yields
\[
H_n(\mathfrak{g}, M) \cong \text{Tor}^*_n(U(\mathfrak{g}), M) \cong \text{Tor}^*_U(\mathfrak{J}, M) \quad \text{for} \quad n \geq 2
\]
and the exact sequence
\[
0 \to H_1(\mathfrak{g}, M) \to \mathfrak{J} \otimes_{U(\mathfrak{g})} M \to M \to M_{\mathfrak{g}} \to 0. \quad (*)
\]

Theorem 2 For any Lie algebra \( \mathfrak{g} \), \( H_1(\mathfrak{g}, k) \cong \mathfrak{g}_{ab} \).

Proof. Putting \( M = k \) in (*) yields the exact sequence
\[
0 \longrightarrow H_1(\mathfrak{g}, k) \longrightarrow \mathfrak{J} \otimes_{U(\mathfrak{g})} k \longrightarrow k \overset{\sim}{\longrightarrow} k_{\mathfrak{g}} \longrightarrow 0
\]
where
\[
\mathfrak{J} \otimes_{U(\mathfrak{g})} k \cong \mathfrak{J} \otimes_{U(\mathfrak{g})} (U(\mathfrak{g})/\mathfrak{J}) \cong \mathfrak{J}/\mathfrak{J}^2 \cong \mathfrak{g}_{ab} \quad \Box
\]

Corollary 3 If \( M \) is any trivial \( \mathfrak{g} \)-module, \( H_1(\mathfrak{g}, M) \cong \mathfrak{g}_{ab} \otimes_k M \).
Proof. Since $M = M_g$, $(*)$ yields $H_1(g, M) \cong \mathfrak{Z} \otimes_{U(g)} M \cong (\mathfrak{Z} \otimes_{U(g)} k) \otimes_k M \cong (\mathfrak{Z} \otimes_{U(g)} U(g)) / \mathfrak{Z} \otimes_k M \cong \mathfrak{Z} / \mathfrak{Z}^2 \otimes_k M \cong g^{ab} \otimes_k M$. \hfill \diamondsuit

Definition 14 If $M$ is a $g$-module, a derivation from $g$ into $M$ is a $k$-linear map $D : g \to M$ such that $D([x, y]) = x(Dy) - y(Dx)$. The set of all such derivations is denoted $\text{Der}(g, M)$; it is a $k$-submodule of $\text{Hom}_k(g, M)$. Note that if $g = M$, then $\text{Der}(g, g)$ is the derivation algebra $\text{Der}(g)$. If $M$ is a trivial $g$-module, then $\text{Der}(g, M) = \text{Hom}_k(g^{ab}, M)$.

Example 6 If $m \in M$, define $D_m(x) = xm x \in g$ is a derivation, the $D_m$ is called an inner derivations of $g$ into $M$, they form a $k$-subalgebra $\text{Der}_{\text{inn}}(g, M)$ of $\text{Der}(g, M)$.

If $\varphi : \mathfrak{Z} \to M$ is a $g$-homomorphism, let $D_\varphi : g \to M$ be defined by $D_\varphi(x) = \varphi(i(x))$. This is a derivation since

$$D_\varphi([x, y]) = \varphi(i(x)i(y) - i(y)i(x)) = \varphi(x(i(y))) - \varphi(y(i(x))).$$

Lemma 1 The map $\varphi \mapsto D_\varphi$ is a natural isomorphism of $k$-modules:

$$\text{Hom}_g(\mathfrak{Z}, M) \cong \text{Der}(g, M).$$

Proof. The map $\varphi \mapsto D_\varphi$ defines a natural homomorphism, so it suffices to show that it is an isomorphism. Note that $U(g) \otimes_k g \to U(g)g = \mathfrak{Z}$ is onto with kernel generated by the terms $(u \otimes [xy] - ux \otimes y + uy \otimes x)$ with $u \in U(g)$ and $x, y \in g$.

Given a derivation $D : g \to M$, consider the map

$$f : U(g) \otimes_k g \to M \quad f(u \otimes x) = u(D(x)).$$

Since $D$ is a derivation, $f(u \otimes [xy] - ux \otimes y + uy \otimes x) = 0$ for all $u, x$ and $y$. Therefore $f$ induced a map $\varphi : g \to M$, which is a left $g$-module map. Since $D_\varphi(x) = \varphi(i(x)) = f(1 \otimes x) = Dx$ lifts $D$ to a element in $\text{Hom}_g(\mathfrak{Z}, M)$. On the other hand, given $D = D_h$ for some $h \in \text{Hom}_g(\mathfrak{Z}, M)$, we have $\varphi(ux) = u(Dx) = uh(x) = h(ux)$ for all $u \in U(g), x \in g$. Hence $\varphi = h$ as map from $\mathfrak{Z} = U(g)g$ to $M$. \hfill \diamondsuit

Theorem 3 $H^1(g, M) \cong \text{Der}(g, M)/\text{Der}_{\text{inn}}(g, M)$.

Proof. If $\varphi : \mathfrak{Z} \to M$ extends to a $g$-map $U(g) \to M$ sending 1 to $m \in M$, then

$$D_\varphi(x) = \varphi(x \cdot 1) = xm = D_m(x).$$

Hence $D_\varphi$ is an inner derivation. This shows that the image of

$$M \to \text{Hom}_g(\mathfrak{Z}, M) = \text{Der}(g, M)$$

is a submodule of inner derivations, as desired. \hfill \diamondsuit

Corollary 4 If $M$ is trivial $g$-module

$$H^1(g, M) \cong \text{Der}(g, M) \cong \text{Hom}_{\text{Lie}}(g, M) \cong \text{Hom}_k(g^{ab}, M).$$
7 The Chevalley-Eilenberg complex

Let $\mathfrak{g}$ be a Lie algebra over $k$. Denote $\wedge^p \mathfrak{g}$ the $p$-th exterior product of $\mathfrak{g}$, which is generated by monomials $x_1 \wedge ... \wedge x_n$ with $x_i \in \mathfrak{g}$. Define chain complex

$V_p(\mathfrak{g}) = U(\mathfrak{g}) \otimes_k \wedge^p \mathfrak{g}$, which is free as a left $U_\mathfrak{g}$-module. Since $\wedge^0(\mathfrak{g}) = k$ and $\wedge^1 = \mathfrak{g}$, so $V_0 = U(\mathfrak{g})$ and $V_1 = U(\mathfrak{g}) \otimes_k \mathfrak{g}$. We define $\varepsilon : V_0(\mathfrak{g}) = U(\mathfrak{g}) \to k$ $i(\mathfrak{g}) \mapsto 0$ to be the augmentation and $d_1 : V_1(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathfrak{g} \to V_0(\mathfrak{g}) = U(\mathfrak{g})$ to be the product map $d(u \otimes x) = ux$ whose image is the ideal $\mathfrak{g}$. We have the exact sequence

$$V_1(\mathfrak{g}) \xrightarrow{d} V_0(\mathfrak{g}) \xrightarrow{\varepsilon} k \xrightarrow{} 0$$

**Definition 15** For $p \geq 2$, let $d : V_p(\mathfrak{g}) \to V_{p-1}(\mathfrak{g})$ be given by $d(u \otimes x_1 \wedge ... \wedge x_p) = \theta_1 + \theta_2$, where $u \in U(\mathfrak{g})$ $x_i \in \mathfrak{g}$ and

$$\theta_1 = \sum_{i=1}^p (-1)^{i+1}ux_i \otimes x_1 \wedge ... \wedge \hat{x}_i \wedge ... \wedge x_p$$

$$\theta_2 = \sum_{i<j} (-1)^{i+j}u \otimes [x_i, x_j] \wedge x_1 \wedge ... \wedge \hat{x}_i \wedge ... \wedge \hat{x}_j \wedge ... \wedge x_p$$

$V_*(\mathfrak{g})$ with this differential is called Chevalley-Eilenberg complex.

If $p = 2$, then $d(u \otimes x \wedge y) = ux \otimes y - uy \otimes x - u \otimes [xy]$.

**Theorem 4** $V_*(\mathfrak{g}) \to^\varepsilon k$ is a projective resolution of the $\mathfrak{g}$-module $k$.

**Corollary 5** (Chevalley-Eilenberg) If $M$ is a right $\mathfrak{g}$-module, then the homology modules $H_*(\mathfrak{g}, M)$ are the homology of the chain complex

$$M \otimes_{U_\mathfrak{g}} V_*(\mathfrak{g}) \cong M \otimes_{U_\mathfrak{g}} U_\mathfrak{g} \otimes_k \wedge^* \mathfrak{g} \cong M \otimes_k \wedge^* \mathfrak{g}.$$  

If $M$ is a left $\mathfrak{g}$-module, then the cohomology modules $H^*(\mathfrak{g}, M)$ are the cohomology of the cochain complex

$$\text{Hom}_\mathfrak{g}(V_*(\mathfrak{g}), M) = \text{Hom}_\mathfrak{g}(U_\mathfrak{g} \otimes \wedge^* \mathfrak{g}, M) \cong \text{Hom}_k(\wedge^* \mathfrak{g}, M).$$

In this complex, an $n$-cochain $f : \wedge^n \mathfrak{g} \to M$ is just an alternating $k$-multilinear function $f(x_1, ..., x_n)$ of $n$ variables in $\mathfrak{g}$ taking values in $M$. The coboundary of such an $n$-cochain is the $n+1$ cochain

$$df(x_1, ..., x_{n+1}) = \sum (-1)^n x_i f(x_1, ..., \hat{x}_i, ..., x_{n+1})$$

$$+ \sum (-1)^{i+j} f([x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+1}).$$
We thus constructed cochain complex \((C^*(\mathfrak{g}, M), d)\) where \(C^*(\mathfrak{g}, M) = \text{Hom}_k(\mathfrak{g}, M)\). Denote by
\[
Z^p(\mathfrak{g}, M) = \{ f \in C^p(\mathfrak{g}, M) | dc = 0 \}
\]
the space of p-cocycles, and by
\[
B^p(\mathfrak{g}, M) = \{ f \in C^p(\mathfrak{g}, M) | \exists f' \in C^{p-1}(\mathfrak{g}, M) \text{ such that } c = dc' \}
\]
the space of p-coboundaries. Then
\[
H^p_{\text{Lie}}(\mathfrak{g}, M) = Z^p(\mathfrak{g}, M) / B^p(\mathfrak{g}, M).
\]
In particular, we have
\[
Z^1(\mathfrak{g}, M) = \{ c \in C^1(\mathfrak{g}, M) | c([x, y]) = x \cdot c(y) - y \cdot c(x) \} = \text{Der}(\mathfrak{g}, M)
\]
\[
B^1(\mathfrak{g}, M) = \{ c \in C^1(\mathfrak{g}, M) | c(x) = x \cdot m \text{ } m \in M \} = \text{Der}_{\text{Inn}}(\mathfrak{g}, M)
\]

**Reinterpret \(H^0(\mathfrak{g}, M)\) and \(H^1(\mathfrak{g}, M)\)**
\[
H^0(\mathfrak{g}, M) = Z^0(\mathfrak{g}, M) = \{ m \in M | dm = 0 \} = \{ m \in M | x \cdot m = 0 \text{ } \forall x \in \mathfrak{g} \}
\]
\[
H^1(\mathfrak{g}, M) = Z^1(\mathfrak{g}, M) / B^1(\mathfrak{g}, M) = \text{Der}(\mathfrak{g}, M) / \text{Der}_{\text{Inn}}(\mathfrak{g}, M).
\]

**Example 7** Consider \(\mathfrak{sl}(2, \mathbb{C}) = \langle x, y, h \rangle\), where
\[
x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
with bracket relation \([x, y] = h, [h, x] = 2x, [h, y] = -2y\). This shows easily that \(H^1(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})) = 0\), i.e. all derivations are inner, and that \(H^1(\mathfrak{sl}(2, \mathbb{C}), k) = 0\).

**Application** If \(\mathfrak{g}\) is a \(n\)-dimensional over \(k\), then \(H^i(\mathfrak{g}, M) = H_i(\mathfrak{g}, M) = 0 \text{ } \forall i > n\). (\(\land'\mathfrak{g} = 0\))

### 8 \(H^2\) and Extensions

**Definition 16** An extension of Lie algebra of \(\mathfrak{g}\) by \(M\) is a short exact sequence
\[
0 \longrightarrow M \xrightarrow{i} \mathfrak{e}_1 \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0 \text{ in which } M \text{ is an abelian Lie algebra.}
\]
Indeed \(M\) is a \(\mathfrak{g}\)-module via \(x \cdot m = [\tilde{g}, m] \text{ in } \mathfrak{e}\) where \(\tilde{g}\) lift of \(g\) in \(\mathfrak{e}\), \(x \in \mathfrak{g}\) \(m \in M\).
Give $\mathfrak{g}$-module $M$, say two extensions are equivalent if $\exists \varphi : e_1 \to e_2$ isomorphism such that the diagram commutes

$$
\begin{array}{ccc}
0 & \overset{i}{\longrightarrow} & M & \overset{\pi}{\longrightarrow} & \mathfrak{g} & \overset{\varphi}{\longrightarrow} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \overset{i'}{\longrightarrow} & M & \overset{\pi'}{\longrightarrow} & \mathfrak{g} & \overset{\varphi'}{\longrightarrow} & 0
\end{array}
$$

Denote $\text{Ext}(\mathfrak{g}, M)$ the set of equivalent classes of extensions.

**Classification Theorem**

Let $M$ be a $\mathfrak{g}$-module. The set $\text{Ext}(\mathfrak{g}, M)$ of equivalent classes extensions of $\mathfrak{g}$ by $M$ is in 1-1 correspondence with $H^2_{\text{Lie}}(\mathfrak{g}, M)$.

**Sketch of proof.** In one direction, associate to a given 2-cocycle $c$ the extension $e_c = M \oplus \mathfrak{g}$ (as vector space) with the bracket

$$
[(a, x), (b, y)] = (x \cdot b - y \cdot a + c(x, y), [x, y]).
$$

In other direction, choose a linear section $s([x, y]) - [s(x), s(y)]$ of $\pi$. The default of $s$ being a Lie algebra morphism gives a cocycle:

$$
c(x, y) = s([x, y]) - [s(x), s(y)] \quad \forall x, y \in \mathfrak{g}
$$

As $\text{Ker}(\pi) = \text{Im}(i)$, $c$ takes values in $i(M) \cong M \circ$

**9 $H^3$ and Crossed Modules**

**Definition 17** A crossed module of Lie algebras is a homomorphism of Lie algebra $\mu : \mathfrak{m} \to \mathfrak{n}$ together with an action of $\mathfrak{n}$ on $\mathfrak{m}$ by derivations, denoted by $\mathfrak{m} \overset{\mu}{\longrightarrow} \mathfrak{n} \cdot \mathfrak{m}$, such that for all $m, m' \in \mathfrak{m}$ and for all $n \in \mathfrak{n}$

(a) $\mu(n \cdot m) = [n, \mu(m)]$

(b) $\mu \cdot m' = [m, m']$

The requirements (a) and (b) imply that the cokernel of $\mu$, denote by $\mathfrak{g}$, is a Lie algebra, that the kernel of $\mu$, denote by $M$, is a central ideal, and that $\mathfrak{g}$ acts on $M$. Therefore a crossed module corresponds to a 4 term exact sequence of Lie algebras

$$
0 \longrightarrow M \longrightarrow \mathfrak{m} \overset{\mu}{\longrightarrow} \mathfrak{n} \longrightarrow \mathfrak{g} \longrightarrow 0.
$$

**Fact.** $H^3(\mathfrak{g}, M)$ is isomorphic to the set of equivalent classes of crossed modules with cokernel $\mathfrak{g}$ and kernel $M$. 

12
10 Semisimple Lie Algebras

Let \( \mathfrak{g} \) be finite dimensional Lie algebra over \( k \) where \( \text{char} \ k = 0 \).

**Proposition 3** If \( \mathfrak{g} \) is finite dimensional and semisimple, then \( \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \). Consequently, \( H_1(\mathfrak{g}, k) = H^1(\mathfrak{g}, k) = 0 \).

Proof. Since \( \mathfrak{g} = \mathfrak{g}_1 \times \ldots \times \mathfrak{g}_k \) where \( \mathfrak{g}_i \) simple Lie algebra and \( [\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i \) (If \( [\mathfrak{g}_i, \mathfrak{g}_i] = 0 \), then \( \mathfrak{g}_i \subseteq \text{Rad}(\mathfrak{g}) = 0 \)), hence \( [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g} \). This gives us \( \mathfrak{g}^{ab} = 0 \). On the other hand, we saw in Corollary 2 and Corollary 4 that \( H_1(\mathfrak{g}, k) \cong \mathfrak{g}^{ab} \) and \( H^1(\mathfrak{g}, k) \cong \text{Hom}_k(\mathfrak{g}^{ab}, k) \). ∎

The Casimir operator

Let \( \mathfrak{g} \) be a semisimple Lie algebra and let \( M \) a \( m \)-dim \( \mathfrak{g} \)-module, if \( \mathfrak{h} \) be the image of the structure map \( \rho : \mathfrak{g} \to \text{Lie}(\text{End}_k(M)) \cong \text{gl}_m(k) \) then \( \rho \cong \mathfrak{h} \times \ker(\rho) \) \( \mathfrak{h} \subseteq \text{gl}_m(k) \). By Cartan’s criterion, \( tr \) is non-degenerate on \( \mathfrak{h} \). Choose \( e_1, \ldots, e_r \) basis of \( \mathfrak{h} \), then there is a dual basis \( e_1, \ldots, e_r \) dual basis of \( \mathfrak{h} \) with respect to \( tr \). Then the element \( c_M := \sum e_i e^i \in U_\mathfrak{g} \) is called the Casimir operator for \( M \), which is independent of choice of basis.

Recall \( c_M \) has the following properties:

1. If \( x \in \mathfrak{g} \) and \( [e_i, x] = c_{ij} e_j \), then \( [x, e^j] = \sum c_{ij} e^j \).
2. \( c_M \in \text{center}(U(\mathfrak{g})) \) and \( c_M \in \mathfrak{z} \).
3. Image of \( c_M \) in \( \text{End}_k(M) \) is \( \frac{1}{m} \text{id}_M \), where \( r = \text{dim}(\mathfrak{h}) \).

**Theorem 5** Let \( \mathfrak{g} \) be a semisimple over field of characteristic 0. If \( M \) is a simple \( \mathfrak{g} \)-module \( M \neq k \), then

\[
H^i_{\text{Lie}}(\mathfrak{g}, M) = H^i_{\text{Lie}}(\mathfrak{g}, M) = 0 \quad \forall i.
\]

Proof. Let \( C \) be the center of \( U(\mathfrak{g}) \). We saw in Corollary 2 that \( H_*(\mathfrak{g}, M) = \text{Tor}_*^{U(\mathfrak{g})}(k, M) \) and \( H^*(\mathfrak{g}, M) = \text{Ext}^*_U(k, M) \) are naturally \( C \)-modules. Multiplication by \( c \in C \) is induced by \( c : k \to k \) as well as \( c : M \to M \). Since the Casimir element \( c_M \) acts by 0 on \( k \) (as \( c_M \in \mathfrak{z} \)) and by the invertible scalar \( r/m \) on \( M \), we must have \( 0 = r/m \) on \( H_*(\mathfrak{g}, M) \) and \( H^*(\mathfrak{g}, M) \). This can only happen if they are zero. ∎

**Corollary 6** (Whitehead’s first lemma) Let \( \mathfrak{g} \) be a semisimple Lie algebra over field of characteristic 0. If \( M \) is any finite dimensional \( \mathfrak{g} \)-module, then \( H^1_{\text{Lie}}(\mathfrak{g}, M) = 0 \). That is, \( \text{Der}(\mathfrak{g}, M) = \text{Der}_{\text{Inn}}(\mathfrak{g}, M) \).
Proof. Induction on $\dim(M)$. If $M$ is simple, then either $M = k$ and $H^1(\mathfrak{g}, k) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = 0$ or else $M \neq k$ and $H^*(\mathfrak{g}, M) = 0$ by the theorem. Otherwise, $M$ contains a proper submodule $L$. By induction, $H^1(\mathfrak{g}, L) = H^1(\mathfrak{g}, M/L) = 0$. Since $0 \to L \to M \to M/L \to 0$ induces exact sequence of cohomology,

$$\ldots H^1(\mathfrak{g}, L) \to H^1(\mathfrak{g}, M) \to H^1(\mathfrak{g}, L/M) \ldots$$

We have $H^1(\mathfrak{g}, M) = 0$. By Theorem 3, $H^1(\mathfrak{g}, M) \cong \text{Der}(\mathfrak{g}, M)/\text{Der}_{\text{inn}}(\mathfrak{g}, M)$.

\[\Box\]

**Theorem 6 (Weyl’s Theorem)** Let $\mathfrak{g}$ be a semisimple Lie algebra over field of characteristic 0. Then every finite dimensional $\mathfrak{g}$-module $M$ is completely reducible i.e. direct sum of $\mathfrak{g}$-module.

Proof. Suppose $M$ is not a direct sum of simple modules. Since $\dim(M)$ is finite, $M$ contains a submodule $M_1$ minimal with respect to this property. Clearly $M_1$ is not simple, so it contains a proper $\mathfrak{g}$-submodule $M_0$. By induction, both $M_0$ and $M_2 = M_1/M_0$ are direct sums of simple $\mathfrak{g}$-modules but $M_1$ is not, so the extension of $M_2$ by $M_0$ is not split. Hence it is represented by a nonzero element of

$$\text{Ext}^1_{U(\mathfrak{g})}(M_2, M_0) \cong H^1_{\text{Lie}}(\mathfrak{g}, \text{Hom}_k(M_2, M_0))$$

by Example 5 and this contradicts Whitehead’s first lemma. \[\Box\]

**Corollary 7 (Whitehead’s Second lemma)** Let $\mathfrak{g}$ be a semisimple Lie algebra over field of characteristic 0. If $M$ is any finite dimensional $\mathfrak{g}$-module, then $H^2_{\text{Lie}}(\mathfrak{g}, M) = 0$.

Proof. Since $H^* \text{ commutes with direct sums, and } M$ is a direct sum of simple $\mathfrak{g}$-modules by Weyl’s theorem, we may assume that $M$ is simple. If $M \neq k$, then $H^2_{\text{Lie}}(\mathfrak{g}, M) = 0$ by Theorem 5, so it suffices to show that $H^2(\mathfrak{g}, k) = 0$. That is, every Lie algebra extension

$$0 \longrightarrow k \longrightarrow \mathfrak{e} \longrightarrow \mathfrak{g} \longrightarrow 0$$

splits. We claim that $\mathfrak{e}$ can be made into a $\mathfrak{g}$-module in such a way that $\pi$ is a $\mathfrak{g}$-map. To see this, let $\tilde{x}$ be any lift of $x \in \mathfrak{g}$ to $\mathfrak{e}$ and define $x \cdot e$ to be $[\tilde{x}, e]$ for $e \in \mathfrak{e}$. This is independent of choice of $\tilde{x}$ because $k$ is in the center of $\mathfrak{e}$. This defines a $\mathfrak{g}$-module, and by construction $\pi(x \cdot e) = [x, \pi(e)]$.

By Weyl’s theorem, $\mathfrak{e}$ and $\mathfrak{g}$ splits as $\mathfrak{g}$-modules, and there is a $\mathfrak{g}$-module homomorphism $\tau : \mathfrak{g} \to \mathfrak{e}$ splitting $\pi$ such that $\mathfrak{e} \cong k \times \mathfrak{g}$ as a $\mathfrak{g}$-module. If we choose $\tilde{x} = \tau(x)$, then it is clear that $\tau$ is a Lie algebra homomorphism and that $\mathfrak{e} \cong k \times \mathfrak{g}$ as a Lie algebra. This proves that $H^2(\mathfrak{g}, k) = 0$. \[\Box\]
Theorem 7 (Levi-Malcev) If \( g \) is a finite dimensional Lie algebra over a field of characteristic zero, then there is a semisimple Lie subalgebra \( \mathfrak{sl} \) of \( g \) (called a Levi factor of \( g \)) such that \( g \) is isomorphic to the semidirect product

\[
\mathfrak{g} \cong (\text{rad}\mathfrak{g}) \rtimes \mathcal{L}.
\]

Proof. Since \( \mathfrak{g}/\text{rad} \mathfrak{g} \) is semisimple, it suffices to show that the following Lie algebra extension splits.

\[
0 \to \text{rad} \mathfrak{g} \to \mathfrak{g} \to \mathfrak{g}/\text{rad} \mathfrak{g} \to 0
\]

If \( \text{rad} \mathfrak{g} \) is abelian, then this extension is classified by \( H^2(\mathfrak{g}/\text{rad} \mathfrak{g}, \text{rad} \mathfrak{g}) \), which vanishes by Whitehead’s second lemma, so every extension splits.

If \( \text{rad} \mathfrak{g} \) is not abelian, we proceed by induction on the derived length of \( \text{rad} \mathfrak{g} \). Let \( \mathfrak{r} \) denote the ideal \( [\mathfrak{g}, \mathfrak{g}] \) of \( \mathfrak{g} \). Since \( \text{rad} \mathfrak{g}/\mathfrak{r} = \text{rad} \mathfrak{g}/\mathfrak{r} \) is abelian, the extension

\[
0 \to (\text{rad} \mathfrak{g})/\mathfrak{r} \to \mathfrak{g}/\mathfrak{r} \to \mathfrak{g}/(\text{rad} \mathfrak{g}) \to 0
\]

splits. Hence there is an ideal \( \mathfrak{h} \) of \( \mathfrak{g} \) containing \( \mathfrak{r} \) such that \( \mathfrak{g}/\mathfrak{r} \cong (\text{rad} \mathfrak{g})/\mathfrak{r} \times \mathfrak{h}/\mathfrak{r} \) and \( \mathfrak{h}/\mathfrak{r} \cong \mathfrak{g}/(\text{rad} \mathfrak{g}) \). Now \( \text{rad} \mathfrak{h} = \text{rad} \mathfrak{g} \cap \mathfrak{h} = \mathfrak{r} \), and \( \mathfrak{r} \) has a smaller derived length than \( \text{rad} \mathfrak{g} \). By induction there is a Lie subalgebra \( \mathcal{L} \) of \( \mathfrak{h} \) such that \( \mathfrak{h} \cong \mathfrak{r} \times \mathcal{L} \) and \( \mathcal{L} \cong \mathfrak{h}/\mathfrak{r} \cong \mathfrak{g}/(\text{rad} \mathfrak{g}) \). Then \( \mathcal{L} \) is our desired Levi factor of \( \mathfrak{g} \).

References